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VARIATIONAL APPROACH AND THE NUMBER OF THE NONTRIVIAL PERIODIC SOLUTIONS FOR A CLASS OF THE SYSTEM OF THE NONTRIVIAL SUSPENSION BRIDGE EQUATIONS

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ABSTRACT. We investigate the multiplicity of the nontrivial periodic solutions for a class of the system of the nonlinear suspension bridge equations with Dirichlet boundary condition and periodic condition. We show that the system has at least two nontrivial periodic solutions by the abstract version of the critical point theory on the manifold with boundary. We investigate the geometry of the sublevel sets of the corresponding functional of the system and the topology of the sublevel sets. Since the functional is strongly indefinite, we use the notion of the suitable version of the Palais-Smale condition.

1. INTRODUCTION

In this paper we investigate the multiplicity of the nontrivial periodic solutions for a class of the system of the nonlinear suspension bridge equations with Dirichlet boundary condition and periodic condition

$$
\begin{aligned}
(1.1)\qquad & (u_1)_{tt} + (u_1)_{xxxx} - F_{r_1}(x,t,u_1,\dots,u_n) = 0 \qquad \text{in } (-\tfrac{\pi}{2},\tfrac{\pi}{2}) \times R,\\
& (u_2)_{tt} + (u_2)_{xxxx} - F_{r_2}(x,t,u_1,\dots,u_n) = 0 \qquad \text{in } (-\tfrac{\pi}{2},\tfrac{\pi}{2}) \times R,\\
& \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
& (u_n)_{tt} + (u_n)_{xxxx} - F_{r_n}(x,t,u_1,\dots,u_n) = 0 \qquad \text{in } (-\tfrac{\pi}{2},\tfrac{\pi}{2}) \times R,\\
& u_i(\pm\tfrac{\pi}{2},t) = (u_i)_{xx}(\pm\tfrac{\pi}{2},t) = 0, \qquad i = 1,\dots,n,\\
& u_i(x,t) = u_i(-x,t) = u_i(x,-t) = u_i(x,t+\pi), \qquad i = 1,\dots,n,
\end{aligned}
$$

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where $F : (-\frac{\pi}{2}, \frac{\pi}{2}) \times R \times R^n \to R$ is a differentiable function with $F(x, t, 0, \dots, 0) = 0$, $F_x(x, t, 0, \dots, 0) = 0$ and $F_t(x, t, 0, \dots, 0) = 0$, and

$$F_{r_i}(x, t, r_1, \dots, r_n) = \frac{\partial F}{\partial r_i}(x, t, r_1, \dots, r_n).$$

Let $u = (u_1, \dots, u_n)$. We assume that F satisfies the following conditions:

(F1) $\displaystyle\lim_{(u_1,\dots,u_n)\to(0,\dots,0)} \frac{F_{r_i}(x,t,u)}{|u_1| + \dots + |u_n|} = 0.$

(F2) $\displaystyle\lim_{|u_1|+\dots+|u_n|\to\infty} \frac{F_{r_i}(x,t,u)}{|u_1| + \dots + |u_n|} = \infty,\ i = 1, \dots, n.$

(F3) $u \cdot F_u(x,t,u) \geq \mu F(x,t,u)\ \forall x,\ t,\ \mu > 2$;

(F4) $|F_{r_1}(x,t,r_1,\dots,r_n)| + \dots + |F_{r_n}(x,t,r_1,\dots,r_n)| \leq \gamma(|r_1|^\nu + \dots + |r_n|^\nu)\ \forall x,$ $t, r_1, \dots, r_n,\ \gamma > 0,\ \nu > 1,\ i = 1, \dots, n.$

As the physical model for these systems we can find crossing n beams with travelling waves supported by cables with a load f as follows:

$$u_{tt} + u_{xxxx} = bu^2 + f(x,t) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0.$$

Choi and Jung ([3],[4],[5]) investigate the existence and multiplicity of solutions for the single nonlinear suspension bridge equation with Dirichlet boundary condition.

Let $u = (u_1, \dots, u_n)$ and

$$F_u(x,t,u) = (F_{u_1}(x,t,u_1,\dots,u_n), \dots, F_{u_n}(x,t,u_1,\dots,u_n))$$

and $|\cdot|$ denote the Euclidean norm in R^n. System (1.1) can be rewritten by

$$\begin{cases} u_{tt} + u_{xxxx} &= F_u(x,t,u), \\ u(\pm\frac{\pi}{2}, t) &= u_{xx}(\pm\frac{\pi}{2}, t) = (0, \dots, 0), \\ u(x, t+\pi) &= u(x,t) = u(-x,t) = u(x,-t), \end{cases} \tag{1.2}$$

where $u_{tt} + u_{xxxx} = ((u_1)_{tt} + (u_1)_{xxxx}, \dots, (u_n)_{tt} + (u_n)_{xxxx})$.

The main result of this paper is the following:

Theorem 1.1. *Assume that the nonlinear term F satisfies the conditions (F1) – (F4). Then system (1.1) has at least two nontrivial periodic solutions.*

As well known the solutions of system (1.1) coincide with the critical points of the functional $I : H \to R \in C^{1,1}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega [-|u_t|^2 + |u_{xx}|^2] dxdt - \int_\Omega F(x,t,u) dxdt, \tag{1.3}$$

where $u = (u_1, \dots, u_n)$, $-|u_t|^2 + |u_{xx}|^2 = \sum_{i=1}^n (-|(u_i)_t|^2 + |(u_i)_{xx}|^2)$. $n \geq 1$, and the space H is introduced in section 2. For the proof of Theorem 1.1 we use a variational method and an abstract version of critical point theory on the manifold with boundary. In the proof we study the geometry and topology of the sub-level sets of I. Since the functional is strongly indefinite, we use the notion of the suitable version of the Palais-Smale condition.

The proof of Theorem 1.1 is organized as follows: In section 2, we approach the variational method for strongly indefinite functional, obtain some results on F and recall the abstract version of the critical point theory on the manifold. In section 3, we prove Theorem 1.1.

2. Variational Approach for the Strongly Indefinite Functional

The eigenvalue problem

$$v_{tt} + v_{xxxx} = \lambda v \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R. \tag{2.1}$$

$$v(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0,$$

$$v(x,t) = v(-x,t) = v(x,-t) = v(x,t+\pi)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions ϕ_{mn}, m, $n > 0$, given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for } \; n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi}\cos 2mt\cos(2n+1)x \qquad \text{for } \; m > 0, n \geq 0.$$

Let Ω be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and E' the Hilbert space defined by

$$E' = \{v \in L^2(\Omega) | \; v \text{ is even in } x \text{ and } t, \; \int_Q v = 0 \; \}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in E'. Let us denote an element v, in E', as

$$v = \sum h_{mn}\phi_{mn}$$

and we define a subspace E of E' as

$$E = \{v \in E' | \; \sum |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

This is a complete normed space with a norm $\|v\| = \left[\sum |\lambda_{mn}| h_{mn}^2\right]^{\frac{1}{2}}$.

Let H be the n cartesian product space of E, i.e.,

$$H = E \times E \times \ldots \times E.$$

The norm in H is given by

$$\|u\|^2 = \|P^+u\|^2 + \|P^-u\|^2, \qquad u = (u_1, \ldots, u_n)$$

where $\|P^+u\|^2 = \sum_{i=1}^n \|P^+u_i\|^2$, $\|P^-u\|^2 = \sum_{i=1}^n \|P^-u_i\|^2$. Let H^+ and H^- be the subspaces of H on which the functional

$$u \mapsto Q(u) = \int_\Omega [-|u_t|^2 + |u_{xx}|^2] dxdt, \qquad u = (u_1, \ldots, u_n)$$

is positive definite and negative definite, respectively. Then

$$H = H^+ \oplus H^-.$$

Let P^+ be the projection from H onto H^+ and P^- the projection from H onto H^-. The functional $I(u)$ can be rewritten by

$$(2.2) \quad I(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 - \int_\Omega F(x,t,u)dxdt = \frac{1}{2}Q(u) - \int_\Omega F(x,t,u)dxdt.$$

Let $(H_n)_n$ be a sequence of closed finite dimensional subspace of H with the following assumptions: $H_n = H_n^- \oplus H_n^+$ where $H_n^+ \subset H^+$, $H_n^- \subset H^-$ for all n (H_n^+ and H_n^- are subspaces of H), $\dim H_n < +\infty$, $H_n \subset H_{n+1}$, $\cup_{n \in N} H_n$ is dense in H.

Since each eigenvalue has a finite multiplicity and $|\lambda_{mn}| \geq 1$ for all m, n, we have some properties for a single equation:

Lemma 2.1. (i) $\|u\| \geq \|u\|_{L^2(\Omega)}$, *where* $\|u\|_{L^2(\Omega)}$ *denotes the* L^2 *norm of* u.
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(\Omega)} = 0$.
(iii) $u_{tt} + u_{xxxx} \in E$ *implies* $u \in E$.

Lemma 2.2. *Suppose that* c *is not an eigenvalue of* L, $Lu = u_{tt} + u_{xxxx}$, *and let* $f \in E'$. *Then we have* $(L-c)^{-1} f \in E$.

Proof. When n is fixed, we define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 8n^2 + 8n + 1,$$
$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$f = \sum h_{mn}\phi_{mn}.$$

Then

$$(L-c)^{-1}f = \sum \frac{1}{\lambda_{mn}-c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L-c)^{-1}f\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}-c)^2} h_{mn}^2 \leq C \sum h_{mn}^2$$

for some C, which means that

$$\|(L-c)^{-1}f\| \leq C_1 \|f\|_{L^2(\Omega)}, \qquad C_1 = \sqrt{C}.$$

□

Now we return to the case of the system. By the following Proposition 2.1, the weak solutions of system (1.1) coincide with the critical points of the associated functional I.

Proposition 2.1. *Assume that F satisfies nthe conditions $(F1)$-$(F4)$. Then the functional $I(u)$ is continuous, Frèchet differentiable in H with Frèchet derivative*

$$\nabla I(u)v = \int_Q [(u_{tt} + u_{xxxx}) \cdot v - F_u(u) \cdot v] dxdt. \tag{2.3}$$

Moreover $\nabla I \in C$. That is $I \in C^1$.

Proof. For $u, v \in H$,

$$\begin{aligned}
&|I(u+v) - I(u) - \nabla I(u)v| \\
&= \left| \frac{1}{2} \int_\Omega (u_{tt} + u_{xxxx} + v_{tt} + v_{xxxx}) \cdot (u+v) dxdt - \int_\Omega F(u+v) dxdt \right. \\
&\quad \left. - \frac{1}{2} \int_\Omega (u_{tt} + u_{xxxx}) \cdot u dxdt + \int_\Omega F(u) dxdt - \int_\Omega (u_{tt} + u_{xxxx} - F_u(u)) \cdot v dxdt \right| \\
&= \left| \frac{1}{2} \int_\Omega [(u_{tt} + u_{xxxx}) \cdot v + (v_{tt} + v_{xxxx}) \cdot u + (v_{tt} + v_{xxxx}) \cdot v] dxdt \right. \\
&\quad \left. - \int_\Omega [F(u+v) - F(u)] dxdt - \int_\Omega [(u_{tt} + u_{xxxx} - F_u(u)) \cdot v] dxdt \right|.
\end{aligned}$$

We have

$$\left| \int_\Omega [F(u+v) - F(u)] dxdt \right| \leq \left| \int_\Omega [F_u(u) \cdot v + o(|v|)] dxdt \right| = O(|v|). \tag{2.4}$$

Thus we have

$$|I(u+v) - I(u) - \nabla I(u)v| = O(|v|^2). \tag{2.5}$$

Next we prove that $I(u)$ is continuous. For $u, v \in H$,

$$\begin{aligned}|I(u+v)-I(u)| &= \Big|\frac{1}{2}\int_\Omega (u_{tt}+u_{xxxx}+v_{tt}+v_{xxxx})\cdot(u+v)dxdt - \int_\Omega F(u+v)dxdt \\ &\quad - \frac{1}{2}\int_\Omega (u_{tt}+u_{xxxx})\cdot u dxdt + \int_\Omega F(u)dxdt\Big| \\ &= \Big|\frac{1}{2}\int_\Omega [(u_{tt}+u_{xxxx})\cdot v + (v_{tt}+v_{xxxx})\cdot u + (v_{tt}+v_{xxxx})\cdot v]dxdt \\ &\quad - \int_\Omega (F(u+v)-F(u))dxdt\Big| = O(|v|).\end{aligned}$$

Similarly, it is easily checked that I is C^1. □

Proposition 2.2. *Assume that F satisfies the conditions (F1)-(F4). Then there exist $a_0 > 0$, $b_0 \in R$ and $\mu > 2$ such that*

$$F(x,t,u) \geq a_0|u|^\mu - b_0, \qquad \forall x, t, u. \tag{2.6}$$

Proof. Let $u \in H$ be such that $|u|^2 \geq R^2$. Let us set $\varphi(\xi) = F(x,t,\xi u)$ for $\xi \geq 1$. Then

$$\varphi(\xi)' = u \cdot F_u(x,t,\xi u) \geq \frac{\mu}{\xi}\varphi(\xi).$$

Multiplying by $\xi^{-\mu}$, we get

$$(\xi^{-\mu}\varphi(\xi))' \geq 0,$$

hence $\varphi(\xi) \geq \varphi(1)\xi^\mu$ for $\xi \geq 1$. Thus we have

$$F(x,t,u) \geq F\left(x,t,\frac{R|u|}{\sqrt{|u|^2}}\right)\left(\frac{\sqrt{|u|^2}}{R}\right)^\mu \geq c_0\left(\frac{\sqrt{|u|^2}}{R}\right)^\mu \geq a_0|u|^\mu - b_0,$$

for some a_0, b_0, where $c_0 = \inf\{F(x,t,u)|\ (x,t) \in \Omega,\ |u|^2 = R^2\}$. □

Proposition 2.3. *Assume that F satisfies the conditions $(F1)$-$(F4)$. Then if $\|u_n\| \to +\infty$ and*

$$\frac{\int_\Omega u_n \cdot F_u(x,t,u_n)dxdt - 2\int_\Omega F(x,t,u_n)dxdt}{\|u_n\|} \to 0,$$

then there exist $(u_{h_n})_n$ and $w \in H$ such that

$$\frac{grad\Big(\int_\Omega F(x,t,u_{h_n})dxdt\Big)}{\|u_{h_n}\|} \to w \ \text{and}\ \frac{u_{h_n}}{\|u_{h_n}\|} \rightharpoonup (0,\dots,0).$$

Proof. By (F3) and Proposition 2.2, for $u \in H$,

$$\int_\Omega [u \cdot F_u(x,t,u)]dxdt - 2\int_\Omega F(x,t,u)dxdt$$
$$\geq (\mu - 2)\int_\Omega F(x,t,u)dxdt \geq (\mu-2)(a_0\|u\|^\mu_{L^\mu} - b_1).$$

By (F4),

$$\left\|\text{grad}\Big(\int_\Omega F(x,t,u)dxdt\Big)\right\| \leq C'\||u|^\nu\|_{L^r}$$

for $r > 1$ and suitable constants C'. To get the conclusion it suffices to estimate $\|\frac{|u|^\nu}{\|u\|}\|_{L^r}$ in terms of $\frac{\|u\|^\mu_{L^\mu}}{\|u\|}$. If $\mu \geq r\nu$, then this is an consequence of *Hölder* inequality. If $\mu < r\nu$, by the standard interpolation arguments, it follows that

$$\left\|\frac{|u|^\nu}{\|u\|}\right\|_{L^r} \leq C\Big(\frac{\|u\|^\mu_{L^\mu}}{\|u\|}\Big)^{\frac{\nu}{\mu}}\|u\|^l,$$

where l is such that $l = -1 + \frac{\nu}{\mu}$. Thus we prove the proposition. □

We need the following multiplicity result in [12], which will be used in the proof of our main theorem.

Theorem 2.1. *Let H be a Hilbert space and let $H = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of H with X_2 of finite dimension. Moreover let $(H_n)_n$ be a sequence of closed subspaces of H with finite dimension and such that for all n,*

$$X_2 \subset H_n, \qquad P_{X_i} \circ P_{H_n} = P_{H_n} \circ P_{X_i} (= P_{X_i \cap H_n}),\ i = 1,2,3,$$

where, for a given subspace X of H, P_X is the orthogonal projection from H onto X. Set

$$C = \{x \in H |\ \|P_{X_2}x\| \geq 1\}$$

and let $f : W \to R$ be a $C^{1,1}$ function defined on a neighborhood W of C. Let $1 < \rho < R$, $R_1 > 0$, we define

$$\Delta_{12} = \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, 1 < \|x_2\| < R\},$$
$$\Sigma_{12} = \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = 1\}$$
$$\cup \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = R\}$$
$$\cup \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| = R_1, 1 \leq \|x_2\| \leq R\},$$
$$S_{23} = \{x \in X_2 \oplus X_3 |\ \|x\| = \rho\}.$$

Let

$$\alpha = \inf f(S_{23}), \qquad \beta = \sup f(\Delta_{12}).$$

(i) *Assume that*

$$\sup f(\Sigma_{12}) < \inf f(S_{23}).$$

(ii) *Assume that the* $(P.S.)^*_c$ *condition holds for* f *on* C, *with respect to the sequence* $(C_n)_n$, $\forall c \in [\alpha, \beta]$.

(iii) *Assume that* $f|_{X_1 \oplus X_3}$ *has no critical points with* $\alpha \le f(u) \le \beta$.

(iv) *Assume* $\beta < +\infty$.

Then there exist two lower critical points u_1, u_2 *for* f *on int* (C) *such that*

$$\inf f(S_{23}) \le f(u_i) \le \sup f(\Delta_{12}), \quad i = 1, 2.$$

3. Proof of Theorem 1.1

Let us set

$$\begin{aligned} B_r &= \{u \in H^+ |\ \|u\| \le r\} \subset H^+, \\ S_r &= \{u \in H^+ |\ \|u\| = r\} \subset H^+, \\ S(\bar{\rho}) &= \{u \in H^+ |\ \|u\| = \bar{\rho}\} \subset H^+, \\ \Delta_R(S(\bar{\rho}), H^-) &= \{u_1 + u_2 |\ u_1 \in H^-, u_2 \in S(\bar{\rho}) \subset H^+, \bar{\rho} > 0, \|u_1 + u_2\| \le R\}, \\ \Sigma_R(S(\bar{\rho}), H^-) &= \{u_1 + u_2 |\ u_1 \in H^-, u_2 \in S(\bar{\rho}) \subset H^+, \bar{\rho} > 0, \|u_1 + u_2\| = R\} \\ &\quad \cup \{u_1 |\ \|u_1\| \le R, u_2 \in S(\bar{\rho}\}. \end{aligned}$$

We have the following variation linking inequality:

Lemma 3.1. *Assume that* F *satisfies the conditions* $(F1) - (F4)$ *and let* $Y = H_n^+$ *be any closed subspace of* H^+. *Then there exist* $\bar{\rho}$, $R > 0$ *and* r *with* $R > r$ *such that*

$$\sup_{u \in \Sigma_R(S(\bar{\rho}), H^-)} I(u) < 0 < \inf_{\substack{u \in H^+ \\ u \in S_r}} I(u) \qquad \text{and}$$

$$\inf_{\substack{u \in H^+ \\ u \in B_r}} I(u) > -\infty, \qquad \sup_{w \in \Delta_R(S(\bar{\rho}), H^-)} I(w) < \infty,$$

where $S(\bar{\rho}) = \{u |\ \|u\| = \bar{\rho}\} \subset Y$ *and* $S_r = \{u |\ \|u\| = r\} \subset H^+$.

Proof. First we will prove that there exists B_r with radius $r > 0$ and $B_r \cap S(\bar{\rho}) \ne \emptyset$ such that $\inf_{\substack{u \in H^+ \\ u \in S_r = \partial B_r}} I(u) > 0$. Let $u \in H^+$. Then we have that $\|P^- u\| = 0$ and

$$I(u) = \frac{1}{2}\|P^+ u\|^2 - \int_\Omega F(x, t, u) dx dt.$$

By $(F1)$ and $(F4)$, $F(x, t, u_1, \ldots, u_n) \le a|u|^\beta$, $a > 0$ and $\beta > 2$. So we have

$$I(u) \geq \frac{1}{2}\|P^+u\|^2 - a\|u\|^{\beta}_{L^2(\Omega)}.$$

Since $\beta > 2$, there exists a small sphere $S_r = \partial B_r$ with radius r contained in H^+ such that for $u \in S_r$, $\inf_{\substack{u\in H^+\\ u\in S_r}} I(u) > 0$ and $\inf_{\substack{u\in H^+\\ u\in B_r}} I(u) > -a\|u\|^{\beta}_{L^2(\Omega)} > -\infty$. Next, we will prove that there exist $\bar{\rho}$, $R > 0$ and $r > 0$, $R > r$ such that $B_r \cap S(\bar{\rho}) \neq \emptyset$ and $\sup_{u\in\Sigma_R(S(\bar{\rho}),H^-)} I(u) < 0$. Let $u \in H^- \oplus H^+$, $P^+u \in S(\bar{\rho}) \subset Y \subset H^+$ and $\bar{\rho}$ is a small number. Then we have

$$I(u) = \frac{1}{2}\bar{\rho}^2 - \frac{1}{2}\|P^-u\|^2 - \int_{\Omega} F(x,t,u)dxdt.$$

By Proposition 2.2, there exist $a_0 > 0$, $b_0 \in R$ and $\mu > 2$ such that $F(x,t,u) \geq a_0|u|^{\mu} - b_0$, $\forall x$, t, u. Thus we have

$$I(u) \leq \frac{1}{2}\bar{\rho}^2 - \frac{1}{2}\|P^-u\|^2 - a_0\|u\|^{\mu}_{L^2(\Omega)} + b_0\pi^2.$$

Since $\mu > 2$, there exist $R > \bar{\rho}$ such that if $u \in \Sigma_R(S(\bar{\rho}), H^-)$, $I(u) < 0$. Thus we have $\sup_{u\in\Sigma_R(S(\bar{\rho}),H^-)} I(u) < 0$. Moreover if $u \in \Delta_R$, then $I(u) \leq< \frac{1}{2}\bar{\rho}^2 + b_0\pi^2 < \infty$. Thus $\sup_{u\in\Delta_R} I(u) < \infty$. □

Let $Y = H_n^+$ for some n, and denote by P_Y the orthogonal projection from H onto Y. Let

$$C = \{w \in H | \ \|P_Y w\| \geq 1\}.$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap H_n$. Let us define a functional $\Psi : Y \to H$ by

$$\Psi(w) = w - \frac{P_Y w}{\|P_Y w\|} = P_{H^-\oplus(H^+\setminus Y)}w + \left(1 - \frac{1}{\|P_Y w\|}\right)P_Y w. \tag{3.1}$$

We have

$$\nabla\Psi(w)(z) = z - \frac{1}{\|P_Y w\|}\left(P_Y z - \langle\frac{P_Y w}{\|P_Y w\|}, z\rangle\frac{P_Y w}{\|P_Y w\|}\right). \tag{3.2}$$

Let us define the functional $\tilde{I} : C \to W$ by

$$\tilde{I} = I \circ \Psi.$$

Then $\tilde{I} \in C^{1,1}_{loc}$. We note that if $\tilde{u}$ is the critical point of $\tilde{I}$ and lies in the interior of C, then $u = \Psi(\tilde{u})$is the critical point of I. We also note that

$$\|grad^-_C\tilde{I}(\tilde{u})\| \geq \|P_{H^-\oplus(H^+\setminus Y)}\nabla I(\Psi(\tilde{u}))\| \qquad \forall\tilde{u} \in \partial C. \tag{3.3}$$

Lemma 3.2. *Assume that F satisfies the condition $(F1)-(F4)$. Then $\tilde{I}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$ for every real number c such that*

$$\inf_{\tilde{w}\in\Psi^{-1}(S_r)} \tilde{I}(\tilde{w}) \le c \le \sup_{\tilde{w}\in\Psi^{-1}(\Delta_R(S(\bar{\rho})),H^-)} \tilde{I}(\tilde{w}).$$

Proof. Let $(k_n)_n$ be a sequence such that $k_n \to +\infty$, $(\tilde{w_n})_n$ be a sequence in C such that $\tilde{w_n} \in C_{k_n}$, $\forall n$, $\tilde{I}(\tilde{w_n}) \to c$ and $grad^-_C \tilde{I}|_{H_{k_n}}(\tilde{w_n}) \to 0$. Set $w_n = \Psi(\tilde{w_n})$, then $w_n \in H_{k_n}$ and $I(w_n) \to c$. We first consider the case in which $w_n \notin H^- \oplus (H^+\backslash Y)$, $\forall n$. Since for n large $P_{H_n} \circ P_Y = P_Y \circ P_{H_n} = P_Y$, we have

$$P_{H_{k_n}} \nabla\tilde{I}(\tilde{w_n}) = P_{H_{k_n}} \Psi'(\tilde{w_n})(\nabla I(w_n)) = \Psi'(\tilde{w_n})(P_{H_{k_n}} \nabla I(w_n)) \longrightarrow 0.$$

By (3.2),

$$P_{H_{k_n}} w_n \to 0 \qquad \text{or}$$

$$P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \nabla I(w_n) \to 0 \quad \text{and} \quad P_Y w_n \to 0.$$

In the first case the claim follows from the limit Palais-Smale condition for I. In the second case $P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \nabla I(w_n) \to 0$. We claim that $(w_n)_n$ is bounded. By contradiction, we suppose that $\|w_n\| \to +\infty$ and set $z_n = \frac{w_n}{\|w_n\|}$. Up to a subsequence $z_n \rightharpoonup z_0$ weakly for some $z_0 \in H^- \oplus (H^+\backslash Y)$. By the asymptotically linearity of $\nabla I(w_n)$ we have

$$\langle \nabla I(w_n), \frac{w_n}{\|w_n\|}\rangle = \langle P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \nabla I(w_n), \frac{w_n}{\|w_n\|}\rangle + \langle \frac{\nabla I(w_n)}{\|w_n\|}, P_Y w_n\rangle \longrightarrow 0.$$

We have

$$\begin{aligned}&\langle \nabla I(w_n), \frac{w_n}{\|w_n\|}\rangle \\ &= \frac{2I(w_n)}{\|w_n\|} + \int_\Omega \left[\frac{2F(x,t,w_n)}{\|w_n\|} - \frac{(F_{r_1}(x,t,w_n),\dots,F_{r_n}(x,t,w_n))\cdot w_n}{\|w_n\|}\right] dxdt,\end{aligned}$$

where $w_n = ((w_n)_1,\dots,(w_n)_n)$, $F_{r_i} = \frac{\partial F}{\partial r_i}$. Passing to the limit we get

$$\lim_{n\to\infty} \int_\Omega \left[\frac{2F(x,t,w_n)}{\|w_n\|} - \frac{(F_{r_1}(x,t,w_n),\dots,F_{r_n}(x,t,w_n))\cdot w_n}{\|w_n\|}\right] dxdt = 0.$$

By Proposition 2.3,

$$\frac{\text{grad} \displaystyle\int_\Omega F(x,t,w_n)dxdt}{\|w_n\|}$$

converges and $z_n \rightharpoonup 0$. Moreover we have

$$\begin{aligned}
&\langle P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \frac{\nabla I(w_n)}{\|w_n\|}, z_n \rangle \\
&= P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \int_\Omega \left[\frac{-(w_n)_t \cdot (z_n)_t + (w_n)_{xx} \cdot (z_n)_{xx}}{\|w_n\|} \right. \\
&\left. - \int_\Omega \frac{(F_{r_1}(x,t,w_n), \dots, F_{r_n}(x,t,w_n)) \cdot z_n}{\|w_n\|} \right] dxdt \\
&= P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \int_\Omega \left[-|z_n|_t^2 + |z_n|_{xx}^2 - \frac{(F_{r_1}(x,t,w_n), \dots, F_{r_n}(x,t,w_n)) \cdot z_n}{\|w_n\|} \right] dxdt.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&\langle P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \frac{\nabla I(w_n)}{\|w_n\|}, P^+ z_n - P^- z_n \rangle \\
&= \|P_{H^+\backslash Y} z_n\|^2 + \|P_{H^-} z_n\|^2 + \int_\Omega \frac{(F_{r_1}(x,t,w_n), \dots, F_{r_n}(x,t,w_n)) \cdot z_n}{\|w_n\|} dxdt.
\end{aligned}$$

Since z_n converges to 0 weakly and

$$\frac{\int_\Omega (F_{r_1}(x,t,w_n), \dots, F_{r_n}(x,t,w_n)) \cdot z_n}{\|w_n\|} \to 0, \quad \|P_{H^+\backslash Y} z_n\|^2 + \|P_{H^-} z_n\|^2 \to 0.$$

Since $\|P_Y z_n\|^2 \to 0$, we get $z_n \to 0$, which yields a contradiction. Hence $(w_n)_n$ is bounded. Up to a subsequence, we can suppose that w_n converges to w weakly for some $w \in H^- \oplus (H^+\backslash Y)$. We claim that w_n converges to w strongly. We have

$$\begin{aligned}
&\langle P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \nabla I(w_n), w_n \rangle \\
&= P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \int_\Omega [((w_n)_{tt} + (w_n)_{xxxx}) \cdot w_n \\
&\quad -(F_{r_1}(x,t,w_n), \dots, F_{r_n}(x,t,w_n)) \cdot w_n] dxdt \longrightarrow 0.
\end{aligned}$$

Since w_n converges to w weakly,

$$\int_\Omega (F_{r_1}(x,t,w_n), \dots, F_{r_n}(x,t,w_n)) \cdot w_n dxdt \to \int_\Omega (F_{r_1}(x,t,w), \dots, F_{r_n}(x,t,w)) \cdot w dxdt$$

and

$$\begin{aligned}
&P_{H^-\oplus(H^+\backslash Y)} P_{H_{k_n}} \int_\Omega [((w_n)_{tt} + (w_n)_{xxxx}) \cdot w_n] dxdt \\
&= \|P_{H^+\backslash Y} P_{H_{k_n}} w_n\|^2 - \|P_{H^-} P_{H_{k_n}} w_n\|^2 \\
&\longrightarrow P_{H^-\oplus(H^+\backslash Y)} \int_\Omega (w_{tt} + w_{xxxx}) \cdot w dxdt = \|P_{H^+\backslash Y} w\|^2 - \|P_{H^-} w\|^2.
\end{aligned}$$

Thus we have $\|P_{H^+\backslash Y}P_{H_{k_n}}w_n\|^2 \to \|P_{H^+\backslash Y}w\|^2$ and $\|P_{H^-}w_n\|^2 \to \|P_{H^-}w\|^2$, so we have $\|P_{H^+\backslash Y}P_{H_{k_n}}w_n\|^2+\|P_{H^-}w_n\|^2 \to \|P_{H^+\backslash Y}w\|^2+\|P_{H^-}w\|^2$. Thus $\|P_{H_{k_n}}w_n\|^2 \to \|w\|^2$. Thus we have that w_n converges to w strongly. Thus we have

$$grad_C^- \tilde{I}(\tilde{w}) = grad_C^- I(w) = \lim_{n\to\infty} P_{H_{k_n}} grad_C^- I(w_n) = \lim_{n\to\infty} P_{H_{k_n}} grad_C^- \tilde{I}(\tilde{w_n}) = 0.$$

So we prove the first case. We consider the case $P_Y w_n = 0$, i.e., $w_n \in H^- \oplus (H^+\backslash Y)$. Then $\tilde{w_n} \in \partial C$, $\forall n$. In this case $w_n = \Psi(\tilde{w_n}) \in H^- \oplus (H^+\backslash Y)$ and $P_{H^-\oplus(H^+\backslash Y)}\nabla I(w_n) \to 0$. Thus by the same argument as the first case we obtain the conclusion. So we prove the lemma. □

Lemma 3.3. *Assume that the F satisfies conditions (F1)-(F4). Then $\tilde{I}$ has no critical point $\tilde{u}$ such that $\tilde{I}(\tilde{u}) = \xi > 0$ and $\tilde{u} \in \partial C$.*

Proof. It suffices to prove that I has no critical point $u = \Psi(\tilde{u})$ such that $I(u) = \xi > 0$ and $u \in X_1 \oplus X_3$. We notice that from Lemma 3.1, for fixed $u_1 \in H^-$, the functional $u_3 \mapsto I(u_1 + u_3)$ is weakly convex in H^+, while, for fixed $u_3 \in H^+$, the functional $u_1 \mapsto I(u_1+u_3)$ is strictly concave in H^-. Moreover $(0,\ldots,0)$ is a critical point in $H^- \oplus H^+$ with $I(0,\ldots,0) = 0$. So if $u = u_1 + u_3$ is another critical point for $I|_{H^-\oplus H^+}$, then we have

$$0 = I(0,\ldots,0) \le I(u_3) \le I(u_1+u_3) \le I(u_1) \le I(0,\ldots,0) = 0.$$

So $I(u_1+u_3) = I(0,\ldots,0) = 0$. □

Proof of Theorem 1.1. First we will find the critical points for the functional $\tilde{I}$. We set:

$$\begin{aligned} S(\bar{\rho}) &= \{w \in H^+ |\ \|w\| = \bar{\rho}\}, \\ \tilde{S}(\bar{\rho}) &= \Psi^{-1}(S(\bar{\rho})), \\ S_r &= \{w \in H^+ |\ \|w\| = r\}, \\ \tilde{S}_r &= \Psi^{-1}(S_r), \\ \tilde{\Sigma_R} &= \Psi^{-1}(\Sigma_R(S(\bar{\rho}), H^-)), \\ \tilde{\Delta_R} &= \Psi^{-1}(\Delta_R(S(\bar{\rho}), H^-)). \end{aligned}$$

Let $Y = H_n^+$ be any closed space of H^+ of finite dimension. By Lemma 3.1, $\tilde{I}$ satisfies the variation linking inequality, i. e., there exist $\bar{\rho}$, $R > 0$ and r with $R > r$ such that $B_r \cap S(\bar{\rho}) \neq \emptyset$ and

$$\sup_{\tilde{w}\in\tilde{\Sigma_R}} \tilde{I}(\tilde{w}) = \sup_{w\in\Sigma_R(S(\bar{\rho}),H^-)} I(w) < \inf_{\substack{w\in Y \\ w\in S_r}} I(w) = \inf_{\tilde{w}\in\tilde{S}_r} \tilde{I}(\tilde{w}) \tag{3.4}$$

and

$$\sup_{\tilde{w}\in\tilde{\Delta_R}} \tilde{I}(\tilde{w}) = \sup_{w\in\Delta_R} I(w) < \infty. \tag{3.5}$$

We note that $(\tilde{\Delta}_R, \tilde{\Sigma}_R)$ has the same topological structure as the pair (Δ_R, Σ_R) in Theorem 2.1 Moreover by Lemma 3.2, $\tilde{I}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$, $C_n = C \cap H_n$ for any c such that

$$\inf_{\tilde{w}\in\tilde{S_r}} \tilde{I}(\tilde{w}) \le c \le \sup_{\tilde{w}\in\tilde{\Delta_R}} \tilde{I}(\tilde{w})$$

Let us set $X_1 = H^-$, $X_2 = Y$ and $X_3 = H^+\backslash Y$. Then $H = X_1 \oplus X_2 \oplus X_3$. By Proposition 2.1, I is $C^1(H, R^1)$, so $\tilde{I}$ is $C^1(H, R^1)$. By (3.4), the condition (i) of Theorem 2.1 is satisfied. By lemma 3.2, $\tilde{I}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$ for every real number c such that

$$\inf_{\tilde{w}\in\Psi^{-1}(S_r)} \tilde{I}(\tilde{w}) \le c \le \sup_{\tilde{w}\in\Psi^{-1}(\Delta_R(S(\bar{\rho})),H^-)} \tilde{I}(\tilde{w}),$$

so the condition (ii) of Theorem 2.1 is satisfied. By Lemma 3.3, the condition (iii) of Theorem 2.1 is satisfied. By (3.5), the condition (iv) of Theorem 2.1 is satisfied. Thus by Theorem 2.1, there exist at least two lower critical points of $\tilde{w}_1$, $\tilde{w}_2$ for $\tilde{I}$ such that $\inf_{\tilde{w}\in\tilde{S_r}} \tilde{I}(\tilde{w}) \le \tilde{I}(\tilde{w}_i) \le \sup_{\tilde{w}\in\tilde{\Delta_R}} \tilde{I}(\tilde{w})$, $i = 1, 2$. Setting $w_i = \Psi(\tilde{w}_i)$, $i = 1, 2$, we have

$$0 < \inf_{w\in S_r} I(w) \le I(w_i) \le \sup_{w\in\Delta_R(S(\bar{\rho}),H^-)} I(w).$$

Thus we prove the theorem. □

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