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CERTAIN SUBGROUPS OF SELF-HOMOTOPY EQUIVALENCES OF THE WEDGE OF TWO MOORE SPACES II.

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ABSTRACT. In the previous work [5] we have determined the group ${\mathcal{E}_{\#}}^{dim+r}(X)$ for $X = M(\mathbf{Z}_q, n+1) \vee M(\mathbf{Z}_q, n)$ for all integers $q > 1$. In this paper, we investigate the group ${\mathcal{E}_{\#}}^{dim+r}(X)$ for $X = M(\mathbf{Z} \oplus \mathbf{Z}_q, n+1) \vee M(\mathbf{Z} \oplus \mathbf{Z}_q, n)$ for all odd numbers $q > 1$.

1. INTRODUCTION

For a based topological space X the set $\mathcal{E}(X)$ of homotopy classes of self-homotopy equivalences forms a group under composition of maps

For a based, 1-connected, finite CW-complex X, let ${\mathcal{E}_{\#}}^{dim+r}(X)$ be the subgroup of homotopy classes which induces the identity on the homotopy groups of X in dimensions $\leq dimX + r$. The group $\mathcal{E}(X)$ and the subgroup ${\mathcal{E}_{\#}}^{dim+r}(X)$ have been studied extensively. For a survey of known results and applications of $\mathcal{E}(X)$, see[2], and for a list of references on the subgroups mentioned above, see [3]. In particular, Arkowitz and Maruyama examined ${\mathcal{E}_{\#}}^{dim+r}(X)$ for Moore spaces X in [4], and we have extended their computation to the case $X = M(\mathbf{Z}_q, n+1) \vee M(\mathbf{Z}_q, n)$ for all positive integers $q > 1$ in [5].

In this paper we calculate the subgroup ${\mathcal{E}_{\#}}^{dim+r}(X)$ for X the wedge of two Moore spaces $X = M(\mathbf{Z} \oplus \mathbf{Z}_q, n+1) \vee M(\mathbf{Z} \oplus \mathbf{Z}_q, n)$ for all odd numbers $q > 1$.

We fix some notations and conventions. We shall work in the category of spaces with base points and maps preserving the base points. If $f : X \longrightarrow Y$ is a map, then $f_{*n} : H_n(X) \longrightarrow H_n(Y)$ and $f_{\#n} : \pi_n(X) \longrightarrow \pi_n(Y)$ denote the induced homology and homotopy homomorphism in dimension n, respectively. In this paper we do not distinguish notationally between a map $X \longrightarrow Y$ and its homotopy class in $[X, Y]$.

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For a finitely generated abelian group G write $G = F \oplus T$, to indicate that F is a free part of G and T is the torsion subgroup of G. Consequently $M(G,n) = M(F,n) \vee M(T,n)$. If G is free-abelian, $M(G,n)$ is just a wedge of n-spheres. Note that when G is finitely-generated, $M(G,n)$ is a finite CW-complex of dim n if G is free-abelian and of dim $n+1$ if G is not free-abelian. Since $M(G,n)$ is a double suspension, the set of homotopy classes $[M(G,n),X]$ can be given abelian group structure with binary operation $'+'$.

Finally, if A is an abelian group, we write

$$\bigoplus^{r} A = A \oplus \cdots \oplus A \quad (r \text{ summands}).$$

We also use $'\oplus'$ to denote cartesian product of sets.

2. Preliminaries

We begin with some well-known results. The first is the universal coefficient theorem for homotopy with coefficients.

Theorem 2.1 ([6, p. 30]). *There is a short exact sequence :*

$$0 \to Ext(G, \pi_{n+1}(X)) \to \pi_n(G;X) \to Hom(G, \pi_n(X)) \to 0,$$

where $\lambda : \pi_n(G;X) \to Hom(G, \pi_n(X)$ *is the homomorphism defined by* $\lambda(f) = f_{\#n} : G \approx \pi_n(M(G,n)) \to \pi_n(X)$.

Proposition 2.2. *If X is $(k-1)$-connected and Y is $(l-1)$-connected, $k, l \geq 2$, and $dimP < k+l-1$, then the projections $X \vee Y \longrightarrow X$ and $X \vee Y \longrightarrow Y$ induce a bijection*

$$[P, X \vee Y] \longrightarrow [P,X] \oplus [P,Y].$$

Proposition 2.2 is a consequence of [7, p. 405] since the inclusion $X \vee Y \longrightarrow X \times Y$ is a $(k+l-1)$-equivalence.

We consider abelian groups G_1 and G_2 and Moore spaces $Y_1 = M(G_1, n_1)$ and $Y_2 = M(G_2, n_2)$. Let $X = Y_1 \vee Y_2 = M(G_1, n_1) \vee M(G_2, n_2)$ and denote by $i_j : Y_j \longrightarrow X$ the inclusions and by $p_j : X \longrightarrow Y_j$ the projections, $j = 1, 2$. If $f : X \longrightarrow X$, then we define $f_{jk} : Y_k \longrightarrow Y_j$ by $f_{jk} = p_j f i_k$ for $j, k = 1, 2$.

Proposition 2.3. *The function θ which assigns to each $f \in [X,X]$, the 2×2 matrix*

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [Y_1, Y_2]$, is a bijection. In addition,

(1) $\theta(f+g)=\theta(f)+\theta(g)$, *so* θ *is an isomorphism* $[X,X]\longrightarrow\bigoplus_{j,k=1,2}[Y_1,Y_2]$.
(2) $\theta(fg)=\theta(f)\theta(g)$, *where* fg *denotes composition in* $[X,X]$ *and* $\theta(f)\theta(g)$ *denotes matrix multiplication.*
(3) *If* $\alpha_r:\pi(Y_1)\oplus\pi_r(Y_2)\longrightarrow\pi_r(Y_1\oplus Y_2)$ *and* $\beta_r:\pi(Y_1)\vee\pi_r(Y_2)\longrightarrow\pi_r(Y_1\oplus Y_2)$ *are the homomorphisms induced by the inclusions and projections, respectively, then*

$$\beta_r f_{\#r}\alpha_r(x,y)=(f_{11\#r}(x)+f_{12\#r}(x),f_{21\#r}(x)+f_{22\#r}(x)),$$

for $x\in\pi_r(Y_1)$ *and* $y\in\pi_r(Y_2)$.

The homotopy groups $\pi_{n+k}(M(G,n))$ and the groups of homotopy classes $[M(G,n+k),M(G,k)]$ have been determined by Araki and Toda [1] when G is the cyclic group $\mathbf{Z}_q$, $(q>1)$ in stable homotopy category. They obtained the following results. See [1] if you want to know that in details.

Proposition 2.4 ([1]). *Let* $q>1$ *be an odd number. Then*
(1) $\pi_n(M(\mathbf{Z}_q,n))\approx\mathbf{Z}_q$.
(2) $\pi_{n+1}(M(\mathbf{Z}_q,n))=0$.
(3) $\pi_{n+2}(M(\mathbf{Z}_q,n))=0$.
(4) $\pi_{n+3}(M(\mathbf{Z}_q,n))\approx\mathbf{Z}_{(q,24)}$.

Proposition 2.5 ([1]). *Let* $q>1$ *be an odd number. Then*
(1) $[(M(\mathbf{Z}_q,n-1)),(M(\mathbf{Z}_q,n))]\approx\mathbf{Z}_q$.
(2) $[(M(\mathbf{Z}_q,n)),(M(\mathbf{Z}_q,n))]\approx\mathbf{Z}_q$.
(3) $[(M(\mathbf{Z}_q,n+1)),(M(\mathbf{Z}_q,n))]=0$.
(4) $[(M(\mathbf{Z}_q,n+2)),(M(\mathbf{Z}_q,n))]\approx\mathbf{Z}_{(q,24)}$.

Proposition 2.6. *Let* $q>1$ *be an odd number. Then*
(1) $[(M(\mathbf{Z}_q,n-2)),S^n)]=0$.
(2) $[(M(\mathbf{Z}_q,n-1)),S^n)]=0$.
(3) $[(M(\mathbf{Z}_q,n)),S^n)]=0$.
(4) $[(M(\mathbf{Z}_q,n+1)),S^n)]=0$.
(5) $[(M(\mathbf{Z}_q,n+2)),S^n)]\approx\mathbf{Z}_{(q,24)}$.

Proof. (1) We know that $[(M(\mathbf{Z}_q,n-2)),S^n)]\approx\pi_{n-2}(\mathbf{Z}_q,S^n)$.
By Theorem 2.1, we obtain the short exact sequence :

$$0\to Ext(\mathbf{Z}_q,\pi_{n-1}(S^n))\to\pi_{n-2}(\mathbf{Z}_q,S^n)\to Hom(\mathbf{Z}_q,\pi_{n-2}(S^n))\to 0.$$

And $Ext(\mathbf{Z}_q,\pi_{n-1}(S^n))=0$ and $Hom(\mathbf{Z}_q,\pi_{n-2}(S^n)=0$.

Therefore $[(M(\mathbf{Z}_q, n-2)), S^n)] = 0$.

(2) We know also that

$$[(M(\mathbf{Z}_q, n-1)), S^n)] \approx \pi_{n-1}(\mathbf{Z}_q, S^n), Ext(\mathbf{Z}_q, \pi_n(S^n)) \approx \mathbf{Z}_q$$

and

$$Hom(\mathbf{Z}_q, \pi_{n-1}(S^n) = 0.$$

By use of the short exact sequence in Theorem 2.1, $[(M(\mathbf{Z}_q, n-1)), S^n)] = 0$.

(3) Since q is an odd number,

$$Ext(\mathbf{Z}_q, \pi_{n+1}(S^n)) \approx Ext(\mathbf{Z}_q, \mathbf{Z}_2) = 0$$

and

$$Hom(\mathbf{Z}_q, \pi_n(S^n) = 0.$$

We obtain $[(M(\mathbf{Z}_q, n)), S^n)] = 0$.

We can show the rest of the proof by the same manner. □

We also need the following theorem.

Theorem 2.7 ([4]). *For the Moore space $X = M(G, n)$,*

(1) $\mathcal{E}_{\#}^{dim}(X) \cong \bigoplus^{(r+s)s} \mathbf{Z}_2$, *where r is the rank of G and s is the number of 2-torsion summands in G.*

(2) $\mathcal{E}_{\#}^{dim+1}(X) = 1$ *if $n > 3$.*

3. MAIN THEOREM

In this section we determine the group $\mathcal{E}_{\#}{}^{dim+r}(X)$ for $X = M(\mathbf{Z} \oplus \mathbf{Z}_q, n+1) \vee M(\mathbf{Z} \oplus \mathbf{Z}_q, n), n \geq 5$ and $q > 1$: *odd*.

We let $M_1 = M(\mathbf{Z}_q, n+1) = S^{n+1} \cup_q e^{n+2}$ and $M_2 = M(\mathbf{Z}_q, n) = S^n \cup_q e^{n+1}$. We know that $M(\mathbf{Z} \oplus \mathbf{Z}_q, n+1) = M(\mathbf{Z}, n+1) \vee M(\mathbf{Z}_q, n+1) = S^{n+1} \vee (S^{n+1} \cup_q e^{n+2})$ and $M(\mathbf{Z} \oplus \mathbf{Z}_q, n) = M(\mathbf{Z}, n) \vee M(\mathbf{Z}_q, n) = S^n \vee (S^n \cup_q e^{n+1})$. And we set $Y_1 = S^{n+1} \vee M_1$ and $Y_2 = S^n \vee M_2$. Then we can denote $X = Y_1 \vee Y_2$. We now let $f \in [X, X]$ and use the notation of Section 2 so that $f_{jk} = p_j f i_k \in [Y_k, Y_j]$ for $j, k = 1, 2$. By Proposition 2.2 and Proposition 2.3, we can identify $f \in \mathcal{E}(X)$ with the 2×2 matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{11} \in \mathcal{E}(Y_1), f_{12} \in [Y_2, Y_1], f_{21} \in [Y_1, Y_2], f_{22} \in \mathcal{E}(Y_2)$. The group structure in $\mathcal{E}(X)$ is then given by matrix multiplication.

Lemma 3.1. $\pi_{n+k}(Y_1 \vee Y_2) \approx \pi_{n+k}(Y_1) \oplus \pi_{n+k}(Y_2)$ *for $k = 0, 1, 2, 3, 4$.*

Proof. The Moore spaces Y_1 and Y_2 are n-connected and $(n-1)$-connected, respectively and $n \geq 5$.

By Proposition 2.1, $[S^{n+k}, Y_1 \vee Y_2] \approx [S^{n+k}, Y_1] \oplus [S^{n+k}, Y_2]$, for $k < n$. □

From Lemma 3.1, it is clear that

$$f_{\#n+k}(x, y) = \begin{pmatrix} f_{11\#n+k} & f_{12\#n+k} \\ f_{21\#n+k} & f_{22\#n+k} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \forall x \in \pi_{n+k}(Y_1), \forall y \in \pi_{n+k}(Y_2),$$

$k = 0, 1, 2, 3, 4$.

The following theorem is the main result in this paper.

Theorem 3.2. *For the space $X = M(\mathbf{Z} \oplus \mathbf{Z}_q, n+1) \vee M(\mathbf{Z} \oplus \mathbf{Z}_q, n)$,*

$$\mathcal{E}_{\#}{}^{\dim}(X) \approx \mathcal{E}_{\#}{}^{\dim+1}(X) \approx \mathbf{Z}_q \oplus \mathbf{Z}_q \ \ (\forall q > 1 : odd).$$

Proof. By Proposition 2.2, $[X, X] = [Y_1, Y_1] \oplus [Y_1, Y_2] \oplus [Y_2, Y_1] \oplus [Y_2, Y_2]$. Now $G = \mathbf{Z} \oplus \mathbf{Z}_q$ has no 2-torsion, $dimX = dimY_1 = n+2$ and $dimY_2 = n+1$. By Theorem 2.7, $\mathcal{E}_{\#}^{dimX}(Y_1) = 1$ and $\mathcal{E}_{\#}^{dimX}(Y_2) = 1$. Let $f \in \mathcal{E}_{\#}^{\dim}(X)$ be given a $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then $f_{11} = 1$ and $f_{22} = 1$. So it suffices that we consider just f_{12} and f_{21}.

First $f_{12} \in [Y_2, Y_1] \approx [S^n, S^{n+1}] \oplus [M_2, S^{n+1}] \oplus [S^n, M_1] \oplus [M_2, M_1]$.

So we can identify $f_{12} \in [Y_2, Y_1]$ with the 2×2 matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, where $g_{11} \in [S^n, S^{n+1}]$, $g_{12} \in [M_2, S^{n+1}]$, $g_{21} \in [S^n, M_1]$, $g_{22} \in [M_2, M_1]$. Then we see that $g_{11} = 0$ and $g_{21} = 0$ obviously.

Now for any element $g_{12} \in [M_2, S^{n+1}], g_{12\#k}(\pi_k(M_2)) = 0, \forall k \leq dimX$. Because $\pi_k(S^{n+1}) = 0, \forall k \leq n$ and $\pi_k(M_2) = 0,\ k = n+1,\ n+2$.

And for any element $g_{22} \in [M_2, M_1]$, $g_{22\#k}(\pi_k(M_2)) = 0, \forall k \leq dimX$. Because $\pi_k(M_1) = 0, \forall k \leq n$ and $\pi_k(M_2) = 0,\ k = n+1,\ n+2$.

By the fact of $[M_2, S^{n+1}] \approx \mathbf{Z}_q =< \pi >$ and $[M_2, M_1] \approx \mathbf{Z}_q =< i\pi >$, we obtain $f_{12} \in \{\begin{pmatrix} 0 & g_{12} \\ 0 & g_{22} \end{pmatrix} | g_{12} \in < \pi >,\ g_{22} \in < i\pi >\} \approx \mathbf{Z}_q \oplus \mathbf{Z}_q$.

Second $f_{21} \in [Y_1, Y_2] \approx [S^{n+1}, S^n] \oplus [M_1, S^n] \oplus [S^{n+1}, M_2] \oplus [M_1, M_2]$.

So we can identify $f_{21} \in [Y_1, Y_2]$ with the 2×2 matrix $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, where $h_{11} \in [S^{n+1}, S^n]$, $h_{12} \in [M_1, S^n]$, $h_{21} \in [S^{n+1}, M_2]$, $h_{22} \in [M_1, M_2]$.

By Proposition 2.4, 2.5 and 2.6, $[M_1, S^n] = [S^{n+1}, M_2] = [M_1, M_2] = 0$. So $h_{12} = h_{21} = h_{22} = 0$.

Now $\eta_{\#n+1} : \pi_{n+1}(S^{n+1}) \longrightarrow \pi_n(S^n)$, $\eta_{\#n+1}(1) = \eta \circ 1 = \eta \neq 0$. So $h_{11} = 0$.

Finally, $f_{21} = 0$.

Therefore

$$\mathcal{E}_{\#}{}^{\dim}(X) \approx \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & g_{12} \\ 0 & g_{22} \end{pmatrix} \atop \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) | g_{12} \in < \pi >, g_{22} \in < i\pi > \right\} \approx \mathbf{Z}_q \oplus \mathbf{Z}_q.$$

By Proposition 2.3, $\pi_{n+3}(M_2) \approx \mathbf{Z}_{(q,24)} =< i\nu >$. $\pi_{\# n+3}(i\nu) = \pi i\nu = 0$ and $(i\pi)_{\# n+3}(i\nu) = i\pi i\nu = 0$. So $\mathcal{E}_{\#}{}^{\dim}(X) \approx \mathcal{E}_{\#}{}^{\dim+1}(X)$. □

We denote by $\mathcal{Z}(X)$ the subset of $[X, X]$ consisting of all homotopy classes which induces the trivial homomorphism on homotopy groups in dimensions less than or equal to n.

Corollary 3.3. *For the space $X = Y_1 \vee Y_2$ and q : odd,*

$$\mathcal{Z}(X) \approx \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & g_{12} \\ 0 & g_{22} \end{pmatrix} \atop \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) | g_{12} \in < \pi >, g_{22} \in < i\pi > \right\}.$$

Proof. Consider the bijection map $T : \mathcal{E}_{\#}{}^{dim}(X) \to \mathcal{Z}(X)$ defined by the translation by the identity map, that is, $T(f) = f - 1$. □

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