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ON THE GENERALIZED HYERS-ULAM STABILITY OF THE CAUCHY-JENSEN FUNCTIONAL EQUATION II

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ABSTRACT. In this paper, we obtain the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation

$$f(x+y,z) - f(x,z) - f(y,z) = 0,$$

$$2f\left(x, \frac{y+z}{2}\right) - f(x,y) - f(x,z) = 0$$

in the spirit of P. Găvruta.

1. INTRODUCTION

In 1940, S.M. Ulam [12] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D.H. Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th.M. Rassias [11] gave a generalization. Since then, the further generalization has been extensively investigated by a number of mathematicians[1, 6, 8, 9].

Throughout this paper, let X and Y be a normed space and a Banach space, respectively. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x) + g(y)$). For a given mapping $f : X \times X \to Y$, we

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define

$$C_1f(x,y,z) := f(x+y,z) - f(x,z) - f(y,z),$$
$$J_2f(x,y,z) := 2f\left(x, \frac{y+z}{2}\right) - f(x,z) - f(y,z)$$

for all $x, y, z \in X$. A mapping $f : X \times X \to Y$ is called a Cauchy-Jensen mapping if f satisfies the functional equations $C_1 f = 0$ and $J_2 f = 0$.

In 2006, Park and Bae [10] obtained the generalized Hyers-Ulam stability of a Cauchy-Jensen mapping in the following theorem.

Theorem A. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be two functions satisfying*

$$\tilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \left[\frac{1}{2^{j+1}}\varphi(2^j x, 2^j y, z) + \frac{1}{3^j}\varphi(x, y, 3^j z)\right] < \infty,$$
$$\tilde{\psi}(x,y,z) := \sum_{j=0}^{\infty} \left[\frac{1}{3^{j+1}}\psi(x, 3^j y, 3^j z) + \frac{1}{2^j}\psi(2^j x, y, z)\right] < \infty$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|C_1 f(x,y,z)\| \le \varphi(x,y,z),$$
$$\|J_2 f(x,y,z)\| \le \psi(x,y,z)$$

for all $x, y, z \in X$. Then there exist two Cauchy-Jensen mappings $F_C, F_J : X \times X \to Y$ such that

$$\|f(x,y) - F_C(x,y)\| \le \tilde{\varphi}(x,x,y),$$
$$\|f(x,y) - f(x,0) - F_J(x,y)\| \le \tilde{\psi}(x,y,-y) + \tilde{\psi}(x,-y,3y)$$

for all $x, y \in X$. The mappings $F_C, F_J : X \times X \to Y$ are given by

$$F_C(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \quad F_J(x,y) := \lim_{j\to\infty} \frac{1}{3^j} f(x, 3^j y)$$

for all $x, y \in X$.

Jun et al. [3, 4, 5] improved the Park and Bae's results.

In this paper, we investigate the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation. Under the different inequality condition, we have obtained the more improved stability results than that of Theorem A.

2. Stability of a Cauchy-Jensen Functional Equation

Jun et al. [5] established the basic properties of a Cauchy-Jensen mapping in the following lemma.

Lemma 1. *Let $f : X \times X \to Y$ be a Cauchy-Jensen mapping. Then*

$$f(x,y) = 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - (4^n - 2^n) f\left(\frac{x}{2^n}, 0\right),$$

$$f(x,y) = \frac{f(2^n x, 2^n y)}{4^n} + (2^n - 1) f\left(\frac{x}{2^n}, 0\right),$$

$$\begin{aligned} f(x,y) &= \frac{f(2^n x, 2^n y)}{4^n} + \left(\frac{1}{2^n} - \frac{1}{4^n}\right) f(2^n x, 0) \\ &= \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{2}\left(\frac{1}{2^n} - \frac{1}{4^n}\right)(f(2^n x, 2^n y) + f(2^n x, -2^n y)) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we have better stability results than that of Theorem A.

Theorem 2. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} \frac{1}{2^j} (\varphi(2^j x, 2^j y, 2^j z) + \psi(2^j x, 2^j y, 2^j z)) < \infty \tag{1}$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|C_1 f(x,y,z)\| \le \varphi(x,y,z), \tag{2}$$

$$\|J_2 f(x,y,z)\| \le \psi(x,y,z) \tag{3}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned} \|f(x,y) - F(x,y)\| \le & \sum_{j=0}^{\infty} \frac{\varphi(2^j x, 2^j x, 0)}{2^{j+1}} \\ & + \sum_{j=0}^{\infty} \frac{\varphi(2^j x, 2^j x, 2^{j+1} y) + 2\psi(2^j x, 2^{j+1} y, 0) + \varphi(2^j x, 2^j x, 0)}{4^{j+1}} \end{aligned} \tag{4}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j \to \infty} \left[\frac{1}{4^j} f(2^j x, 2^j y) + \frac{1}{2^j} f(2^j x, 0)\right]$$

for all $x, y \in X$.

Proof. By (2) and (3), we get

$$\begin{aligned}&\left\|\frac{1}{2^j}f(2^jx,0)-\frac{1}{2^{j+1}}f(2^{j+1}x,0)\right\|\\&=\frac{1}{2^{j+1}}\|C_1f(2^jx,2^jx,0)\|\le\frac{1}{2^{j+1}}\varphi(2^jx,2^jx,0),\end{aligned}$$

and

$$\begin{aligned}&\left\|\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0))-\frac{1}{4^{j+1}}(f(2^{j+1}x,2^{j+1}y)-f(2^{j+1}x,0))\right\|\\&=\left\|\frac{C_1f(2^jx,2^jx,2^{j+1}y)-2J_2f(2^jx,2^{j+1}y,0)-C_1f(2^jx,2^jx,0)}{4^{j+1}}\right\|\\&\le\frac{\varphi(2^jx,2^jx,2^{j+1}y)+2\psi(2^jx,2^{j+1}y,0)+\varphi(2^jx,2^jx,0)}{4^{j+1}}\end{aligned}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\left\|\frac{f(2^lx,0)}{2^l}-\frac{f(2^mx,0)}{2^m}\right\|\le\sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^jx,2^jx,0) \tag{5}$$

and

$$\begin{aligned}&\left\|\frac{1}{4^l}(f(2^lx,2^ly)-f(2^lx,0))-\frac{1}{4^m}(f(2^mx,2^my)-f(2^mx,0))\right\|\\&\le\sum_{j=l}^{m-1}\frac{\varphi(2^jx,2^jx,2^{j+1}y)+2\psi(2^jx,2^{j+1}y,0)+\varphi(2^jx,2^jx,0)}{4^{j+1}}\end{aligned} \tag{6}$$

for all $x, y \in X$. By (1), the sequences $\{\frac{1}{2^j}(f(2^jx,0)-f(0,0))\}$ and $\{\frac{1}{4^j}(f(2^jx,2^jy)-f(2^lx,0))\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{\frac{1}{2^j}(f(2^jx,0)-f(0,0))\}$ and $\{\frac{1}{4^j}(f(2^jx,2^jy)-f(2^lx,0))\}$ converge for all $x, y \in X$. Define $F_1, F_2 : X \times X \to Y$ by

$$F_1(x,y):=\lim_{j\to\infty}\frac{1}{2^j}f(2^jx,0),$$

and

$$F_2(x,y):=\lim_{j\to\infty}\frac{1}{4^j}[f(2^jx,2^jy)-f(2^jx,0)]=\lim_{j\to\infty}\frac{1}{4^j}f(2^jx,2^jy)$$

for all $x, y \in X$. By (2), (3) and the definitions of F_1 and F_2, we get

$$\begin{aligned}&C_1F_1(x,y,z)=\lim_{j\to\infty}\frac{1}{2^j}C_1f(2^jx,2^jy,0)=0,\\&J_2F_1(x,y,z)=0,\end{aligned}$$

$$C_1F_2(x,y,z) = \lim_{j\to\infty} \frac{1}{4^j}[C_1(2^jx,2^jy,2^jz) - C_1(2^jx,2^jy,0)] = 0,$$
$$J_2F_2(x,y,z) = \lim_{j\to\infty} \frac{1}{4^j}J_2(2^jx,2^jy,2^jz) = 0$$

for all $x,y,z \in X$ and so F_1, F_2 are Cauchy-Jensen mappings. Putting $l = 0$ and taking $m \to \infty$ in (5) and (6), one can obtain the inequalities

$$\|f(x,0) - F_1(x,y)\| \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}}\varphi(2^jx,2^jx,0)$$

and

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \leq \sum_{j=0}^{\infty} \frac{\varphi(2^jx,2^jx,2^{j+1}y) + 2\psi(2^jx,2^{j+1}y,0) + \varphi(2^jx,2^jx,0)}{4^{j+1}}$$

for all $x,y,z \in X$. From the above inequalities, it follows that F is a Cauchy-Jensen mapping satisfying (4), where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y).$$

Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (4). By (1) and Lemma 1, we have

$$\begin{aligned}\|F(x,y) - F'(x,y)\| &= \left\|\frac{1}{4^n}(F-F')(2^nx,2^ny) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right)(F-F')(2^nx,0)\right\| \\ &\leq \frac{1}{4^n}\|(F-f)(2^nx,2^ny)\| + \frac{1}{2^n}\|(F-f)(2^nx,0)\| \\ &\quad + \frac{1}{4^n}\|(f-F')(2^nx,2^ny)\| + \frac{1}{2^n}\|(f-F')(2^nx,0)\| \\ &\leq \sum_{j=n}^{\infty} \frac{1}{2^{j-2}}\varphi(2^jx,2^jx,0) + \sum_{j=n}^{\infty} \frac{\psi(2^jx,0,0)}{2^j} \\ &\quad + \sum_{j=n}^{\infty} \frac{2\varphi(2^jx,2^jx,2^{j+1}y) + 4\psi(2^jx,2^{j+1}y,0)}{4^{j+1}}\end{aligned}$$

for all $x,y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x,y \in X$. Thus such a Cauchy-Jensen mapping $F : X \times X \to Y$ is unique. □

Remark. If $\psi : X \times X \times X \to [0,\infty)$ in Theorem 2 satisfies the inequality

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^jx,2^jy,2^jz) + \sum_{j=0}^{\infty} \frac{1}{4^j}\psi(2^jx,2^jy,2^jz) < \infty$$

for all $x, y, z \in X$ instead of (1), then we obtain the result in Theorem 2 except the uniqueness of F.

Theorem 3. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=1}^{\infty} 4^j \left(\varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \psi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right)\right) < \infty$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2) *and* (3) *for all $x, y, z \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ satisfying*

$$\begin{aligned} \|f(x,y) - F(x,y)\| &\le \sum_{j=1}^{\infty} 2^{j-1} \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, 0\right) \\ &+ \sum_{j=1}^{\infty} 4^{j-1} \left(\varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{y}{2^{j-1}}\right) + 2\psi\left(\frac{x}{2^j}, \frac{y}{2^{j-1}}, 0\right) + \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, 0\right)\right) \end{aligned} \tag{7}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} \left[4^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - (4^j - 2^j) f\left(\frac{x}{2^j}, 0\right)\right]$$

for all $x, y \in X$.

Proof. By the similar method in the proof of Theorem 2, we get two Cauchy-Jensen mappings F_1, F_2 defined by

$$\begin{aligned} F_1(x,y) &:= \lim_{j\to\infty} 2^j f\left(\frac{x}{2^j}, 0\right) \\ F_2(x,y) &:= \lim_{j\to\infty} 4^j \left[f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - f\left(\frac{x}{2^j}, 0\right)\right] \end{aligned}$$

for all $x, y \in X$ and F is a Cauchy-Jensen mapping satisfying (7) where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y).$$

Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (7). Using Lemma 1, we have

$$\|F(x,y) - F'(x,y)\| \le \left\| 4^n (F - F')\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| + 4^n \left\| (F - F')\left(\frac{x}{2^n}, 0\right)\right\|$$

$$\leq \sum_{j=1}^{\infty} 4^j \left(\varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{y}{2^{j-1}}\right) + 2\psi\left(\frac{x}{2^j}, \frac{y}{2^{j-1}}, 0\right) + 2\psi\left(\frac{x}{2^j}, 0, 0\right)\right)$$
$$+ \sum_{j=1}^{\infty} 4^{j+2} \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, 0\right)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F : X \times X \to Y$ is unique. □

Theorem 4. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \sum_{j=0}^{\infty} \frac{1}{4^j}(\varphi(2^j x, 2^j y, 2^j z) + \psi(2^j x, 2^j y, 2^j z)) < \infty$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (2) and (3) for all $x, y, z \in X$. Then there exists a Cauchy-Jensen mapping $F : X \times X \to Y$ satisfying

$$\|f(x,y) - F(x,y)\| \leq \sum_{j=1}^{\infty} 2^{j-1} \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, 0\right)$$
$$+ \sum_{j=0}^{\infty} \frac{\varphi(2^j x, 2^j x, 2^{j+1} y) + 2\psi(2^j x, 2^{j+1} y, 0) + \varphi(2^j x, 2^j x, 0)}{4^{j+1}} \tag{8}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j \to \infty} \frac{1}{4^j} \left[f(2^j x, 2^j y) - f(2^j x, 0) + 2^j f\left(\frac{x}{2^j}, 0\right)\right]$$

for all $x, y \in X$.

Proof. We can obtain F_1 as in Theorem 3 and F_2 as in Theorem 2, so F is a Cauchy-Jensen mapping satisfying (8) where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y).$$

□

Applying Theorem 2, we easily get the following corollary.

Corollary 5. *Let p, q, θ be nonnegative real numbers with $p, q < 1$ and $p \neq 1$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1 f(x,y,z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$
$$\|J_2 f(x,y,z)\| \leq \theta(\|x\|^q + \|y\|^q + \|z\|^q)$$

for all $x, y, z \in X$. Then there exists a Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned} \|f(x,y) - F(x,y)\| \le \frac{2\theta}{|2^q - 4|}\|x\|^q + \left(\frac{4\theta}{|2^p - 4|} + \frac{2\theta}{|2^p - 2|}\right)\|x\|^p \\ + \frac{2^{q+1}\theta}{|2^q - 4|}\|y\|^q + \frac{2^p\theta}{|2^p - 4|}\|y\|^p \end{aligned} \tag{9}$$

for all $x, y \in X$.

Applying Theorem 4, we easily get the another corollary.

Corollary 6. *Let $1 < p < 2, q < 2$ and let f, θ be as in Corollary 5. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ satisfying (9) for all $x, y \in X$.*

Applying Theorem 3, we easily get the following corollary.

Corollary 7. *Let $p, q > 2$ and let f, θ be as in Corollary 5. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ satisfying (9) for all $x, y \in X$.*

3. STABILITY OF A CAUCHY-JENSEN MAPPING ON THE PUNCTURED DOMAIN.

The following lemma [7] requires to prove Theorem 10. Throughout this section, let A be a subset of X satisfying the following condition:
for every $x \neq 0$, there exists a positive integer n_x such that $nx \notin A$ for all $|n| \geq n_x$ and $nx \in A$ for all $|n| < n_x$.

Lemma 8. *Let $f : X \times X \to Y$ be a mapping such that*

$$C_1 f(x,y,z) = 0, \quad J_2 f(x,y,z) = 0$$

for all $x, y, z \in X \backslash A$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$F(x,y) = f(x,y)$$

for all $x, y \in X \backslash A$. In particular

$$F(x,y) = f(x,y)$$

for all $(x,y) \in (X \times X) \backslash (A \times A)$.

Corollary 9. *Let $f : X \times X \to Y$ be a mapping such that*

$$C_1 f(x,y,z) = 0, \quad J_2 f(x,y,z) = 0$$

for all $x, y, z \in X \backslash \{0\}$ and $f(0,0) = 0$. Then f is a Cauchy-Jensen mapping.

Theorem 10. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying (1) for all $x, y, z \in X \backslash A$. Let $f : X \times X \to Y$ be a mapping satisfying (2) and (3) for all $x, y, z \in X \backslash A$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le \sum_{j=0}^{\infty} \frac{\Phi_1(2^j x, 2^j y)}{2^j} + \sum_{j=0}^{\infty} \frac{\Phi_2(2^j x, 2^j y)}{4^j} \tag{10}$$

for all $x, y \in X \backslash A$, where Φ_1, Φ_2 are given by

$$\Phi_1(x,y) = \frac{1}{2}\psi(x,y,-y) + \frac{1}{4}\psi(2x,y,-y) + \frac{1}{4}\varphi(x,x,y) + \frac{1}{4}\varphi(x,x,-y),$$
$$\Phi_2(x,y) = \frac{1}{4}(\psi(x,3y,y) + \psi(x,3y,-y) + \psi(x,y,-y) + \varphi(x,x,2y)) + \frac{1}{2}\Phi_1(x,y)$$

The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} \frac{f(2^j x, 2^j y)}{4^j} + \lim_{j\to\infty} \frac{f(2^j x, 0)}{2^j}$$

for all $x, y \in X$.

Proof. By (2) and (3), we get

$$\begin{aligned}
&\left\| \frac{f(2^j x, 0)}{2^j} - \frac{f(2^{j+1}x, 0)}{2^{j+1}} \right\| \\
&= \frac{1}{2^{j+2}} \|2J_2 f(2^j x, 2^j y, -2^j y) - J_2 f(2^{j+1}x, 2^j y, -2^j y) \\
&\quad - C_1 f(2^j x, 2^j x, 2^j y) - C_1 f(2^j x, 2^j x, -2^j y)\| \\
&\le \frac{1}{2^j}\Phi_1(2^j x, 2^j y), \\
&\left\| \frac{f(2^j x, 2^j y) - f(2^j x, 0)}{4^j} - \frac{f(2^{j+1}x, 2^{j+1}y) - f(2^{j+1}x, 0)}{4^{j+1}} \right\| \\
&= \frac{1}{4^{j+1}} \Big\| - J_2 f(2^j x, 3\cdot 2^j y, 2^j y) + J_2 f(2^j x, 3\cdot 2^j y, -2^j y) \\
&\quad - J_2 f(2^j x, 2^j y, -2^j y) - C_1 f(2^j x, 2^j x, 2^{j+1} y) - J_2 f(2^j x, 2^j y, -2^j y) \\
&\quad + \frac{1}{2} J_2 f(2^{j+1}x, 2^j y, -2^j y) + \frac{1}{2} C_1 f(2^j x, 2^j x, 2^j y) + \frac{1}{2} C_1 f(2^j x, 2^j x, -2^j y) \Big\| \\
&\le \frac{1}{4^j}\Phi_2(2^j x, 2^j y)
\end{aligned}$$

for all $x, y \in X \backslash A$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\left\| \frac{f(2^l x, 0)}{2^l} - \frac{f(2^m x, 0)}{2^m} \right\| \le \sum_{j=l}^{m-1} \frac{1}{2^j}\Phi_1(2^j x, 2^j y), \tag{11}$$

$$\left\|\frac{f(2^l x, 2^l y) - f(2^l x, 0)}{4^l} - \frac{f(2^m x, 2^m y) - f(2^m x, 0)}{4^m}\right\| \leq \sum_{j=l}^{m-1} \frac{1}{4^j}\Phi_2(2^j x, 2^j y) \tag{12}$$

for all $x, y \in X \backslash A$. By (1), the sequence $\{\frac{f(2^j x,0)}{2^j}\}$, $\{\frac{f(2^j x,2^j y)-f(2^j x,0)}{4^j}\}$ are Cauchy for all $x, y \in X\backslash A$. Since Y is complete, the sequences $\{\frac{f(2^j x,0)}{2^j}\}$, $\{\frac{f(2^j x,2^j y)-f(2^j x,0)}{4^j}\}$ converge for all $x, y \in X\backslash A$. Define $F_1 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} \frac{f(2^j x, 0)}{2^j}$$

for all $x, y \in X$ and define $F_2' : X \times (X\backslash\{0\}) \to Y$ by

$$F_2'(x, y) := \lim_{j\to\infty} \frac{f(2^j x, 2^j y) - f(2^j x, 0)}{4^j} = \lim_{j\to\infty} \frac{f(2^j x, 2^j y)}{4^j}$$

for all $y \in X$ and $x \neq 0$. Since

$$\lim_{j\to\infty}\left(\frac{f(0, 2^j y)}{4^j} - \frac{f(2^j x, 2^j y) + f(-2^j x, 2^j y)}{4^j}\right) = 0,$$

we can define $F_2 : X \times X \to Y$ by

$$F_2(x, y) = \lim_{j\to\infty} \frac{f(2^j x, 2^j y)}{4^j}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (11) and (12), one can obtain the inequalities

$$\|f(x, 0) - F_1(x, y)\| \leq \sum_{j=0}^{\infty} \frac{1}{2^j}\Phi_1(2^j x, 2^j y),$$
$$\|f(x, y) - f(x, 0) - F_2(x, y)\| \leq \sum_{j=0}^{\infty} \frac{1}{4^j}\Phi_2(2^j x, 2^j y)$$

for all $x, y \in X\backslash A$. From (2), (3) and the definitions of F_1 and F_2, we get

$$\begin{aligned} C_1F_1(x, y, z) &= \lim_{j\to\infty} \frac{1}{2^{j+1}}[J_2 f(2^j(x + y), 2^j z, -2^j z) - J_2 f(2^j x, 2^j z, -2^j z) \\ &\quad - J_2 f(2^j y, 2^j z, -2^j z) + C_1 f(2^j x, 2^j y, 2^j z) \\ &\quad + C_1 f(2^j x, 2^j y, -2^j z)] = 0, \\ J_2F_1(x, y, z) &= 0, \\ C_1F_2(x, y, z) &= \lim_{j\to\infty} \frac{C_1 f(2^j x, 2^j y, 2^j z)}{4^j} = 0, \\ J_2F_2(x, y, z) &= \lim_{j\to\infty} \frac{J_2 f(2^j x, 2^j y, 2^j z)}{4^j} = 0 \end{aligned}$$

for all $x, y, z \in X\backslash\{0\}$. By Corollary 8, F_1, F_2 are Cauchy-Jensen mappings. Hence, F is a Cauchy-Jensen mapping satisfying (10), where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (10). By (1) and Lemma 1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&= \frac{1}{2}\left\| \left(\frac{1}{2^n} + \frac{1}{4^n}\right)(F - F')(2^n x, 2^n y) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right)(F - F')(2^n x, -2^n y) \right\| \\
&\le \frac{1}{2^n}\|(F - f)(2^n x, 2^n y)\| + \frac{1}{2^n}\|(F - f)(2^n x, -2^n y)\| \\
&\quad + \frac{1}{2^n}\|(f - F')(2^n x, 2^n y)\| + \frac{1}{2^n}\|(f - F')(2^n x, -2^n y)\| \\
&\le \sum_{j=n}^{\infty} \frac{\Phi_1(2^j x, 2^j y) + \Phi_2(2^j x, 2^j y) + \Phi_1(2^j x, -2^j y) + \Phi_2(2^j x, -2^j y)}{2^{j-1}}
\end{aligned}$$

for all $x, y \in X\backslash A$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X\backslash A$. By Lemma 8, $F(x,y) = F'(x,y)$ for all $x, y \in X$ as desired. □

Corollary 11. *Let $p < 1$ and $\theta > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1 f(x,y,z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$
$$\|J_2 f(x,y,z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$ with $\|x\|, \|y\|, \|z\| \ge 1$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}
\|f(x,y) - F(x,y)\| \le &\left(\frac{(6 + 2^p)}{2(2 - 2^p)} + \frac{16 + 2^p}{2(4 - 2^p)}\right)\theta\|x\|^p \\
&+ \left(\frac{4}{2 - 2^p} + \frac{8 + 2^p + 2 \cdot 3^p}{4 - 2^p}\right)\theta\|y\|^p
\end{aligned}$$

for all $\|x\|, \|y\| \ge 1$. If $\|x\|, \|y\| \ge 1$ and $p \le 0$, then $2^p, 3^p, \|x\|^p, \|y\|^p \le 1$. So, we get the following corollary.

Corollary 12. *Let $p \le 0$ and let f, θ be as in Corollary 11. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le 14\theta$$

for all $\|x\|, \|y\| \ge 1$.

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