International Journal of Fuzzy Logic and Intelligent Systems, vol. 9, no. 1, March 2009 pp. 16-19

Existence and Uniqueness of Solutions for the Fuzzy Differential Equations in n-Dimension Fuzzy Vector Space

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Abstract

In this paper, we study the existence and uniqueness of solutions for the fuzzy differential equations in n-dimension fuzzy vector space $(E_N)^n$ using by Banach fixed point theorem.

Key words : Existence and uniqueness, fuzzy differential equations, n-dimension fuzzy vector space, Banach fixed point theorem.

1. Introduction

Many authors have studied several concepts of fuzzy systems. Diamond and Kloeden [2] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \;\; x(0) = x_0,$$

where $x(\cdot), u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [4] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently, Park, Park and Kwun [6] find the sufficient condition of nonlocal controllability for the semilinear fuzzy integrodifferential equation with nonlocal initial condition.

In this paper, we study the the existence and uniqueness of solutions for the following fuzzy differential equations:

$$\frac{dx_i(t)}{dt} = a_i(t)x_i(t) + f_i(t, x_i(t)) \text{ on } E_N^i \qquad (1)$$

$$x_i(0) = x_{0_i} \in E_N^i \qquad (i = 1, 2, \cdots, n) \qquad (2)$$

where $a_i : [0,T] \to E_N^i$ is fuzzy coefficient, E_N^i is the set of all upper semi-continuously convex fuzzy numbers with $E_N^i \neq E_N^j$ $(i \neq j)$, $f : [0,T] \times E_N^i \to E_N^i$ is a nonlinear regular fuzzy function and $x_{0_i} \in E_N^i$ is initial condition.

Manuscript received Oct. 11, 2008; revised Feb. 19, 2009.
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This study was supported by research funds from Dong-A University.

2. Preliminaries

A fuzzy subset (in short a fuzzy set) of R^n is function $u : R^n \to [0,1]$. For each such fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$, its α-level set.

Let u, v be fuzzy subset of R^n. It is well know that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0,1]$ implies $u = v$.

Let E^n denote the collection of all fuzzy sets of R^n that satisfy the following conditions:

1. u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_o) = 1$;
2. u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for any $x, y \in R^n, 0 \leq \lambda \leq 1$;
3. $u(x)$ is upper semi-continuous, i.e., $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n$ $(k = 0, 1, 2, \cdots)$, $x_k \to x_0$;
4. $[u]^0$ is compact.

And we call $u \in E^n$ a n-dimension fuzzy number.

For any $u_i \in E$, $i = 1, 2, \cdots, n$, we call the ordered one-dimension fuzzy number class $u_1, u_2, \cdots, u_n$

(i.e., the Cartesian product of one-dimension fuzzy number $u_1, u_2, \cdots, u_n$) a n-dimension fuzzy vector, denote it as $(u_1, u_2, \cdots, u_n)$, and call the collection of all n-dimension fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}$) n-dimension fuzzy vector space, and denote it as $(E)^n$.

Definition 2.1[8] If $u \in E^n$, and $[u]^\alpha$ is a hyperrectangle, i.e., $[u]^\alpha$ can be represented by $\prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$, i.e., $[u_{1l}^\alpha, u_{1r}^\alpha] \times [u_{2l}^\alpha, u_{2r}^\alpha] \times \cdots \times [u_{nl}^\alpha, u_{nr}^\alpha]$ for every $\alpha \in [0,1]$, where $u_{il}^\alpha, u_{ir}^\alpha \in R$ with $u_{il}^\alpha \leq u_{ir}^\alpha$ when $\alpha \in (0,1]$, $i = 1, 2, \cdots, n$, then we call u a fuzzy n-cell number. And we denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

Theorem 2.1[8] For any $u \in L(E^n)$ with $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$, there exists a unique $(u_1, u_2, \cdots, u_n) \in (E)^n$ such that $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \cdots, n$ and $\alpha \in [0,1])$.

Conversely, for any $(u_1, u_2, \cdots, u_n) \in (E)^n$ with $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \cdots, n$ and $\alpha \in [0,1])$, there exists a unique $u \in L(E^n)$ such that $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$.

Note 2.1 Theorem 2.1 indicates that fuzzy n-cell numbers and n-dimension fuzzy vectors can be represented each other, so $L(E^n)$ and $(E)^n$ may be regarded as identity. If $(u_1, u_2, \cdots, u_n) \in (E)^n$ is the unique n-dimension fuzzy vector determined by $u \in L(E^n)$, then we denote $u = (u_1, u_2, \cdots, u_n)$.

Definition 2.2[8] The complete metric D_L on $(E_N^i)^n$ is defined by

$$\begin{aligned} D_L(u, v) &= \sup_{0<\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0<\alpha\leq 1} \max_{1\leq i<n} \{|u_{il}^\alpha - v_{il}^\alpha, \ u_{ir}^\alpha - v_{ir}^\alpha|\} \end{aligned}$$

for any $u, v \in (E_N^i)^n$, which satisfy $d_L(u + w, v + w) = d_L(u, v)$.

Definition 2.3 Let $u, v \in C([0, T] : (E_N^i)^n)$

$$H_1(u, v) = \sup_{0\leq t\leq T} D_L(u(t), v(t)).$$

Definition 2.4[8] The derivative $x'(t)$ of a fuzzy process $x \in (E_N^i)^n$ is defined by

$$[x'(t)]^\alpha = \prod_{i=1}^n [(x_{il}^\alpha)'(t), (x_{ir}^\alpha)'(t)]$$

provided that is equation defines a fuzzy $x'(t) \in (E_N^i)^n$.

Definition 2.5[8] The fuzzy integral $\int_b^a x(t)dt$, $a, b \in [0, T]$ is defined by

$$\Big[\int_b^a x(t)dt\Big]^\alpha = \prod_{i=1}^n \Big[\int_b^a x_{il}^\alpha(t)dt, \int_b^a x_{ir}^\alpha(t)dt\Big]$$

provided that the Lebesgue integrals on the right exist.

3. Existence and Uniqueness

In this section we consider the existence and uniqueness of the fuzzy solution for the equation(1)-(2).

We defined by

$$a = (a_1, a_2, \cdots, a_n),$$

$$x = (x_1, x_2, \cdots, x_n),$$

$$f = (f_1, f_2, \cdots, f_n),$$

and

$$x_0 = (x_{0_1}, x_{0_2}, \cdots, x_{0_n})$$

then

$$a, \ x, \ f, \ x_0 \ \in \ (E_N^i)^n$$

Instead of (1)-(2), we consider the following fuzzy integral equations in $(E_N^i)^n$.

$$\frac{dx(t)}{dt} = a(t)x(t) + f(t, x(t)) \tag{3}$$

$$x(0) = x_0 \tag{4}$$

with fuzzy coefficient $a : [0, T] \to (E_N^i)^n$, initial value $x_0 \in (E_N^i)^n$, and given nonlinear regular fuzzy function $f : [0, T] \times (E_N^i)^n \to (E_N^i)^n$ is satisfy global Lipschitz condition.

Definition 3.1 The fuzzy process $x : I \to (E_N^i)^n$ is a fuzzy solution of the equation (3)-(4) without inhomogeneous term if and only if

$$\begin{aligned} (x_{il}^\alpha)'(t) &= \min\{a_{ij}^\alpha(t) x_{ik}^\alpha(t) : j, k = l, r\} \\ (x_{ir}^\alpha)'(t) &= \max\{a_{ij}^\alpha(t) x_{ik}^\alpha(t) : j, k = l, r\} \\ x_{il}^\alpha(0) &= x_{0_{il}}^\alpha, \ \ x_{ir}^\alpha(0) = x_{0_{ir}}^\alpha. \end{aligned}$$

Theorem 3.1 For every $x_0 \in (E_N^i)^n$

$$\frac{dx(t)}{dt} = a(t)x(t), \qquad x(0) = x_0$$

has a unique fuzzy solution $x \in C([0, T] : (E_n^i)^n)$.

Proof. Assume that the value x_0 and $a(t)$ are positive fuzzy numbers.
From the definition of fuzzy solution,

$$(x_{il}^{\alpha})'(t) = a_{il}^{\alpha}(t)x_{il}^{\alpha}(t),$$
$$(x_{ir}^{\alpha})'(t) = a_{ir}^{\alpha}(t)x_{ir}^{\alpha}(t)$$

and

$$x_{il}^{\alpha}(t) = \exp\Big(\int_0^t a_{il}^{\alpha}(s)ds\Big)x_{0_{il}}^{\alpha},$$
$$x_{ir}^{\alpha}(t) = \exp\Big(\int_0^t a_{ir}^{\alpha}(s)ds\Big)x_{0_{ir}}^{\alpha}.$$

Define $S(t) \in (E_n^i)^n$ and

$$\begin{aligned}&[S(t)]^{\alpha}\\ = &\prod_{i=1}^{n}[S_i(t)]^{\alpha} = \prod_{i=1}^{n}[S_{il}^{\alpha}(t), S_{ir}^{\alpha}(t)]\\ = &\prod_{i=1}^{n}\Big[\exp\Big(\int_0^t a_{il}^{\alpha}(s)ds\Big), \exp\Big(\int_0^t a_{ir}^{\alpha}(s)ds\Big)\Big]\end{aligned}$$

where $s_{ij}^{\alpha}(t)$ $(i = 1, 2, \cdots, n, j = l, r)$ is continuous.
Therefore

$$\begin{aligned}[x(t)]^{\alpha} &= \prod_{i=1}^{n}[x_{il}^{\alpha}(t), x_{ir}^{\alpha}(t)]\\ &= \prod_{i=1}^{n}[S_{il}^{\alpha}(t)x_{0_{il}}^{\alpha}, S_{ir}^{\alpha}(t)x_{0_{ir}}^{\alpha}] = [S(t)x_0]^{\alpha}.\end{aligned}$$

From the definition of fuzzy derivative,

$$\begin{aligned}&[x_i'(t)]^{\alpha}\\ = &[(x_{il}^{\alpha})'(t), (x_{ir}^{\alpha})'(t)]\\ = &[(S_{il}^{\alpha}x_{0_{il}}^{\alpha})'(t), (S_{ir}^{\alpha}x_{0_{ir}}^{\alpha})'(t)],\ i = 1, 2, \cdots, n.\end{aligned}$$

Therefore

$$\begin{aligned}&(x_{il}^{\alpha})'(t) = (S_{il}^{\alpha}x_{0_{il}}^{\alpha})'(t) = (S_{il}^{\alpha})'(t)x_{0_{il}}^{\alpha}\\ = &a_{il}^{\alpha}(t)S_{il}^{\alpha}(t)x_{0_{il}}^{\alpha} = a_{il}^{\alpha}(t)x_{il}^{\alpha}(t)\\ &(x_{ir}^{\alpha})'(t) = (S_{ir}^{\alpha}x_{0_{ir}}^{\alpha})'(t) = (S_{ir}^{\alpha})'(t)x_{0_{ir}}^{\alpha}\\ = &a_{ir}^{\alpha}(t)S_{ir}^{\alpha}(t)x_{0_{ir}}^{\alpha} = a_{ir}^{\alpha}(t)x_{ir}^{\alpha}(t)\end{aligned}$$

for each $i = 1, 2, \cdots, n$.
Hence

$$\begin{aligned}[x'(t)]^{\alpha} &= \prod_{i=1}^{n}[(x_{il}^{\alpha})'(t), (x_{ir}^{\alpha})'(t)]\\ &= \prod_{i=1}^{n}[a_{il}^{\alpha}(t)x_{il}^{\alpha}(t), a_{ir}^{\alpha}(t)x_{ir}^{\alpha}(t)]\\ &= [a(t)x(t)]^{\alpha}.\end{aligned}$$

Since a n-dimension fuzzy vector $x(t)$ may be decomposed into its level sets through the resolution identity, $x(t) = S(t)x_0$ is fuzzy solution.

(H1) Assume that there exists a finite constant $k > 0$ such that

$$d_L([f(s, x(s))]^{\alpha}, [f(s, y(s))]^{\alpha}) \le k d_L([x(s)]^{\alpha}, [y(s)]^{\alpha})$$

for all $x(s), y(s) \in (E_N^i)^n$.

(H2) That is there exists a constant $c > 0$ such that $|S_{ij}^{\alpha}(t)| \le c$ for all $t \in [0, T]$.

The equation (3)-(4) is related to the following fuzzy integral equations:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds \quad (5)$$
$$x(0) = x_0 \quad (6)$$

Theorem 3.2 Let $T > 0$, satisfies hypotheses (H1) and (H2), for every $x_0 \in (E_N^i)^n$, (5)-(6) has a unique fuzzy solution $x \in C([0, T] : (E_N^i)^n)$.

Proof. For $x(t) \in (E_N^i)^n$, $t \in [0, T]$, define

$$(G_0x)(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds.$$

Thus, $G_0x : [0.T] \to (E_N^i)^n$is continuous, and $G_0 : C([0, T] : (E_N^i)^n) \to C([0, T] : (E_N^i)^n)$. For $x, y \in C([0, T] : (E_N^i)^n)$,

$$\begin{aligned}&d_L\big([(G_0x)(t)]^{\alpha}, [(G_0y)(t)]^{\alpha}\big)\\ = &d_L\Big(\Big[S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds\Big]^{\alpha},\\ &\Big[S(t)x_0 + \int_0^t S(t-s)f(s, y(s))ds\Big]^{\alpha}\Big)\\ = &d_L\Big([S(t)x_0]^{\alpha} + \Big[\int_0^t S(t-s)f(s, x(s))ds\Big]^{\alpha},\\ &[S(t)x_0]^{\alpha} + \Big[\int_0^t S(t-s)f(s, y(s))ds\Big]^{\alpha}\Big)\\ = &d_L\Big(\Big[\int_0^t S(t-s)f(s, x(s))ds\Big]^{\alpha},\\ &\Big[\int_0^t S(t-s)f(s, y(s))ds\Big]^{\alpha}\Big)\\ \le &\int_0^t d_L\big([S(t-s)f(s, x(s))]^{\alpha},\\ &[S(t-s)f(s, y(s))]^{\alpha}\big)ds\end{aligned}$$

$$\begin{aligned}
&= \int_0^t \max_{1\le i\le n} \Big\{ |S_{il}^{\alpha}(t-s)f_{il}^{\alpha}(s,x(s)) \\
&\qquad -S_{il}^{\alpha}(t-s)f_{il}^{\alpha}(s,y(s))|, \\
&\quad |S_{ir}^{\alpha}(t-s)f_{ir}^{\alpha}(s,x(s)) \\
&\qquad -S_{ir}^{\alpha}(t-s)f_{ir}^{\alpha}(s,y(s))| \Big\} ds \\
&= \int_0^t \max_{1\le i\le n} \Big\{ |S_{il}^{\alpha}(t-s)\big(f_{il}^{\alpha}(s,x(s)) - f_{il}^{\alpha}(s,y(s))\big)|, \\
&\quad |S_{ir}^{\alpha}(t-s)\big(f_{ir}^{\alpha}(s,x(s)) - f_{ir}^{\alpha}(s,y(s))\big)| \Big\} ds \\
&\le c\int_0^t \max_{1\le i\le n} \Big\{ |f_{il}^{\alpha}(s,x(s)) - f_{il}^{\alpha}(s,y(s))|, \\
&\quad |f_{ir}^{\alpha}(s,x(s)) - f_{ir}^{\alpha}(s,y(s))| \Big\} ds \\
&= c\int_0^t d_L\Big([f(s,x(s))]^{\alpha}, [f(s,y(s))]^{\alpha}\Big) ds \\
&\le ck\int_0^t d_L\big([x(s)]^{\alpha}, [y(s)]^{\alpha}\big) ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
& D_L\big((G_0x)(t), (G_0y)(t)\big) \\
&= \sup_{0<\alpha\le 1} d_L\big([(G_0x)(t)]^{\alpha}, [(G_0y)(t)]^{\alpha}\big) \\
&\le ck \sup_{0<\alpha\le 1} \int_0^t d_L\big([x(s)]^{\alpha}, [y(s)]^{\alpha}\big) ds \\
&\le ck \int_0^t D_L\big(x(s), y(s)\big) ds.
\end{aligned}$$

Hence

$$\begin{aligned}
H_1(G_0x, G_0y) &= \sup_{0\le t\le T} D_L\big((G_0x)(t), (G_0y)(t)\big) \\
&\le ck \sup_{0\le t\le T} \int_0^t D_L\big(x(s), y(s)\big) ds \\
&\le ckT\ H_1(x,y).
\end{aligned}$$

We take sufficiently small T, $ckT < 1$. Hence G_0 is a contraction mapping.
By the Banach fixed point theorem, (5)-(6) has a unique fixed point $x \in C([0.T] : (E_N^i)^n)$.

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