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The Properties of Fuzzy Relations

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Abstract

We investigate the properties of fuzzy relations and $\odot$-equivalence relation on a stsc quantale lattice L and a commutative cqm-lattice. In particular, we find $\odot$-equivalence relations induced by fuzzy relations.

Key words : stsc-quantales, commutative cqm-lattice, $\odot$-equivalence relations

1. Introduction and preliminaries

Quantales were introduced by Mulvey [11,12] as the non-commutative generalization of the lattice of open sets in topological spaces. Recently, quantales have arisen in an analysis of the semantics of linear logic systems developed by Girard [4], which supports part of foundation of theoretic computer science. Recently, Höhle [6-8,13] developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices. Bělohlávek [1-3] investigate the properties of fuzzy relations and similarities on a residual lattice.

In this paper, we investigate the properties of fuzzy relations and $\odot$-equivalence relation on a stsc-quantale lattice and a commutative cqm-lattice. In particular, we find $\odot$-equivalence relations induced by fuzzy relations.

Definition 1.1. [6-8, 11-13] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) if it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(Q2) $(L, \odot)$ is a commutative semigroup;

(Q3) $a = a \odot 1$, for each $a \in L$;

(Q4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Remark 1.2. [6-8](1) A completely distributive lattice is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantale.

(3) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,
$(x \odot y) \leq z$ iff $x \leq (y \to z)$.

Lemma 1.3. [6-8,13] Let $(L, \leq, \odot)$ be a stsc-quantale with a strong negation $x^* = x \to 0$. Let $x, y, z, x_i, y_i \in L$ for all $i \in \Gamma$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(5) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$.

(6) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.

(7) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(8) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$.

(9) $y \odot z \leq x \to (x \odot y \odot z)$ and $x \odot (x \odot y \to z) \leq y \to z$.

(10) $(x \odot y)^* = x \to y^*$.

(11) $(x \to y) \odot (y \to z) \leq x \to z$.

(12) $x \to y = 1$ iff $x \leq y$.

(13) $x \to y = y^* \to x^*$.

Definition 1.4. Let X and Y be nonempty sets. A map $R : X \times Y \to L$ is called a fuzzy relation.

Definition 1.5. [1-3], [6-8,13] Let X be a set. A function $R : X \times X \to L$ is called:

(R1) reflexive if $R(x,x) = 1$ for all $x \in X$,

(R2) symmetric if $R(x,y) = R(y,x)$, for all $x, y \in X$,

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(R3) transitive if $R(x,y)\odot R(y,z)\le R(x,z)$, for all $x,y,z\in X$.

If R satisfies (R1) and (R2), R is an $\odot$-quasi-equivalence relation. If an $\odot$-quasi-equivalence relation R satisfies (R2), then R is an $\odot$-equivalence relation.

2. Fuzzy Relations

Theorem 2.1. Let X be a set and $A,B\in L^X$. We define R_{A*B}, for each $*\in\{\odot,\rightarrow,\leftarrow,\oplus,\leftrightarrow\}$ as follows:

$$R_{A*B}(x,y)=A(x)*B(y).$$

We have the following properties.

(1) $R_{A\odot B},R_{A\rightarrow A},R_{A\leftarrow A},R_{A\leftrightarrow A}$ are transitive.

(2) $R_{A\oplus B^*}=R_{A\leftarrow B}=R^s_{B\rightarrow A}$ where $R^s(x,y)=R(y,x)$.

(3) $R_{A\oplus A^*}$ is reflexive, if $A\le B$, then $R_{A\rightarrow B}$ and $R_{B\leftarrow A}$ are reflexive.

(4) $R_{A\odot A},R_{A\oplus A},R_{A\leftrightarrow A}$ are symmetric.

(5) $R_{A\rightarrow A}$ and $R_{A\leftarrow A}$ are quasi-equivalence relation. Moreover, $R_{A\leftrightarrow A}$ is an equivalence relation.

(6) If $X=\{x_1,...,x_n\}$, then $R_{A*B}=A*B^t$ defined as

$$R_{A*B}=\begin{pmatrix}A(x_1)\\ .\\ A(x_1)\end{pmatrix}*(B(x_1)...B(x_n))$$

$$=\begin{pmatrix}A(x_1)*B(x_1) & ... & A(x_1)*B(x_n)\\ ... & ... & ...\\ A(x_n)*B(x_1) & ... & A(x_n)*B(x_n)\end{pmatrix}$$

Proof. (1) It follows from

$$A(x)\odot B(y)\odot A(y)\odot B(z)\le A(x)\odot B(z)$$

by lemma 1.3 (11),

$$(A(x)\rightarrow A(y))\odot(A(y)\rightarrow A(z))\le(A(x)\rightarrow A(z))$$

$$(A(y)\rightarrow A(x))\odot(A(z)\rightarrow A(y))\le(A(z)\rightarrow A(x))$$

(2) By Lemma 1.3 (10). $R_{A\oplus B^*}(x,y)=(A(x)^*\odot B(y)^{**})^*=B(y)\rightarrow A(x)=R_{A\leftarrow B}(x,y)=R^s_{B\rightarrow A}(y,x)$.

(3) For $A\le B$, by Lemma 1.3 (12), $R_{A\rightarrow B}(x,x)=A(x)\rightarrow B(x)=1$. Other cases are similarly proved.

(4) Since operations $\odot,\oplus$ and $\leftrightarrow$ are commutative, it is trivial.

(5) It follows from (1) and (3).

(6) It is trivial.

□

Example 2.2. Define a binary operation $\odot$ (called Łukasiewicz conjection) on $[0,1]$ by

$$x\odot y=\max\{0,x+y-1\},\ x\rightarrow y=\min\{1-x+y,1\}$$

$$x\oplus y=\min\{1,x+y\}.$$

Then $([0,1],\vee,\odot,0,1)$ is a stsc-quantale (ref.[6-8]). Let $X=\{x_1,x_2,x_3\}$ be a set and $A(x_1)=0.9,A(x_2)=0.6,A(x_3)=0.8$. We regard A as $(0.9,0.6,0.8)^t$. By Theorem 2.1(6), we obtain

$$R_{A\odot A}=\begin{pmatrix}0.8&0.5&0.7\\0.5&0.2&0.4\\0.7&0.4&0.6\end{pmatrix}R_{A\rightarrow A}=\begin{pmatrix}1.0&0.7&0.9\\1.0&1.0&1.0\\1&0.8&1.0\end{pmatrix}$$

$$R_{A\leftarrow A}=\begin{pmatrix}1.0&1.0&1.0\\0.7&1.0&0.8\\0.9&1.0&1.0\end{pmatrix}R_{A\oplus A}=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

$$R_{A\leftrightarrow A}=\begin{pmatrix}1.0&0.7&0.9\\0.7&1.0&0.8\\0.9&0.8&1.0\end{pmatrix}$$

Theorem 2.3. Let $R_1\in L^{X\times Y}$ and $R_2\in L^{Y\times Z}$ be fuzzy relations. The compositions of R_1 and R_2 are defined as

$$R_1\circ R_2(x,z)=\bigvee_{y\in Y}R_1(x,y)\odot R_2(y,z)$$

$$(R_1\Rightarrow R_2)(x,z)=\bigwedge_{y\in Y}(R_1(x,y)\rightarrow R_2(y,z))$$

$$(R_1\Leftarrow R_2)(x,z)=\bigwedge_{y\in Y}(R_2(y,z)\rightarrow R_1(x,y))$$

$$(R_1\Leftrightarrow R_2)(x,z)=\bigwedge_{y\in Y}(R_1(x,y)\leftrightarrow R_2(y,z))$$

$$(R_1\oplus R_2)(x,z)=\bigwedge_{y\in Y}(R_1(x,y)\oplus R_2(y,z))$$

$$R^s_1(y,x)=R_1(x,y)$$

where $x\oplus y=(x^*\odot y^*)^*$. Then we have the following properties.

(1) $(R_1\circ R_2)^s=R^s_2\circ R^s_1$.

(2) $(R_1\circ R_2)^*=R_1\Rightarrow R^*_2=R_2\Rightarrow R^*_1=R^*_1\oplus R^*_2$.

(3) $(R_1\Rightarrow R_2)^s=R^s_2\Leftarrow R^s_1=(R^s_2)^*\Rightarrow(R^s_1)^*$.

(4) $(R_1\Leftarrow R_2)^s=R^s_2\Rightarrow R^s_1$.

(5) $(R_1\Leftrightarrow R_2)^s=R^s_2\Leftrightarrow R^s_1$.

Proof. (1)

$$\begin{aligned}(R_1\circ R_2)^s(z,x)&=(R_1\circ R_2)(x,z)\\&=\bigvee_{y\in V}(R_1(x,y)\odot R_2(y,z))\\&=\bigvee_{y\in V}(R^s_2(z,y)\odot R^s_1(y,x))\\&=R^s_2\circ R^s_1(z,x).\end{aligned}$$

(2) By Lemma 1.3 (4,7), we have

$$\begin{aligned}(R_1 \circ R_2)^*(x,z) &= \Big(\bigvee_{y\in V}(R_1(x,y)\odot R_2(y,z))\Big) \to 0\\ &= \bigwedge_{y\in V}\Big(R_1(x,y)\to(R_2(y,z)\to 0)\Big)\\ &= (R_1 \Rightarrow R_2^*)(x,z)\\ &= (R_2 \Rightarrow R_1^*)(x,z) = R_1^* \ominus R_2^*(x,z).\end{aligned}$$

(3)

$$\begin{aligned}(R_1 \Rightarrow R_2)^s(z,x) &= (R_1 \Rightarrow R_2)(x,z)\\ &= \bigwedge_{y\in Y}(R_1(x,y)\to R_2(y,z))\\ &= \bigwedge_{y\in Y}(R_1^s(y,x)\to R_2^s(z,y))\\ &= (R_2^s \Leftarrow R_1^s)(z,x)\\ &= \bigwedge_{y\in Y}(R_2^*(y,z)\to R_1^*(x,y))\\ &= \bigwedge_{y\in Y}((R_2^s)^*(z,y)\to (R_1^s)^*(y,x))\\ &= ((R_2^s)^* \Rightarrow (R_1^s)^*)(z,x).\end{aligned}$$

(4) It is similarly proved as in (3).

(5)

$$\begin{aligned}(R_1 \Leftrightarrow R_2)^s &= (R_1 \Rightarrow R_2)^s \wedge (R_1 \Leftarrow R_2)^s\\ &= (R_2^s \Leftarrow R_1^s)\wedge(R_2^s \Rightarrow R_1^s)\\ &= (R_2^s \Leftrightarrow R_1^s).\end{aligned}$$

□

Theorem 2.4. Let $R \in L^{X\times X}$ be a fuzzy relation. We have the following properties.

(1) If R is reflexive, then $R\circ R$ is reflexive, $R \le (R\circ R)$, $(R \Rightarrow R) \le R$, $(R^s \Rightarrow R) \le R$, $(R \Leftarrow R) \le R$ and $(R \Leftarrow R^s) \le R$.

(2) $(R\circ R)^* = (R^* \oplus R^*)$. If R^* is reflexive, then $R \oplus R \le R$.

(3) R is symmetric iff $(R \Rightarrow R)$ is reflexive iff $(R \Leftarrow R)$ is reflexive.

(4) If R is symmetric, then $R\circ R$ is symmetric, $(R \Leftarrow R)^s = R \Rightarrow R$, $(R \Rightarrow R)^s = R^* \Rightarrow R^*$ and $R \Leftrightarrow R$ is symmetric and reflexive.

(4) R is symmetric iff $(R \Rightarrow R)$ is reflexive iff $(R \Leftarrow R)$ is reflexive.

(5) $R^s\circ R \le R$ iff $R \le (R \Rightarrow R)$. Moreover, $R\circ R^s \le R$ iff $R \le (R \Leftarrow R)$.

(6) R is transitive iff $R\circ R \le R$ iff $R \le (R^s \Rightarrow R)$ iff $R \le (R \Leftarrow R^s)$. Moreover, R^* is transitive iff $R \le R\oplus R$.

(7) If $R^{*s}\circ R^* \le R^*$, then $R \le R\oplus R$.

(8) If R is an $\odot$-quasi-equivalence relation, then $R = (R\circ R) = (R^s \Rightarrow R) = (R \Leftarrow R^s)$ and $R^* = R^* \ominus R^*$.

(9) $R^s \circ R$ and $R\circ R^s$ are symmetric.

(10) $R^s\circ R \le R$ and R is reflexive iff R is an $\odot$-equivalence relation iff $(R \Rightarrow R)$ and R are reflexive and $R \le (R \Rightarrow R)$ iff $(R \Leftarrow R)$ and R are reflexive and $R \le (R \Leftarrow R)$.

(11) If $R^{*s}\circ R^* \le R^*$ and R^* is reflexive, then $R = R\oplus R$.

(12) If R is an $\odot$-equivalence relation, then $R = (R\circ R) = (R \Rightarrow R) = (R \Leftarrow R)$ and $R^* = R^* \oplus R^*$.

(13) If R is reflexive and symmetric, then $R \Leftrightarrow R$ is an $\odot$-equivalence relation.

(14) Let R be reflexive and symmetric. We define

$$R^\infty(x,y) = \bigvee_{n\in N} R^n(x,y)$$

Where $R^n = \overbrace{R\circ R\ldots\circ R}^{n}$. Then R^∞ is an $\odot$-equivalence relation.

(15) $(R \Leftrightarrow R^s)$ and $(R^s \Leftrightarrow R)$ are $\odot$-equivalence relations.

Proof. (1) Since $R\circ R(x,x) \ge R(x,x)\odot R(x,x) = 1$, $R\circ R$ is reflexive.

$$\begin{aligned}(R \Rightarrow R)(x,z) &= \bigwedge_{y\in X}(R(x,y)\to R(y,z))\\ &\le (R(x,x)\to R(x,z)) = R(x,z)\end{aligned}$$

Other cases are similarly proved.

(2) Since R^* is reflexive, by (1), $R^* \le R^*\circ R^*$. Thus $R \ominus R = (R^*\circ R^*)^* \le R$.

(3) It easily proved because

$$\begin{aligned}&(R \Rightarrow R)(x,x) = \bigwedge_{y\in X}(R(x,y)\to R(y,x)) = 1\\ &\text{iff } R(x,y) \le R(y,x) \text{ (by Lemma 1.3 (12))}.\end{aligned}$$

(4) $(R\circ R)^s = R^s\circ R^s = R\circ R$. $(R \Leftarrow R)^s = (R^s \Rightarrow R^s) = (R \Rightarrow R)$.

$$\begin{aligned}(R \Leftrightarrow R)^s &= (R \Rightarrow R)^s \wedge (R \Leftarrow R)^s\\ &= (R \Leftarrow R)\wedge(R \Rightarrow R) = (R \Leftrightarrow R).\end{aligned}$$

(5) It easily proved because

$$\begin{aligned}&R^s(x,y)\odot R(y,z) \le R(x,z) \text{ iff } R(y,z) \le R(y,x)\to R(x,z)\\ &R(x,y)\odot R^s(y,z) \le R(x,z) \text{ iff } R(x,y) \le R(z,y)\to R(x,z).\end{aligned}$$

(6) and (7) follow from (5).

(8) It easily proved from (1) and (6).

(9) It follows from $(R^s\circ R)^s = R^s\circ R$ and $(R\circ R^s)^s = R\circ R^s$.

(10) ($\Rightarrow$) Since R is reflexive, $R \le R^s\circ R$. Thus $R = R^s\circ R$. By (9), R is symmetric. Since $R = R^s$ and $R\circ R = R$, R is transitive.

($\Rightarrow$) Let R be an $\odot$-equivalence relation. By (8), $(R^s \Rightarrow R) = (R \Rightarrow R) = R$.

($\Rightarrow$) Let $(R \Rightarrow R)$ and R be reflexive and $R \le (R \Rightarrow R)$. Then R is symmetric. Thus $(R \Leftarrow R)$ is reflexive and $(R \Leftarrow R) = (R^s \Rightarrow R^s)^s = R^s = R$.

($\Leftarrow$) $(R \Leftarrow R)$ and R are reflexive and $R \le R \Leftarrow R$. Then $R^s\circ R = R\circ R^s \le R$.

(13) $(R \Leftrightarrow R)(x,x) = \bigwedge_{z\in U}(R(x,z)\leftrightarrow R(z,x)) = \bigwedge_{z\in U}(R(x,z)\leftrightarrow R(x,z)) = 1$.

$$\begin{aligned}&R(x,p)\odot(R(x,p)\to R(p,y))\odot(R(p,y)\to R(p,z))\\ &\le R(p,z)\\ &\Leftrightarrow (R(x,p)\to R(p,y))\odot(R(p,y)\to R(p,z))\\ &\le R(x,p)\to R(p,z)\end{aligned}$$

Similarly, $R(z,p) \rightarrow R(p,y)) \odot (R(p,y) \rightarrow R(p,x)) \leq R(z,p) \rightarrow R(p,x)$. Hence $(R \Leftrightarrow R)(x,y) \odot (R \Leftrightarrow R)(y,z) \leq (R \Leftrightarrow R)(x,z)$.

(14) Suppose there exist $x, y, z \in X$ such that

$$R^{\infty}(x,y) \circ R^{\infty}(y,z) \not\leq R^{\infty}(x,z).$$

By the definition of $R^{\infty}(x,y)$, there exists $x_i \in X$ such that

$$R(x,x_1) \odot R(x_1,x_2) \odot \ldots \odot R(x_n,y) \circ R^{\infty}(y,z) \not\leq R^{\infty}(x,z).$$

By the definition of $R^{\infty}(y,z)$, there exists $y_j \in X$ such that

$$R(x,x_1) \odot R(x_1,x_2) \odot \ldots \odot R(x_n,y)$$

$$\odot R(y,y_1) \odot R(y_1,y_2) \odot \ldots \odot R(y_n,z) \not\leq R^{\infty}(x,z).$$

It is a contradiction for the definition of $R^{\infty}(x,z)$.

(15) Let $R = (a_{ij})$ and $(R \Leftrightarrow R^s) = (b_{ij})$ be $n \times n$ be matrices.

Since $b_{ii} = \bigwedge_{m \in N}(a_{im} \leftrightarrow a_{im}) = 1$, $(R \Leftrightarrow R^s) = (b_{ij})$ is reflexive.

Since $(a_{im} \rightarrow a_{jm}) \odot (a_{jm} \rightarrow a_{km}) \leq (a_{im} \rightarrow a_{km})$, it implies

$$\begin{aligned} &(R \Leftrightarrow R^s)(x_i,x_j) \odot (R \Leftrightarrow R^s)(x_j,x_k) = b_{ij} \odot b_{jk} \\ &\leq \bigwedge_{m \in N}((a_{im} \rightarrow a_{jm}) \odot (a_{jm} \rightarrow a_{km})) \\ &\leq \bigwedge_{m \in N}(a_{im} \rightarrow a_{km}) \end{aligned}$$

Similarly, $(R \Leftrightarrow R^s)(x_i,x_j) \odot (R \Leftrightarrow R^s)(x_j,x_k) \leq \bigwedge_{m \in N}(a_{km} \rightarrow a_{im})$. Hence $(R \Leftrightarrow R^s)(x_i,x_j) \odot (R \Leftrightarrow R^s)(x_j,x_k) \leq (R \Leftrightarrow R^s)(x_i,x_k)$.

□

Example 2.5. Define a binary operation $\odot$ as same in Example 2.2. Let $R_i \in L^{X \times X}$ on $X = \{a,b\}$ as follows:

$$R_1 = \begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 1.0 \end{pmatrix} R_2 = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.5 \end{pmatrix}$$

(1) $(R_1 \Rightarrow R_2)^s = (R_2^s)^* \Rightarrow (R_1^s)^*$ from

$$\begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 1.0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.5 \end{pmatrix}$$

$$\begin{pmatrix} 0.3 & 0.1 \\ 0.4 & 0.5 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.0 & 1.0 \\ 0.6 & 0.0 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.9 \\ 0.6 & 0.5 \end{pmatrix}$$

$(R_1 \Leftrightarrow R_2) \neq (R_2 \Leftrightarrow R_1)$ from

$$\begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 1.0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.6 \\ 0.3 & 0.4 \end{pmatrix}$$

$$\begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.5 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 1.0 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{pmatrix}$$

(2) Since R_1 is reflexive $R_1 \circ R_1 = R_1$, R_1 is transitive and $(R_1 \Rightarrow R_1) \leq R_1$. But $R_1^s \circ R_1 \not\leq R_1$, $R_1 \not\leq (R_1 \Rightarrow R_1)$, $R_1 \not\leq (R_1 \Leftarrow R_1)$,$(R_1 \Leftarrow R_1) \leq R_1$ and $R_1^s \Rightarrow R_1 = R_1$ from:

$$R_1^s \circ R_1 = \begin{pmatrix} 1.0 & 0.4 \\ 0.4 & 1.0 \end{pmatrix} (R_1 \Rightarrow R_1) = \begin{pmatrix} 0.6 & 0.4 \\ 0.0 & 1.0 \end{pmatrix}$$

$$(R_1^s \Rightarrow R_1) = \begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 1.0 \end{pmatrix} (R_1 \Leftarrow R_1) = \begin{pmatrix} 1.0 & 0.4 \\ 0.0 & 0.6 \end{pmatrix}$$

(3) Since R_2 is not reflexive $R_2 \circ R_2 \leq R_2$, R_2 is transitive and $R_2^s \circ R_2 \not\leq R_2$, $R_2 \not\leq R_2 \Rightarrow R_2$, $R_2 \circ R_2^s \not\leq R_2$ and $R_2 \not\leq R_2 \Leftarrow R_2$ from:

$$R_2^s \circ R_2 = \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} R_2 \circ R_2^s = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

$$R_2 \Rightarrow R_2 = \begin{pmatrix} 1.0 & 0.9 \\ 0.6 & 0.7 \end{pmatrix} R_2 \Leftarrow R_2 = \begin{pmatrix} 0.7 & 1.0 \\ 0.6 & 1.0 \end{pmatrix}$$

(4) Let $R \in L^{X \times X}$ as follows;

$$R = \begin{pmatrix} 0.5 & 0.8 & 0.3 \\ 0.8 & 0.4 & 0.6 \\ 0.3 & 0.6 & 0.9 \end{pmatrix}$$

$$R \Leftarrow R = (R \Rightarrow R)^s = R^* \Rightarrow R^*$$

$$R \Leftarrow R = \begin{pmatrix} 1.0 & 0.7 & 0.4 \\ 0.6 & 1.0 & 0.7 \\ 0.8 & 0.5 & 1.0 \end{pmatrix}$$

(5) Let $R \in L^{X \times X}$ as follows;

$$R = \begin{pmatrix} 1.0 & 0.4 & 0.9 \\ 0.4 & 1.0 & 0.1 \\ 0.9 & 0.1 & 1.0 \end{pmatrix}$$

$$R^{\infty} = (R \circ R) = \begin{pmatrix} 1.0 & 0.4 & 0.9 \\ 0.4 & 1.0 & 0.3 \\ 0.9 & 0.3 & 1.0 \end{pmatrix}$$

$$(R \Leftrightarrow R) = (R \Leftrightarrow R) \circ (R \Leftrightarrow R) = \begin{pmatrix} 1.0 & 0.2 & 0.7 \\ 0.2 & 1.0 & 0.1 \\ 0.7 & 0.1 & 1.0 \end{pmatrix}$$

(6) Let E be an identity relation and $R \in L^{X \times X}$ as

$$R = \begin{pmatrix} 0.5 & 0.7 & 0.8 \\ 0.5 & 0.2 & 0.6 \\ 0.9 & 1.0 & 0.3 \end{pmatrix} (R \Leftrightarrow R^s) = \begin{pmatrix} 1.0 & 0.5 & 0.5 \\ 0.5 & 1.0 & 0.2 \\ 0.5 & 0.2 & 1.0 \end{pmatrix}$$

$$(R^s \Leftrightarrow R) = \begin{pmatrix} 1.0 & 0.7 & 0.4 \\ 0.7 & 1.0 & 0.6 \\ 0.4 & 0.6 & 1.0 \end{pmatrix}$$

Let S be a reflexive and symmetric relation defined as:

$$S = (R \vee R^s \vee E) = \begin{pmatrix} 1.0 & 0.7 & 0.9 \\ 0.7 & 1.0 & 1.0 \\ 0.9 & 1.0 & 1.0 \end{pmatrix}$$

We obtain two $\odot$-equivalence relations $S \Leftrightarrow S$ and S^∞ as follows:

$$(S \Leftrightarrow S) = \begin{pmatrix} 1.0 & 0.7 & 0.7 \\ 0.7 & 1.0 & 0.8 \\ 0.7 & 0.8 & 1.0 \end{pmatrix}$$

$$S^\infty = S \circ S = \begin{pmatrix} 1.0 & 0.9 & 0.9 \\ 0.9 & 1.0 & 1.0 \\ 0.9 & 1.0 & 1.0 \end{pmatrix}$$

Two reflexive and symmetric relations $T = (R \wedge R^s) \vee E$ and $W = (R \circ R^s) \vee E$ induce $\odot$-equivalence relations

$$T = (R \wedge R^s) \vee E = \begin{pmatrix} 1.0 & 0.5 & 0.8 \\ 0.5 & 1.0 & 0.6 \\ 0.8 & 0.6 & 1.0 \end{pmatrix}$$

$$T = T \circ T = (T \Leftrightarrow T).$$

$$W = (R \circ R^s) \vee E = \begin{pmatrix} 1.0 & 0.4 & 0.7 \\ 0.4 & 1.0 & 0.4 \\ 0.7 & 0.4 & 1.0 \end{pmatrix}$$

$$W = W \circ W = (W \Leftrightarrow W).$$

Definition 2.6. Let $(L, \odot)$ be a stsc-quantale. A function $T : L \to L$ is called an equivalence transformation map if it satisfies the following conditions:
(1) $T(1) = 1$,
(2) if $x \le y$, then $T(x) \le T(y)$,
(3) $T(x) \odot T(y) \le T(x \odot y)$.

Theorem 2.7. Let R be an $\odot$-equivalence relation and T an equivalence transformation map. Then $T \circ R$ is an $\odot$-equivalence relation.

Proof. Since $T(R(x,x)) = T(1) = 1$, $T \circ R$ is reflexive. Moreover, $T(R(x,y)) = T(R(y,x))$ and $T \circ R$ is transitive because

$$\begin{aligned} T(R(x,y)) \circ T(R(y,z)) &= T(R(x,y) \odot R(y,z)) \text{ (by (3))} \\ &\le T(R(x,z)) \text{ (by (2))}. \end{aligned}$$

□

Example 2.8. Define a binary operation $\odot$ as same in Example 2.2. Define $T : [0,1] \to [0,1]$ as $T(x) = x^2$. Then T is an equivalence transformation because

$$(T(x) + T(y) - 1) \vee 0 \le T((x + y - 1) \vee 0).$$

Since R is an $\odot$-equivalence relation, we obtain $\odot$-equivalence relation $T \circ R$ as follows:

$$R = \begin{pmatrix} 1.0 & 0.7 & 0.5 \\ 0.7 & 1.0 & 0.6 \\ 0.5 & 0.6 & 1.0 \end{pmatrix} \; T \circ R = \begin{pmatrix} 1.00 & 0.49 & 0.25 \\ 0.49 & 1.00 & 0.36 \\ 0.25 & 0.36 & 1.00 \end{pmatrix}$$

Theorem 2.9. (1) If R_i is an $\odot$-equivalence relation for each $i \in I$, then $\bigwedge_{i \in I} R_i$ is an $\odot$-equivalence relation.

(2) Let R and S be $\odot$-quasi-equivalence relations. $R \vee S$ is an $\odot$-quasi-equivalence relation iff $R \circ S \subset R \vee S$ and $S \circ R \subset R \vee S$.

(3) Let R and S be $\odot$-equivalence relations. $R \circ S$ is an $\odot$-equivalence relation iff $R \circ S = S \circ R$.

Proof. (1) is easily proved.

(2) $(\Rightarrow)$

$$\begin{aligned} R \circ S(x,z) &= \bigvee_y (R(x,y) \odot S(y,z)) \\ &\le \bigvee_y ((R \vee S)(x,y) \odot (R \vee S)(y,z)) \\ &\le (R \vee S)(x,z). \end{aligned}$$

$(\Leftarrow)$ We only show that $R \vee S$ is transitive.

$$\begin{aligned} &(R \vee S)(x,y) \odot (R \vee S)(y,z) \\ &= (R(x,y) \vee S(x,y)) \odot (R(y,z) \vee S(y,z)) \\ &= (R(x,y) \odot R(y,z) \vee (S(x,y) \odot R(y,z)) \\ &\quad \vee (R(x,y) \odot S(y,z)) \vee (S(x,y) \odot S(y,z)) \\ &\le R(x,z) \vee (S \circ R)(x,z) \vee (R \circ S)(x,z) \vee S(x,z) \\ &\le (R \vee S)(x,z) \end{aligned}$$

(3) $(\Rightarrow)$ Since $R \circ S$ is an $\odot$-equivalence relation,

$$\begin{aligned} R \circ S(x,z) &= R \circ S(z,x) \\ &= \bigvee_{y \in X} (R(z,y) \odot S(y,x)) \\ &= \bigvee_{y \in X} (R(y,z) \odot S(x,y)) \\ &= S \circ R(x,z) \end{aligned}$$

$(\Leftarrow)$ We only show that $R \circ S$ is transitive from:

$$\begin{aligned} &R \circ S(x,y) \odot R \circ S(y,z) \\ &= \bigvee_{y_1 \in X} [R(x,y_1) \odot S(y_1,y)] \\ &\odot \bigvee_{z_1 \in X} [S(y,z_1) \odot R(z_1,z)] \\ &= \bigvee_{y_1 \in X} \bigvee_{z_1 \in X} \Big([R(x,y_1) \odot S(y_1,y)] \\ &\odot [S(y,z_1) \odot R(z_1,z)] \Big) \\ &= \bigvee_{y_1 \in X} \bigvee_{z_1 \in X} \Big([R(x,y_1) \\ &\odot (S(y_1,y) \odot S(y,z_1)) \odot R(z_1,z)] \Big) \\ &\le \bigvee_{y_1 \in X} \bigvee_{z_1 \in X} \Big([R(x,y_1) \odot S(y_1,z_1) \odot R(z_1,z)] \Big) \\ &= \bigvee_{y_1 \in X} \bigvee_{z_1 \in X} \Big(R(x,y_1) \odot \\ &\bigvee_{z_1 \in X} [S(y_1,z_1) \odot R(z_1,z)] \Big) \\ &= \bigvee_{z_1 \in X} \Big(\bigvee_{y_1 \in X} [R(x,y_1) \odot R(y_1,z_1)] \odot S(z_1,z)] \Big) \\ &= \bigvee_{z_1 \in X} \Big(R(x,z_1) \odot S(z_1,z)] \Big) \\ &= R \circ S(x,z) \end{aligned}$$

□

Example 2.10. Define a binary operation $\odot$ as same in Example 2.2.

(1) Let R and S be $\odot$-quasi-equivalence relations as follows:

$$R = \begin{pmatrix} 1.0 & 0.8 & 0.7 \\ 0.7 & 1.0 & 0.6 \\ 0.5 & 0.9 & 1.0 \end{pmatrix} \quad S = \begin{pmatrix} 1.0 & 0.7 & 0.8 \\ 0.9 & 1.0 & 0.7 \\ 0.6 & 0.5 & 1.0 \end{pmatrix}$$

$$R \circ S = \begin{pmatrix} 1.0 & 0.8 & 0.8 \\ 0.9 & 1.0 & 0.7 \\ 0.8 & 0.9 & 1.0 \end{pmatrix}$$

$$R \vee S = S \circ R = \begin{pmatrix} 1.0 & 0.8 & 0.8 \\ 0.9 & 1.0 & 0.7 \\ 0.6 & 0.9 & 1.0 \end{pmatrix}$$

Since $R \circ S \not\leq R \vee S$, $R \vee S$ is not an $\odot$-quasi-equivalence relation.

(2) Let R and S be $\odot$-equivalence relations as follows:

$$R = \begin{pmatrix} 1.0 & 0.7 & 0.3 \\ 0.7 & 1.0 & 0.0 \\ 0.3 & 0.0 & 1.0 \end{pmatrix} \quad S = \begin{pmatrix} 1.0 & 0.0 & 0.1 \\ 0.0 & 1.0 & 0.8 \\ 0.1 & 0.8 & 1.0 \end{pmatrix}$$

$$R \circ S = \begin{pmatrix} 1.0 & 0.7 & 0.5 \\ 0.7 & 1.0 & 0.8 \\ 0.3 & 0.8 & 1.0 \end{pmatrix} \quad S \circ R = \begin{pmatrix} 1.0 & 0.7 & 0.3 \\ 0.7 & 1.0 & 0.8 \\ 0.5 & 0.8 & 1.0 \end{pmatrix}$$

Since $R \circ S \neq S \circ R$, $R \circ S$ is not an $\odot$-equivalence relation because it is neither symmetric nor transitive as follows:

$$0.5 = R \circ S(x, z) \neq R \circ S(z, x) = 0.3.$$

$$0.5 = R \circ S(z, y) \odot R \circ S(y, x) \not\leq R \circ S(z, x) = 0.3.$$

(3) Let R and S be $\odot$-equivalence relations as follows:

$$R = \begin{pmatrix} 1.0 & 0.8 & 0.5 \\ 0.8 & 1.0 & 0.7 \\ 0.5 & 0.7 & 1.0 \end{pmatrix} \quad S = \begin{pmatrix} 1.0 & 0.6 & 0.8 \\ 0.6 & 1.0 & 0.8 \\ 0.8 & 0.8 & 1.0 \end{pmatrix}$$

$$R \circ S = \begin{pmatrix} 1.0 & 0.8 & 0.8 \\ 0.8 & 1.0 & 0.8 \\ 0.8 & 0.8 & 1.0 \end{pmatrix} \quad S \circ R = \begin{pmatrix} 1.0 & 0.8 & 0.8 \\ 0.8 & 1.0 & 0.8 \\ 0.8 & 0.8 & 1.0 \end{pmatrix}$$

Since $R \circ S = S \circ R = R \vee S$, $R \circ S = R \vee S$ is an $\odot$-equivalence relation.

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