Robust Adaptive Control for a Class of Nonlinear Systems with Complex Uncertainties

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Abstract – This paper considers a robust adaptive stabilization problem for a class of uncertain nonlinear systems which include an unknown virtual control coefficient, an unknown constant parameter, and a time-varying disturbance whose bound is unknown. We propose a new estimator for an unknown virtual control coefficient and present a robust adaptive backstepping design procedure which results in a smooth state feedback control law, a new two-dimensional parameter update law, and a C^1 Lyapunov function which is positive definite and proper.

Keywords: Robust control, Adaptive backstepping, Unknown virtual coefficient, Nonlinear parameterization

1. Introduction

In this paper, we consider the robust adaptive stabilization problem for uncertain systems described by

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x,\theta) + g_1(x,d(t)) \\
&\vdots \\
\dot{x}_k &= x_{k+1} + f_k(x,\theta) + g_k(x,d(t)) \\
\dot{x}_{k+1} &= b_{k+1}x_{k+2} + f_{k+1}(x,\theta) + g_{k+1}(x,d(t)) \\
\dot{x}_{k+2} &= x_{k+3} + f_{k+2}(x,\theta) + g_{k+2}(x,d(t)) \\
&\vdots \\
\dot{x}_n &= u + f_n(x,\theta) + g_n(x,d(t)),
\end{aligned} \tag{1}$$

where $x=[x_1,\dots,x_n]^T$ is the system state, $u\in\mathbb{R}$ is the control input, $\theta\in\mathbb{R}^p$ is an unknown constant vector, $f_i:\mathbb{R}^{n+p}\to\mathbb{R}$, $i=1,\dots,n$, are C^1 functions with $f_i(0,\dots,0,\theta)=0$, $d(t)$ is an unknown piecewise continuous disturbance or parameter belonging to an unknown compact set $\Omega\subset\mathbb{R}^q$, $g_i:\mathbb{R}^{n+q}\to\mathbb{R}$, $i=1,\dots,n$, are C^1 functions with $g_i(0,\dots,0,d(t))=0$, and $b_k\in\mathbb{R}$ is an unknown constant, called an unknown virtual control coefficient. We assume the sign of b_k is known.

Global robust adaptive regulation problem for uncertain nonlinear systems with unknown parameters and disturbances is one of the most important subjects of control engineering and many researchers have made an effort to solve this problem. Among the various topics in robust and adaptive control, we concentrate on the system (1) with an unknown virtual control coefficient, a nonlinearly parameterized uncertainty, and a time-varying disturbance ranging over an unknown compact set. The key features of this paper are that we introduce a new parameter estimator for the unknown virtual control coefficient and that we consider uncertain systems with disturbances whose bounds are unknown.

When there is some uncertainty related to the control direction, the designer cannot know how the control input affects the behavior of the system. Because of this obstacle, the problem has gained particular interest for linear systems [1]-[6] and nonlinear systems [7]-[11]. Among them, the result [11] is regarded as a standard solution and one of the main assumptions made is that the sign of the unknown virtual control coefficient is completely known. In [11], a two-dimensional update law for the virtual control coefficient is proposed for the system (1) with linear parameterization, i.e., $f_i(x,\theta)=\tilde{f}_i(x_1,\dots,x_i)\theta$ and $g_i(\cdot)=0$, $i=1,\dots,n$. On the other hand, we concentrate on designing a one-dimensional update law for the unknown virtual control coefficient for the system (1) and define the estimator as $\tilde{b}_k=|b_k|-\exp(\gamma\hat{b}_k)$ to exploit the fact $\exp(\gamma\hat{b}_k)>0$. This construction is one of the main ideas of the paper and has the advantage of decreasing the dimension of the update law.

The adaptive backstepping controller proposed in [11] requires that all uncertain parameters appear in the linear way. However, systems such as biochemical processes and dynamics with friction have unknown parameters that enter the system nonlinearly. To tackle the adaptive control problem for such cases, some results have been provided under several restrictive conditions on the unknown parameters: boundedness of the nonlinear parameters in [12] and the convex/concave parameterization in [13]. Compared to these, global adaptive regulation proposed in [14]-[15] does not rely on such restrictive conditions. To handle the uncertain terms $f_i(\cdot)$ with unknown parameters, we adopt the decomposition method in [14]-[15] where the system with $g_i(\cdot)=0$ is considered.

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Received 24 October 2008; Accepted 22 January 2009

When dealing with a controlled plant, one often faces time-varying disturbances. Although many results have been suggested to solve the robust control problem [16]-[18], they assume that the disturbances are bounded and their bounds are known, that is, the time-varying disturbances belong to a known compact set. Unlike the previous results, we suppose that the bound of the disturbance is unknown to handle a wider class of systems. Based on the backstepping scheme combined with a domination method for system uncertainties, we estimate the bound of the disturbance. This is another main idea of this paper.

This paper is organized as follows. We introduce some assumptions and key tools in Section 2. In Section 3, our main result is presented and a constructive design and a recursive proof are provided by developing a C^1 Lyapunov function, a smooth virtual controller and tuning functions at each step. In Sections 4 and 5, an illustrative example and concluding remarks are given.

2. Assumptions and Key Tools

In the following, $\|\cdot\|$ denotes the usual Euclidean norm. We present some assumptions for uncertainties.

Assumption 1. *There is an unknown constant* $0 \le \bar{d} < \infty$ *such that*

$$\|d(t)\| \le \bar{d}, \ \forall t \ge 0. \tag{2}$$

Assumption 2. *There are non-negative smooth functions* $\bar{f}_i(x_{[i]},\theta)$ *and* $\bar{g}_i(x_{[i]},d(t))$ *such that*

$$|f_i(x,\theta)| \le \left(\sum_{j=1}^{i}|x_j|\right)\bar{f}_i(\cdot), \ |g_i(x,d(t))| \le \left(\sum_{j=1}^{i}|x_j|\right)\bar{g}_i(\cdot) \tag{3}$$

where $x_{[i]} = [x_1,\ldots,x_i]^T$.

Next, we recall an inequality from [14] which will be frequently used throughout the paper.

- For any real-valued continuous function $f(x,y)$ where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, there are smooth scalar functions $a(x) \ge 1$, $b(y) \ge 1$ such that

$$f(x,y) \le a(x)b(y). \tag{4}$$

Using (4), we can deduce that

$$\begin{aligned} \bar{f}_i(x_{[i]},\theta) &\le \tilde{f}_i(x_{[i]})a_i(\theta) \\ \bar{g}_i(x_{[i]},d(t)) &\le \tilde{g}_i(x_{[i]})b_i(\bar{d}), \end{aligned} \tag{5}$$

where $\tilde{f}_i(\cdot) \ge 1$, $a_i(\cdot) \ge 1$, $\tilde{g}_i(\cdot) \ge 1$, $b_i(\cdot) \ge 1$ are smooth functions.

Based on (5), we define

$$\begin{aligned} &\Theta_1 = \sum_{i=1}^{n}\left(a_i(\theta)+b_i(\bar{d})\right) \ge 1, \ \Theta = \Theta_1^2 \ge \Theta_1 \\ &\tilde{\Theta} = \Theta - \hat{\Theta}(t), \end{aligned} \tag{6}$$

where $\hat{\Theta}(t)$ represents the estimate of Θ.

3. Global Robust Adaptive Regulation

Theorem 1. *Suppose that the sign of* b_k *of the system (1) is known. Then, there exists a dynamic smooth controller*

$$\begin{aligned} u &= u(x,\xi) \\ \dot{\xi} &= \Omega(x,\xi), \ \ \xi \in \mathbb{R}^2, \end{aligned} \tag{7}$$

which renders the closed-loop system globally stable in the sense of Lyapunov and makes all trajectories of the closed loop system satisfy that

$$\lim_{t\to\infty} x(t) = 0, \ \forall (x(0),\xi(0)) \in \mathbb{R}^n \times \mathbb{R}^2.$$

Proof of Theorem 1: We divide the proof into two parts, one for the upper subsystem (of dimension k) and the other for the lower subsystem. The proof for the first part is standard and thus moved to the Appendix, while the second part involves the unknown virtual control coefficient which should be dealt with carefully. The results of the first part are virtual controls

$$\begin{aligned} &x_1^* \equiv 0 && \bar{x}_1 = x_1 - x_1^* \\ &x_2^* = -\alpha_1(x_1,\hat{\Theta}) && \bar{x}_2 = x_2 - x_2^* \\ &\ \vdots && \ \vdots \\ &x_{k+1}^* = -\alpha_k(x_{[k]},\hat{\Theta}) && \bar{x}_{k+1} = x_{k+1} - x_{k+1}^*, \end{aligned} \tag{8}$$

tuning functions

$$\begin{aligned} &\Pi_{1,1} \equiv 0 \\ &\Pi_{2,1}(x_{[2]},\hat{\Theta}) = \Pi_{2,1}(\cdot) + \gamma\bar{x}_2\tfrac{\partial x_2^*}{\partial\hat{\Theta}} \\ &\ \vdots \\ &\Pi_{k,1}(x_{[k]},\hat{\Theta}) = \Pi_{k-1,1}(\cdot) + \gamma\bar{x}_k\tfrac{\partial x_k^*}{\partial\hat{\Theta}} \\ &\Pi_{1,2}(x_1) = x_1^2\sigma_1(\cdot) \\ &\Pi_{2,2}(x_{[2]},\hat{\Theta}) = \Pi_{1,2}(\cdot) + \bar{x}_2^2\sigma_2(\cdot) \\ &\ \vdots \\ &\Pi_{k,2}(x_{[k]},\hat{\Theta}) = \Pi_{k-1,2}(\cdot) + \bar{x}_k^2\sigma_k(\cdot), \end{aligned}$$

where $\sigma_i(\cdot) \ge 0$, $i = 1,\ldots,k$, are smooth functions, and the Lyapunov function

$$V_k(x_{[k]},\tilde{\Theta}) = \sum_{i=1}^{k}\frac{\bar{x}_i^2}{2} + \frac{\tilde{\Theta}^2}{2\gamma}.$$

Readers are referred to the Appendix for detailed derivation. From now on, we start the second part of the proof, i.e., the case in which the unknown virtual control coefficient appears.

Step $k+1$: At this step, since the unknown virtual control coefficient b_{k+1} appears, we need to design its estimator. We define

$$\tilde{b}_{k+1} = |b_{k+1}| - \exp(\gamma_1\hat{b}_{k+1}) \tag{9}$$

and choose a C^1, positive definite, and proper Lyapunov function as

$$V_{k+1}(x_{[k+1]},\tilde{\Theta},\tilde{b}_{k+1}) = V_k(x_{[k]},\tilde{\Theta}) + \frac{\bar{x}_{k+1}^2}{2} + \frac{\tilde{b}_{k+1}^2}{2\gamma_2}$$

where γ_1 and γ_2 are tuning gains. The design of (9) is one of our key ideas. Different from [11], the design reduces the dimension of the parameter update law for the unknown virtual control coefficient, that is, a one-dimensional update law instead of a two-dimensional one.

We compute the time derivative of V_{k+1} like

$$\dot{V}_{k+1} = \dot{V}_k + \bar{x}_{k+1}(b_{k+1}x_{k+2} + f_{k+1}(\cdot) + g_{k+1}(\cdot)) - \bar{x}_{k+1}\left(\sum_{i=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_i}\dot{x}_i + \frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\dot{\hat{\Theta}}\right) - \frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\tilde{b}_{k+1}\dot{\hat{b}}_{k+1}. \tag{10}$$

Using (3), (5), (8), and Young's inequality, we have

$$\bar{x}_{k+1}(f_{k+1}(\cdot) + g_{k+1}(\cdot)) - \bar{x}_{k+1}\sum_{i=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_i}(f_i(\cdot) + g_i(\cdot)) \le \sum_{i=1}^{k}\frac{\bar{x}_i^2}{n} + \bar{x}_{k+1}^2\sigma_{k+1}(x_{[k+1]}, \hat{\Theta})\Theta, \tag{11}$$

where $\sigma_{k+1}(\cdot) \ge 0$ is a smooth function. A detailed argument can be founded in the Appendix.

Substituting (11) into (10), we arrive at

$$\begin{aligned}\dot{V}_{k+1} \le &-\sum_{i=1}^{k}\left(c_i + 1 - \frac{k+1-i}{n}\right)\bar{x}_i^2 + \left(\tilde{\Theta} + \Pi_{k+1,1}(\cdot)\right)\left(\Pi_{k+1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) + \bar{x}_k\bar{x}_{k+1} \\ &+ b_{k+1}\bar{x}_{k+1}\bar{x}_{k+2} - \bar{x}_{k+1}\sum_{i=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_i}x_{i+1} + \bar{x}_{k+1}^2\sigma_{k+1}(\cdot)\hat{\Theta} \\ &- \bar{x}_{k+1}\left(\gamma\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\Pi_{k,2}(\cdot) + \bar{x}_{k+1}\sigma_{k+1}(\cdot)\Pi_{k+1,1}(\cdot)\right) \\ &- \frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\tilde{b}_{k+1}\dot{\hat{b}}_{k+1},\end{aligned}$$

where

$$\Pi_{k+1,1}(x_{[k+1]}, \hat{\Theta}) = \Pi_{k,1}(\cdot) + \gamma\bar{x}_{k+1}\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}$$

$$\Pi_{k+1,2}(x_{[k+1]}, \hat{\Theta}) = \Pi_{k,2}(\cdot) + \bar{x}_{k+1}^2\sigma_{k+1}(\cdot).$$

Since $|b_{k+1}|/\exp(\gamma_1\hat{b}_{k+1}) = \tilde{b}_{k+1}/\exp(\gamma_1\hat{b}_{k+1}) + 1$ by (9), we design a smooth virtual control as

$$\begin{aligned}x_{k+2}^* = &\frac{\operatorname{sgn}(b_{k+1})}{\exp(\gamma_1\hat{b}_{k+1})}\Bigg[-\bar{x}_{k+1}\left(c_{k+1} + 1 + \sigma_{k+1}(\cdot)\hat{\Theta}\right) \\ &- \bar{x}_k + \sum_{i=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_i}x_{i+1} + \gamma\frac{\partial x_{k+1}^*}{\partial\hat{\Theta}}\Pi_{k,2}(\cdot) \\ &+ \bar{x}_{k+1}\sigma_{k+1}(\cdot)\Pi_{k+1,1}(\cdot)\Bigg] \\ =: &-\frac{\operatorname{sgn}(b_{k+1})}{\exp(\gamma_1\hat{b}_{k+1})}\alpha_{k+1}(x_{[k+1]}, \hat{\Theta}),\end{aligned} \tag{12}$$

$$\bar{x}_{k+2} = x_{k+2} - x_{k+2}^*.$$

With these components, it is easy to have

$$\begin{aligned}\dot{V}_{k+1} \le &-\sum_{i=1}^{k+1}\left(c_i + 1 - \frac{k+1-i}{n}\right)\bar{x}_i^2 + \left(\tilde{\Theta} + \Pi_{k+1,1}(\cdot)\right)\left(\Pi_{k+1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\ &- \left(\Psi_{k+1,1}(\cdot) + \frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\dot{\hat{b}}_{k+1}\right)\left(\tilde{b}_{k+1} + \Psi_{k+1,2}\right) \\ &+ b_{k+1}\bar{x}_{k+1}\bar{x}_{k+2},\end{aligned}$$

where

$$\Psi_{k+1,1}(x_{[k+1]}, \hat{\Theta}) = \frac{\bar{x}_{k+1}\alpha_{k+1}(\cdot)}{\exp(\gamma_1\hat{b}_k)}, \quad \Psi_{k+1,2} = 0.$$

Step $k+2$: Choose the Lyapunov function as $V_{k+2}(x_{[k+2]}, \tilde{\Theta}, \tilde{b}_{k+1}) = V_{k+1}(x_{[k+1]}, \tilde{\Theta}, \tilde{b}_{k+1}) + \bar{x}_{k+2}^2/2$. It can be deduced from the result of step $k+1$ that

$$\begin{aligned}\dot{V}_{k+2} = &\dot{V}_{k+1} + \bar{x}_{k+2}(x_{k+3} + f_{k+2}(\cdot) + g_{k+2}(\cdot)) \\ &- \bar{x}_{k+2}\sum_{i=1}^{k}\frac{\partial x_{k+2}^*}{\partial x_i}\dot{x}_i - \bar{x}_{k+2}\frac{\partial x_{k+2}^*}{\partial x_{k+1}}(b_{k+1}x_{k+2} + f_{k+1}(\cdot) + g_{k+1}(\cdot)) \\ &- \bar{x}_{k+2}\frac{\partial x_{k+2}^*}{\partial\hat{\Theta}}\dot{\hat{\Theta}} - \bar{x}_{k+2}\frac{\partial x_{k+2}^*}{\partial\hat{b}_{k+1}}\dot{\hat{b}}_{k+1}.\end{aligned} \tag{13}$$

Similarly to (11), we can find a smooth function $\sigma_{k+2}(\cdot) \ge 0$ such that

$$\bar{x}_{k+2}(f_{k+2}(\cdot) + g_{k+2}(\cdot)) - \bar{x}_{k+2}\sum_{i=1}^{k+1}\frac{\partial x_{k+2}^*}{\partial x_i}(f_i(\cdot) + g_i(\cdot)) \le \sum_{i=1}^{k+1}\frac{\bar{x}_i^2}{n} + \bar{x}_{k+2}^2\sigma_{k+2}(x_{[k+2]}, \hat{\Theta})\Theta. \tag{14}$$

By the definition of $\tilde{b}_{k+1}$, we obtain

$$\begin{aligned}&b_{k+1}\bar{x}_{k+2}\left(\bar{x}_{k+1} - \frac{\partial x_{k+2}^*}{\partial x_{k+1}}x_{k+2}\right) \\ &= \left(\tilde{b}_{k+1} + \exp(\gamma_1\hat{b}_{k+1})\right)\bar{x}_{k+2}\operatorname{sgn}(b_{k+1})\left(\bar{x}_{k+2} - \frac{\partial x_{k+2}^*}{\partial x_{k+1}}x_{k+2}\right) \\ &=: \left(\tilde{b}_{k+1} + \exp(\gamma_1\hat{b}_{k+1})\right)\bar{x}_{k+2}\eta_{k+2}(x_{[k+2]}, \hat{\Theta}, \hat{b}_{k+1}).\end{aligned} \tag{15}$$

Substituting the estimates (14) and (15) into (13), we conclude that

$$\begin{aligned}\dot{V}_{k+2} \le &-\sum_{i=1}^{k+1}\left(c_i + 1 - \frac{2-i}{n}\right)\bar{x}_i^2 + (\tilde{\Theta} + \Pi_{k+2,1}(\cdot))\left(\Pi_{k+2,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\ &- \left(\Psi_{k+2,1}(\cdot) + \frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\dot{\hat{b}}_{k+1}\right)\left(\tilde{b}_{k+1} + \Psi_{k+2,2}(\cdot)\right) \\ &+ \bar{x}_{k+2}x_{k+3} - \bar{x}_{k+2}\sum_{i=1}^{k}\frac{\partial x_{k+2}^*}{\partial x_i}x_{i+1} + \bar{x}_{k+2}^2\sigma_{k+2}(\cdot)\hat{\Theta} \\ &+ \bar{x}_{k+2}\exp(\gamma_1\hat{b}_{k+1})\eta_{k+2}(\cdot) \\ &- \bar{x}_{k+2}\left(\gamma\frac{\partial x_{k+2}^*}{\partial\hat{\Theta}}\Pi_{k+1,2}(\cdot) + \bar{x}_{k+2}\sigma_{k+2}(\cdot)\Pi_{k+2,1}(\cdot)\right) \\ &- \bar{x}_{k+2}\left(\eta_{k+2}(\cdot)\Psi_{k+1,2} + \psi_{k+2}(\cdot)\Psi_{k+2,1}(\cdot)\right),\end{aligned}$$

where

$$\psi_{k+2}(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}) = \frac{\partial x^*_{k+2}}{\partial \hat{b}_{k+1}} \frac{\gamma_2}{\gamma_1 \exp(\gamma_1 \hat{b}_{k+1})}$$

$$\Pi_{k+2,1}(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}) = \Pi_{k+1,1}(\cdot) + \bar{x}^2_{k+2}\sigma_{k+2}(\cdot)$$

$$\Pi_{k+2,2}(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}) = \Pi_{k+1,2}(\cdot) + \bar{x}_{k+2}(\cdot)\frac{\partial x^*_{k+2}}{\partial \hat{\Theta}}$$

$$\Psi_{k+2,1}(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}) = \Psi_{k+1,1}(\cdot) - \bar{x}_{k+2}\eta_{k+2}(\cdot)$$

$$\Psi_{k+2,2}(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}) = \Psi_{k+1,2}(\cdot) + \bar{x}_{k+2}\psi_{k+2}(\cdot).$$

Then we can design a smooth controller x^*_{k+3} as follows:

$$\begin{aligned} x^*_{k+3} &= -\bar{x}_{k+2}\left(c_{k+2} + 1 + \sigma_{k+2}(\cdot)\hat{\Theta}\right) - \bar{x}_{k+1} + \sum_{i=1}^{k}\frac{\partial x^*_{k+2}}{\partial x_i}x_{i+1} \\ &\quad - \bar{x}_{k+2}\sigma_{k+2}(\cdot)\hat{\Theta} - \exp(\gamma_1\hat{b}_{k+1})\eta_{k+2}(\cdot) \\ &\quad + \left(\gamma\frac{\partial x^*_{k+2}}{\partial\hat{\Theta}}\Pi_{k+1,2}(\cdot) + \bar{x}_{k+2}\sigma_{k+2}(\cdot)\Pi_{k+2,1}(\cdot)\right) \\ &\quad + \left(\eta_{k+2}(\cdot)\Psi_{k+1,2}(\cdot) + \psi_{k+2}(\cdot)\Psi_{k+2,1}(\cdot)\right) \\ &=: -\alpha_{k+2}\left(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}\right), \\ \bar{x}_{k+3} &= x_{k+3} - x^*_{k+3}. \end{aligned} \tag{16}$$

Inductive Step $(l \ge k+2)$: Suppose that, at step l, there exist a C^1 Lyapunov function $V_l(x_{[l]},\tilde{\Theta},\tilde{b}_{k+1})$, which is positive definite and proper, smooth virtual controls

$$\begin{aligned} x^*_{k+3} &= -\alpha_{k+2}(x_{[k+2]},\hat{\Theta},\hat{b}_{k+1}) \quad & \bar{x}_{k+3} &= x_{k+3} - x^*_{k+3} \\ &\vdots & &\vdots \\ x^*_{l+1} &= -\alpha_l(x_{[l]},\hat{\Theta},\hat{b}_{k+1}) & \bar{x}_{l+1} &= x_{l+1} - x^*_{l+1} \end{aligned} \tag{17}$$

and tuning functions

$$\Pi_{l,1}(x_{[l]},\hat{\Theta},\hat{b}_{k+1}) = \Pi_{l-1,1}(\cdot) + \bar{x}^2_l\sigma_l(\cdot)$$

$$\Pi_{l,2}(x_{[l]},\hat{\Theta},\hat{b}_{k+1}) = \Pi_{l-1,2}(\cdot) + \bar{x}_l\frac{\partial x^*_l}{\partial\hat{\Theta}}$$

$$\Psi_{l,1}(x_{[l]},\hat{\Theta},\hat{b}_{k+1}) = \Psi_{l-1,1}(\cdot) - \bar{x}_l\eta_l(\cdot)$$

$$\Psi_{l,2}(x_{[l]},\hat{\Theta},\hat{b}_{k+1}) = \Psi_{l-1,2}(\cdot) + \bar{x}_l\psi_l(\cdot),$$

such that

$$\begin{aligned} \dot{V}_l &\le -\sum_{i=1}^{l}\left(c_i + 1 - \frac{l-i}{n}\right)\bar{x}^2_i + \left(\tilde{\Theta} + \Pi_{l,1}(\cdot)\right)\left(\Pi_{l,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\ &\quad - \left(\Psi_{l,1}(\cdot) + \frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\dot{\hat{b}}_{k+1}\right)\left(\tilde{b}_{k+1} + \Psi_{l,2}(\cdot)\right) \\ &\quad + \bar{x}_l\bar{x}_{l+1}. \end{aligned} \tag{18}$$

We claim that (18) also holds at step $l+1$. To get this result, we first consider the Lyapunov function

$$V_{l+1}(x_{[l+1]},\tilde{\Theta},\tilde{b}_{k+1}) = V_l(x_{[l]},\tilde{\Theta},\tilde{b}_{k+1}) + \frac{\bar{x}^2_{l+1}}{2}.$$

Then, its time derivative along the system (1) is

$$\begin{aligned} \dot{V}_{l+1} = \dot{V}_l + \bar{x}_{l+1}\Bigg[& x_{l+2} + f_{l+1}(\cdot) - \sum_{i=1}^{k}\frac{\partial x^*_{l+1}}{\partial x_i}(x_{i+1} + f_i(\cdot) + g_i(\cdot)) \\ & - \frac{\partial x^*_{l+1}}{\partial x_{k+1}}(b_{k+1}x_{k+2} + f_{k+1}(\cdot) + g_{k+1}(\cdot)) \\ & - \sum_{i=k+2}^{l}\frac{\partial x^*_{l+1}}{\partial x_i}(x_{i+1} + f_i(\cdot) + g_i(\cdot)) - \frac{\partial x^*_{l+1}}{\partial\hat{\Theta}}\dot{\hat{\Theta}} - \frac{\partial x^*_{l+1}}{\partial\hat{b}_{k+1}}\dot{\hat{b}}_{k+1}\Bigg]. \end{aligned}$$

Note that, since x_{k+1}- subsystem includes b_{k+1}, we have to handle this term carefully.

Estimating uncertain terms, we obtain

$$\begin{aligned} &\bar{x}_{l+1}(f_{l+1}(\cdot) + g_{l+1}(\cdot)) - \bar{x}_{l+1}\sum_{i=1}^{l}\frac{\partial x^*_{l+1}}{\partial x_i}(f_i(\cdot) + g_i(\cdot)) \\ &\le \sum_{i=1}^{l}\frac{\bar{x}^2_i}{n} + \bar{x}^2_{l+1}\sigma_{l+1}(x_{[l+1]},\hat{\Theta})\,\Theta, \end{aligned} \tag{19}$$

where $\sigma_{l+1}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}) \ge 0$ is a smooth function.

To handle the terms including b_{k+1}, we compute

$$\begin{aligned} &-b_{k+1}\bar{x}_{l+1}\frac{\partial x^*_{l+1}}{\partial x_{k+1}}x_{k+2} \\ &= \left(\tilde{b}_{k+1} + \exp(\gamma_1\hat{b}_{k+1})\right)\bar{x}_{l+1}\eta_{l+1}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}), \end{aligned} \tag{20}$$

where $\eta_{l+1}(\cdot) := \mathrm{sgn}(b_{k+1})\left(-x_{k+2}\frac{\partial x^*_{l+1}}{\partial x_{k+1}}\right)$ is a smooth function.

Since $\exp(\gamma_1\hat{b}_{k+1})$ is a positive smooth function, there exists a smooth function $\psi_{l+1}(x_{[l]},\hat{\Theta},\hat{b}_{k+1})$ such that

$$-\bar{x}_{l+1}\frac{\partial x^*_{l+1}}{\partial\hat{b}_{k+1}}\dot{\hat{b}}_{k+1} = -\bar{x}_{l+1}\psi_{l+1}(\cdot)\frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\dot{\hat{b}}_{k+1}. \tag{21}$$

From (19), (20), and (21), it follows that

$$\begin{aligned} \dot{V}_{l+1} &\le -\sum_{i=1}^{l}\left(c_i + 1 - \frac{l+1-i}{n}\right)\bar{x}^2_i + \left(\tilde{\Theta} + \Pi_{l+1,1}(\cdot)\right)\left(\Pi_{l+1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\ &\quad - \left(\Psi_{l+1,1}(\cdot) + \frac{\gamma_1}{\gamma_2}\exp(\gamma_1\hat{b}_{k+1})\dot{\hat{b}}_{k+1}\right)\left(\tilde{b}_{k+1} + \Psi_{l+1,2}(\cdot)\right) + \bar{x}_l\bar{x}_{l+1} \\ &\quad + \bar{x}_{l+1}x_{l+2} - \bar{x}_{l+1}\sum_{i=1}^{k}\frac{\partial x^*_{l+1}}{\partial x_i}x_{i+1} - \bar{x}_{l+1}\sum_{i=k+2}^{l}\frac{\partial x^*_{l+1}}{\partial x_i}x_{i+1} \\ &\quad + \bar{x}^2_{l+1}\sigma_{l+1}(\cdot)\hat{\Theta} + \bar{x}_{l+1}\exp(\gamma_1\hat{b}_{k+1})\eta_{l+1}(\cdot) \\ &\quad - \bar{x}_{l+1}\left(\gamma\frac{\partial x^*_{l+1}}{\partial\hat{\Theta}}\Pi_{l,2}(\cdot) + \bar{x}_{l+1}\sigma_{l+1}(\cdot)\Pi_{l+1,1}(\cdot)\right) \\ &\quad - \bar{x}_{l+1}\left(\eta_{l+1}(\cdot)\Psi_{l,2}(\cdot) + \psi_{l+1}(\cdot)\Psi_{l+1,1}(\cdot)\right), \end{aligned} \tag{22}$$

where

$$\psi_{l+1}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}) = \frac{\partial x^*_{l+1}}{\partial\hat{b}_{k+1}}\frac{\gamma_2}{\gamma_1\exp(\gamma_1\hat{b}_{k+1})}$$

$$\Pi_{l+1,1}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}) = \Pi_{l,1}(\cdot) + \bar{x}^2_{l+1}\sigma_{l+1}(\cdot)$$

$$\Pi_{l+1,2}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}) = \Pi_{l,2}(\cdot) + \bar{x}_{l+1}\frac{\partial x^*_{l+1}}{\partial\hat{\Theta}}$$

$$\Psi_{l+1,1}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}) = \Psi_{l,1}(\cdot) - \bar{x}_{l+1}\eta_{l+1}(\cdot)$$

$$\Psi_{l+1,2}(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}) = \Psi_{l,2}(\cdot) + \bar{x}_{l+1}\psi_{l+1}(\cdot).$$

With the help of (22), the smooth virtual controller is defined as

$$x_{l+2}^{*}=-\bar{x}_{l+1}\left(c_{l+1}+1+\sigma_{l+1}(\cdot)\hat{\Theta}\right)-\bar{x}_{l}+\sum_{i=1}^{k}\frac{\partial x_{l+1}^{*}}{\partial x_{i}}x_{i+1}$$
$$+\sum_{i=k+2}^{l}\frac{\partial x_{l+1}^{*}}{\partial x_{i}}x_{i+1}-\bar{x}_{l+1}\sigma_{l+1}(\cdot)\hat{\Theta}-\exp(\gamma_{1}\hat{b}_{k+1})\eta_{l+1}(\cdot)$$
$$+\left(\gamma\frac{\partial x_{l+1}^{*}}{\partial\hat{\Theta}}\Pi_{l,2}(\cdot)+\bar{x}_{l+1}\sigma_{l+1}(\cdot)\Pi_{l+1,1}(\cdot)\right)$$
$$+\left(\eta_{l+1}(\cdot)\Psi_{l,2}(\cdot)+\psi_{l+1}(\cdot)\Psi_{l+1,1}(\cdot)\right) \tag{23}$$
$$=:-\alpha_{l+1}\left(x_{[l+1]},\hat{\Theta},\hat{b}_{k+1}\right),$$
$$\bar{x}_{l+2}=x_{l+2}-x_{l+2}^{*}.$$

The definition of (23) leads to

$$\dot{V}_{l+1}\le-\sum_{i=1}^{l+1}\left(c_{i}+1-\frac{l+1-i}{n}\right)\bar{x}_{i}^{2}+\left(\tilde{\Theta}+\Pi_{l+1,1}(\cdot)\right)\left(\Pi_{l+1,2}(\cdot)-\frac{\dot{\hat{\Theta}}}{\gamma}\right)$$
$$-\left(\Psi_{l+1,1}(\cdot)+\frac{\gamma_{1}}{\gamma_{2}}\exp(\gamma_{1}\hat{b}_{k+1})\dot{\hat{b}}_{k+1}\right)\left(\tilde{b}_{k+1}+\Psi_{l+1,2}(\cdot)\right)$$
$$+\bar{x}_{l+1}\bar{x}_{l+2}.$$

This completes the inductive proof. Following the inductive steps above, we can decide at step n, that there are a smooth feedback control

$$u(x,\hat{\Theta},\hat{b}_{k+1})=x_{n+1}^{*}=-\alpha_{n}(x,\hat{\Theta},\hat{b}_{k+1}) \tag{24}$$

and an adaptive update law

$$\dot{\xi}=\begin{bmatrix}\dot{\hat{\Theta}}\\ \dot{\hat{b}}_{k+1}\end{bmatrix}=\begin{bmatrix}\Pi_{n,1}(x,\hat{\Theta},\hat{b}_{k+1})\\ -\dfrac{\gamma_{1}}{\gamma_{2}\exp(\gamma_{1}\hat{b}_{k+1})}\Psi_{n,1}(x,\hat{\Theta},\hat{b}_{k+1})\end{bmatrix} \tag{25}$$

which render

$$\dot{V}_{n}\le-\sum_{i=1}^{n}c_{i}\bar{x}_{i}^{2}. \tag{26}$$

From (24)-(26), we can conclude that the closed-loop system is globally stable. Also, by LaSalle's invariance principle, all the bounded trajectories of the system ultimately converge to the largest invariant set in $\left\{(x,\hat{\Theta},\hat{b}_{k+1}):\ \dot{V}_{n}(x,\hat{\Theta},\hat{b}_{k+1})=0\right\}$ as $t\to\infty$. Finally, we arrive at

$$\lim_{t\to\infty}\|x(t)\|=0$$

which completes the proof of Theorem 1.

4. Example

In this section, we consider a system given by

$$\dot{x}_{1}=x_{2}+\ln(1+x_{1}^{2}\theta_{1}^{2})+\frac{d_{1}x_{1}\sin t}{1+x_{2}^{2}}$$
$$\dot{x}_{2}=b_{2}x_{3}+\theta_{2}x_{1}+x_{2}\ln(1+d_{2}^{2}\cos^{2}t)$$
$$\dot{x}_{3}=u,$$

where the unknown constants θ_1, θ_2 and the time-varying disturbances have the flowing relations:

$$\left|\ln(1+x_{1}^{2}\theta_{1}^{2})\right|\le\left|x_{1}\theta_{1}\right|\le\left|x_{1}\right|\sqrt{1+\theta_{1}^{2}},\quad\left|\frac{x_{1}d_{1}\sin t}{1+x_{2}^{2}}\right|\le\left|d_{1}\right|\left|x_{1}\right|$$
$$\left|x_{2}\ln(1+d_{2}^{2}\cos^{2}t)\right|\le\left|x_{2}\right|\left|d_{2}\cos t\right|\le\left|d_{2}\right|\left|x_{2}\right|$$

and we assume that b_2 is an unknown positive constant. While the x_1-subsystem includes a nonlinearly parameterized term and a time-varying disturbance, the x_2-subsystem includes a linearly parameterized term and a time-varying disturbance. Note that we assume the bounds of the disturbances are unknown and the first disturbance includes $1+x_2^2$ as a denominator where we handle this term by the domination method.

To estimate unknown constants, we define

$$\max\left\{\sqrt{1+\theta_{1}^{2}},\sqrt{1+\theta_{2}^{2}},\sqrt{1+d_{1}^{2}},\sqrt{1+d_{2}^{2}}\right\}$$
$$\le\sqrt{1+\theta_{1}^{2}+\theta_{2}^{2}+d_{1}^{2}+d_{2}^{2}}=:\Theta_{1},\quad\Theta:=\Theta_{1}^{2}. \tag{28}$$

Now we design the control u using backstepping with the domination approach and the new estimation method.

As proceeded in the proof of Theorem 1, with the continuously differentiable, proper, and positive definite Lyapunov function $V_1=x_1^2/2+\tilde{\Theta}^2/2\gamma$, we have

$$x_{2}^{*}=-x_{1}\left(c_{1}+1+\hat{\Theta}\right),\quad\bar{x}_{2}=x_{2}-x_{2}^{*}$$
$$\dot{V}_{1}\le-(c_{1}+1)x_{1}^{2}+x_{1}\bar{x}_{2}+\left(\tilde{\Theta}+\Pi_{1,1}\right)\left(\Pi_{1,2}(x_{1})-\frac{\dot{\hat{\Theta}}}{\gamma}\right)$$

where $\Pi_{1,1}=0$ and $\Pi_{1,2}=x_1^2$

Next, let the Lyapunov function $V_2=V_1+\bar{x}_2^2/2a_2+\tilde{b}_2^2/2\gamma_2$ where a_2 is a scaling constant and $\tilde{b}_2=|b_2|-\exp(\gamma_1\hat{b}_2)$. Following the procedure of step $k+1$, we obtain

$$x_{3}^{*}=\frac{\operatorname{sgn}(b_{2})}{\exp(\gamma_{1}\hat{b}_{2})}\left[-a_{2}\bar{x}_{2}\left(c_{2}+1+\sigma_{2}(\cdot)\hat{\Theta}\right)-a_{2}x_{1}+\frac{\partial x_{2}^{*}}{\partial x_{1}}x_{2}\right.$$
$$\left.+\gamma\frac{\partial x_{2}^{*}}{\partial\hat{\Theta}}\Pi_{2,2}(\cdot)\right]=:-\frac{\operatorname{sgn}(b_{2})}{\exp(\gamma_{1}\hat{b}_{2})}\alpha_{2}(x_{[2]},\hat{\Theta}),$$
$$\bar{x}_{3}=x_{3}-x_{3}^{*}$$

And

$$\Pi_{2,1}(x_{[2]},\hat{\Theta})=\Pi_{1,1}+\frac{\gamma\bar{x}_{2}}{a_{2}}\frac{\partial x_{2}^{*}}{\partial\hat{\Theta}},\quad\Pi_{2,2}(x_{[2]},\hat{\Theta})$$
$$=\Pi_{1,2}(x_{1})+\bar{x}_{2}^{2}\sigma_{2}(\cdot)$$

and $\Psi_{2,1}(x_{[2]},\hat{\Theta})=\frac{\bar{x}_2\alpha_2(\cdot)}{\exp(\gamma_1\hat{b}_2)}$, $\Psi_{2,2}=0$.

Finally, we choose the Lyapunov function $V_3=V_2+\bar{x}_3^2/2a_3$ where a_3 is a scaling constant.

After deducing

$$\frac{\Theta_{1}\left|\bar{x}_{3}\right|}{a_{3}}\left[\left|x_{1}\right|\left|\frac{\partial x_{3}^{*}}{\partial x_{1}}\right|+\left(\left|x_{1}\right|+\left|x_{2}\right|\right)\left|\frac{\partial x_{3}^{*}}{\partial x_{2}}\right|\right]\le\frac{x_{1}^{2}+\bar{x}_{2}^{2}}{3}+\bar{x}_{3}^{2}\Theta\sigma_{3}(x_{[3]},\hat{\Theta})$$

Where

$$\sigma_3(\cdot)=\tfrac{3}{2a_3^2}\left(\tfrac{\partial x_3^*}{\partial x_1}\right)^2+\tfrac{3}{2a_3^2}(c_1+2+\hat{\Theta})^2\left(\tfrac{\partial x_3^*}{\partial x_2}\right)^2+\tfrac{3}{4a_3^2}\left(\tfrac{\partial x_3^*}{\partial x_2}\right)^2$$

and $b_2\bar{x}_3\left(\tfrac{\bar{x}_2}{a_2}-\tfrac{\partial x_3^*}{\partial x_2}\tfrac{x_3}{a_3}\right)=\left(\tilde{b}_2+\exp(\gamma_1\hat{b}_2)\right)\bar{x}_3\eta_3(x_{[3]},\hat{\Theta})$, $\eta_3(\cdot)=\mathrm{sgn}(b_2)\left(\tfrac{\bar{x}_2}{a_2}-\tfrac{\partial x_3^*}{\partial x_2}\tfrac{x_3}{a_3}\right)$ we design the control and tuning functions as

$$\begin{aligned}u=&-a_3\bar{x}_3(c_3+1+\sigma_3(\cdot)\hat{\Theta})-a_3\exp(\gamma_1\hat{b}_2)\eta_3(\cdot)\\&+\frac{\partial x_3^*}{\partial x_1}x_2+a_3\left(\frac{\gamma}{a_3}\frac{\partial x_3^*}{\partial\hat{\Theta}}\Pi_{3,2}(\cdot)+\bar{x}_3\sigma_3(\cdot)\Pi_{3,1}(\cdot)\right)\\&-\left(\frac{\gamma_2}{\gamma_1\exp(\gamma_1\hat{b}_2)}\frac{\partial x_3^*}{\partial\hat{b}_2}\right)\Psi_{3,1}(\cdot)\end{aligned}$$

$$\dot{\hat{\Theta}}=\gamma\,\Pi_{3,2}(\cdot),\quad \dot{\hat{b}}_2=-\frac{\gamma_2\Psi_{3,1}(\cdot)}{\exp(\gamma_1\hat{b}_2)},$$

which is followed by

$$\dot{V}_3\le -c_1x_1^2-c_2\bar{x}_2^2-c_3\bar{x}_3^2. \tag{29}$$

The tuning functions have the structure of

$$\begin{aligned}\Pi_{3,1}(x_{[3]},\hat{\Theta},\hat{b}_2)&=\Pi_{2,1}(\cdot)+\frac{\gamma\bar{x}_3}{a_3}\frac{\partial x_3^*}{\partial\hat{\Theta}},\\ \Pi_{3,2}(x_{[3]},\hat{\Theta},\hat{b}_2)&=\Pi_{2,2}(\cdot)+\bar{x}_3^2\sigma_3(\cdot),\\ \Psi_{3,1}(x_{[3]},\hat{\Theta},\hat{b}_2)&=\Psi_{2,1}(\cdot)-\bar{x}_3\eta_3(\cdot),\\ \Psi_{3,2}(x_{[3]},\hat{\Theta},\hat{b}_2)&=\Psi_{2,2}(\cdot)+\frac{\gamma_2}{a_3\gamma_1}\frac{\bar{x}_3}{\exp(\gamma_1\hat{b}_2)}\frac{\partial x_3^*}{\partial\hat{b}_2}.\end{aligned}$$

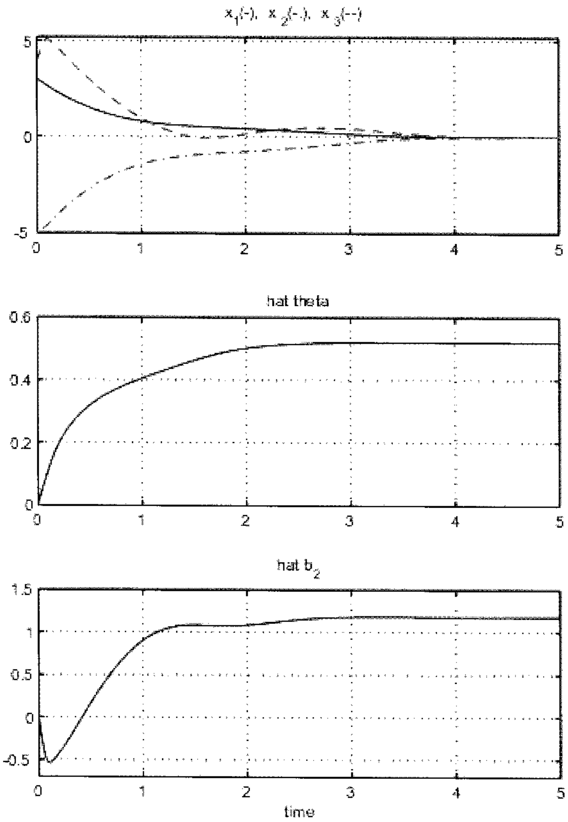

Fig. 1. Simulation result for example

Fig. 1 shows the simulation result. The parameters are $c_1=.5,\ c_2=1,\ c_3=1,\gamma=\gamma_1=.1,\ \gamma_2=4,\ \theta_1=.5, d_1=2,$ $\theta_2=1,\ d_2=2,\ b_2=2,\ a_2=1,\ a_3=10$ and the initial conditions are $x(0)=[3,-5,4]^T,\ \hat{\Theta}(0)=0,\ \hat{b}_2(0)=0$. From the figure, we know that all states are bounded and converge to zero while $\hat{\Theta}(t)$ and $\hat{b}_2(t)$ are bounded. Especially, $\hat{b}_2(t)$ varies very slowly because its dynamics has $\exp(\gamma_1\hat{b}_2)$ as a denominator.

5. Conclusion

In this paper, a smooth robust adaptive controller has been constructed for a class of strict feedback systems with an unknown virtual control coefficient, nonlinearly parameterized uncertainties and time-varying disturbances. The controller and the adaptive update law are obtained by proposing the novel definition of estimation error for the virtual control coefficient, and employing the feedback domination design [14] for the uncertainties and disturbance whose bound is unknown. The result reduces the dimension of the parameter update laws and proposes the method to handle uncertain nonlinear systems that have not been considered.

Appendix

Step 1: Consider the Lyapunov function $V_1(x_{[1]},\tilde{\Theta})=x_1^2/2+\tilde{\Theta}^2/2\gamma$. Then, by (5) and (6), we have

$$\begin{aligned}\dot{V}_1&\le x_1(x_2+f_1(\cdot)+g_1(\cdot))-\frac{\tilde{\Theta}\dot{\hat{\Theta}}}{\gamma}\\&\le x_1x_2+x_1^2(\tilde{f}_1(\cdot)+\tilde{g}_1(\cdot))\Theta-\frac{\tilde{\Theta}\dot{\hat{\Theta}}}{\gamma}\\&\le x_1x_2+x_1^2\sigma_1(x_1)\hat{\Theta}+\tilde{\Theta}\left(x_1^2\sigma_1(x_1)-\frac{\dot{\hat{\Theta}}}{\gamma}\right),\end{aligned}$$

where $\sigma_1(\cdot):=\tilde{f}_1(x_1)+\tilde{g}_1(x_1)$.

From this result, we get the first virtual control

$$\begin{aligned}x_2^*&=-\left(c_1+1+\sigma_1(\cdot)\hat{\Theta}\right)x_1=:-\alpha_1(x_{[1]},\hat{\Theta}),\\ \bar{x}_2&=x_2-x_2^*,\end{aligned}\tag{30}$$

which yields

$$\dot{V}_1\le-(c_1+1)x_1^2+x_1\bar{x}_2+\left(\tilde{\Theta}+\Pi_{1,1}\right)\left(\Pi_{1,2}(\cdot)-\frac{\dot{\hat{\Theta}}}{\gamma}\right),$$

where $\Pi_{1,1}=0,\ \Pi_{1,2}(x_1)=x_1^2\sigma_1(x_1)$

Step 2: Let the Lyapunov function $V_2(x_{[2]},\tilde{\Theta})=V_1(\cdot)$ $+\bar{x}_2/2$. With this selection, we obtain

$$\dot{V}_2 = \dot{V}_1 + \bar{x}_2\dot{x}_2 - \bar{x}_2\left(\frac{\partial x_2^*}{\partial x_1}\dot{x}_1 + \frac{\partial x_2^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}}\right). \quad (31)$$

To separate uncertainties, we need to estimate the terms involving $f_i(\cdot), i=1,2,$ and $g_i(\cdot), i=1,2.$ By (3), (5), (30), and Young's inequality ($(|ab| \le |a|^p/p + |b|^q/q,\ 1/p + 1/q = 1,\ 0 < p, q \in \mathbb{R})$), we obtain

$$\begin{aligned}
&\bar{x}_2(f_2(\cdot)+g_2(\cdot)) - \bar{x}_2\frac{\partial x_2^*}{\partial x_1}(f_1(\cdot)+g_1(\cdot)) \\
&\le |\bar{x}_2|\left(|x_1| + |\bar{x}_2 - \alpha_1(\cdot)x_1|\right)\left(\tilde{f}_2(\cdot)+\tilde{g}_2(\cdot) + \left|\frac{\partial x_2^*}{\partial x_1}\right|\right. \\
&\quad \left.\times\left(\tilde{f}_1(\cdot)+\tilde{g}_1(\cdot)\right)\right)\Theta_1 \\
&\le \frac{x_1^2}{n} + \bar{x}_2^2\sigma_2(x_{[2]},\hat{\Theta})\Theta,
\end{aligned} \quad (32)$$

where $\sigma_2(\cdot)$ is a non-negative smooth function.

Applying (32) into (31) results in

$$\begin{aligned}
\dot{V}_2 \le& -\left(c_1 + 1 - \frac{1}{n}\right)x_1^2 + \left(\tilde{\Theta} + \Pi_{1,1}\right)\left(\Pi_{1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\
&+ x_1\bar{x}_2 + \bar{x}_2x_3 - \bar{x}_2\left(\frac{\partial x_2^*}{\partial x_1}x_2 + \frac{\partial x_2^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}}\right) \\
&+ \bar{x}_2^2\sigma_2(\cdot)\left(\tilde{\Theta} + \hat{\Theta}\right).
\end{aligned}$$

To get the tuning functions [11], we compute

$$\begin{aligned}
&\left(\tilde{\Theta} + \Pi_{1,1}\right)\left(\Pi_{1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) - \bar{x}_2\frac{\partial x_2^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}} + \bar{x}_2^2\sigma_2(\cdot)\tilde{\Theta} \\
&= \left(\tilde{\Theta} + \Pi_{1,1}\right)\left(\Pi_{1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) + \left(\gamma\bar{x}_2\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)\left(\Pi_{1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\
&\quad - \gamma\bar{x}_2\frac{\partial x_2^*}{\partial \hat{\Theta}}\Pi_{1,2}(\cdot) + \bar{x}_2^2\sigma_2(\cdot)\tilde{\Theta} \\
&= \left(\tilde{\Theta} + \Pi_{2,1}(\cdot)\right)\left(\Pi_{1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) - \gamma\bar{x}_2\frac{\partial x_2^*}{\partial \hat{\Theta}}\Pi_{1,2}(\cdot) \\
&\quad + \left(\tilde{\Theta} + \Pi_{2,1}(\cdot)\right)\bar{x}_2^2\sigma_2(\cdot) - \bar{x}_2^2\sigma_2(\cdot)\Pi_{2,1}(\cdot) \\
&= \left(\tilde{\Theta} + \Pi_{2,1}(\cdot)\right)\left(\Pi_{2,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) - \gamma\bar{x}_2\frac{\partial x_2^*}{\partial \hat{\Theta}}\Pi_{1,2}(\cdot) \\
&\quad - \bar{x}_2^2\sigma_2(\cdot)\Pi_{2,1}(\cdot),
\end{aligned}$$

where

$$\Pi_{2,1}(x_{[2]},\hat{\Theta}) = \Pi_{1,1} + \gamma\bar{x}_2\frac{\partial x_2^*}{\partial \hat{\Theta}},$$

$$\Pi_{2,2}(x_{[2]},\hat{\Theta}) = \Pi_{1,2}(\cdot) + \bar{x}_2^2\sigma_2(\cdot).$$

This result leads to

$$\begin{aligned}
\dot{V}_2 \le& -\left(c_1 + 1 - \frac{1}{n}\right)x_1^2 + \left(\tilde{\Theta} + \Pi_{2,1}(\cdot)\right)\left(\Pi_{2,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\
&+ x_1\bar{x}_2 + \bar{x}_2x_3 - \bar{x}_2\frac{\partial x_2^*}{\partial x_1}x_2 + \bar{x}_2^2\sigma_2(\cdot)\hat{\Theta} \\
&- \bar{x}_2\left(\gamma\frac{\partial x_2^*}{\partial \hat{\Theta}}\Pi_{1,2}(\cdot) + \bar{x}_2\sigma_2(\cdot)\Pi_{2,1}(\cdot)\right)
\end{aligned}$$

and we design the virtual control

$$\begin{aligned}
x_3^* =& -\bar{x}_2\left(c_2 + 1 + \sigma_2(\cdot)\hat{\Theta}\right) - x_1 + \frac{\partial x_2^*}{\partial x_1}x_2 + \gamma\frac{\partial x_2^*}{\partial \hat{\Theta}}\Pi_{1,2}(\cdot) \\
&+ \bar{x}_2\sigma_2(\cdot)\Pi_{2,1}(\cdot) \\
=:& -\alpha_2(x_{[2]},\hat{\Theta}), \qquad \bar{x}_3 = x_3 - x_3^*.
\end{aligned}$$

From now on, we shall use an inductive argument. For step *1* to $m-1$ $(3 \le m \le k)$, after a recursive design procedure, we have a set of smooth virtual controls

$$\begin{array}{ll}
x_1^* \equiv 0 & \bar{x}_1 = x_1 - x_1^* \\
x_2^* = -\alpha_1(x_{[1]},\hat{\Theta}) & \bar{x}_2 = x_2 - x_2^* \\
\vdots & \vdots \\
x_m^* = -\alpha_{m-1}(x_{[m-1]},\hat{\Theta}) & \bar{x}_m = x_m - x_m^*
\end{array} \quad (33)$$

and, with the Lyapunov function $V_{m-1} = V_1 + \frac{1}{2}\sum_{i=2}^{m}\bar{x}_i^2$, we have the result

$$\begin{aligned}
\dot{V}_{m-1} \le& -\sum_{i=1}^{m-1}\left(c_i + 1 - \frac{m-1-i}{n}\right)\bar{x}_i^2 + \left(\tilde{\Theta} + \Pi_{m-1,1}(\cdot)\right)\left(\Pi_{m-1,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma}\right) \\
&+ \bar{x}_{m-1}\bar{x}_m,
\end{aligned}$$

where $\Pi_{m-1,1}(x_{[m-1]},\hat{\Theta})$ and $\Pi_{m-1,2}(x_{[m-1]},\hat{\Theta})$ are tuning functions for Θ.

Step m: We choose a C^1 Lyapunov function as $V_m = V_{m-1} + \bar{x}_m^2/2$. Its time derivative along the system (1) is

$$\dot{V}_m = \dot{V}_{m-1} + \bar{x}_m\dot{x}_m - \bar{x}_m\left(\sum_{i=1}^{m-1}\frac{\partial x_m^*}{\partial x_i}\dot{x}_i + \frac{\partial x_m^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}}\right).$$

Using (3), (5), (33), and Young's inequality, we estimate the uncertain terms as

$$\begin{aligned}
&\bar{x}_m(f_m(\cdot)+g_m(\cdot)) - \bar{x}_m\sum_{i=1}^{m-1}\frac{\partial x_m^*}{\partial x_i}(f_i(\cdot)+g_i(\cdot)) \\
&\le |\bar{x}_m|\sum_{i=1}^{m}|x_i|\left(\tilde{f}_m(\cdot)+\tilde{g}_m(\cdot)\right)\Theta_1 \\
&\quad + |\bar{x}_m|\sum_{i=1}^{m-1}\left|\frac{\partial x_m^*}{\partial x_i}\right||x_i|\left(\tilde{f}_i(\cdot)+\tilde{g}_i(\cdot)\right)\Theta_1 \\
&\le \sum_{i=1}^{m-1}\frac{\bar{x}_i^2}{n} + \bar{x}_m^2\sigma_m(x_{[m]},\hat{\Theta})\Theta,
\end{aligned} \quad (34)$$

where $\sigma_m(\cdot) \ge 0$ is a smooth function.

By $\tilde{\Theta} = \Theta - \hat{\Theta}$ and

$$\Pi_{m,1}(x_{[m]},\hat{\Theta}) = \Pi_{m-1,1}(\cdot) + \gamma \bar{x}_m \frac{\partial x_m^*}{\partial \hat{\Theta}}$$

$$\Pi_{m,2}(x_{[m]},\hat{\Theta}) = \Pi_{m-1,2}(\cdot) + \bar{x}_m^2 \sigma_m(\cdot),$$

the virtual control is designed as

$$\begin{aligned} x_{m+1}^* &= -\bar{x}_m(c_m + 1 + \sigma_m(\cdot)\hat{\Theta}) - \bar{x}_{m-1} + \sum_{i=1}^{m-1} \frac{\partial x_m^*}{\partial x_i} x_{i+1} \\ &\quad + \left(\gamma \frac{\partial x_m^*}{\partial \hat{\Theta}} \Pi_{m-1,2}(\cdot) + \bar{x}_m \sigma_m(\cdot) \Pi_{m,1}(\cdot) \right) \\ &=: -\alpha_m(x_{[m]},\hat{\Theta}), \\ \bar{x}_{m+1} &= x_{m+1} - x_{m+1}^* \end{aligned} \tag{36}$$

which is followed by

$$\dot{V}_m \le -\sum_{i=1}^{m} \left(c_i + 1 - \frac{m-i}{n} \right) \bar{x}_i^2 + \left(\tilde{\Theta} + \Pi_{m,1}(\cdot) \right) \left(\Pi_{m,2}(\cdot) - \frac{\dot{\hat{\Theta}}}{\gamma} \right)$$
$$+ \bar{x}_m \bar{x}_{m+1}.$$

This completes the proof of step m.

Acknowledgements

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-314-D00160).

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