Correction of the Approximation Error in the Time-Stepping Finite Element Method

Byung-taek Kim*, Byoung-hun Yu*, Myoung-hyun Choi† and Ho-hyun Kim*

Abstract – This paper proposes a correction method for the error inherently created by time-step approximation in finite element analysis (FEA). For a simple *RL* and *RLC* linear circuit, the error in time-step analysis is analytically investigated, and a correction method is proposed for a non-linear system as well as a linear one. Then, for a practical inductor model, linear and non-linear time-step analyses are performed and the calculation results are corrected by the proposed methods. The accuracy of the corrected results is confirmed by comparing the electric input and output powers.

Keywords: Correction method, Finite element method, Non-linear, Time-step approximation

1. Introduction

The finite element method (FEM) has become increasingly popular in the analysis of electric machines, and there is no doubt that the time-step FEM is a very powerful tool for acquiring accurate solutions, particularly for non-linear systems [1]. However, the method still possesses an error. This error is composed of one within the FEM itself, and another created in time-step approximation. This error is easily found in practical FE-analysis as, for instance, the total electric input power always differs from the total output power including losses. This logical fallacy confuses the calculation of the power factor, efficiency etc. [2].

In this paper, the error from time-step approximation in the steady state of a linear system is examined using an analytic approach. Through this examination, a correction method obtaining the exact solution is proposed regardless of the step size. Since most real systems are non-linear, a new correction method is additionally suggested and tested for non-linear *RL/RLC* systems. For each system, the input and output powers are compared in order to prove the validity of the method.

2. Examination of Error in Time-Step Calculation FORMAT

The differential term in voltage equation in (1) for an RL circuit can be expanded by the Taylor series including an error value E_{resid}, shown as (2), where the asterisk * means the exact solution.

$$v(t) = R \cdot i^*(t) + L\frac{di^*(t)}{dt} \tag{1}$$

$$i^*(t) = i^*(t-\Delta t) + \frac{di^*(t)}{dt}\Delta t + E_{resid}(\Delta t) \tag{2}$$

When the step size Δt gets much smaller, the error E_{resid} becomes negligible, so the voltage equation is written as (3) in time-step form with an approximate solution *i(t)*.

$$v(t) = R \cdot i(t) + L\frac{i(t) - t(t-\Delta t)}{\Delta t} \tag{3}$$

Therefore, to obtain accurate solutions, the step size should be very small. However, in FEA consuming large amounts of time for simulation, it is difficult to take a very small size, so in most cases it depends on the experience of engineers. In several past studies, the paper suggested by S. L. Ho deals with a method of choice of step size, but it focuses only on the current error reduction without considering the relation between the input and the output power [2][3].

To show the error in time-step analysis, let's assume a linear *RL* circuit which has 1Ω resistance and j10Ω reactance. The current drawn by 220V, 60Hz is analyzed by using (3), and in the steady state the average of input (vi) and output powers (i^2r) are obtained respectively by using (4) and (5).

$$P_{in} = \frac{1}{T}\sum_{t=t_0}^{t_0+T} v(t)\cdot i(t)\cdot \Delta t \tag{4}$$

$$P_{out} = \frac{1}{T}\sum_{t=t_0}^{t_0+T} i^2(t)\cdot R\cdot \Delta t \tag{5}$$

For a linear system, the exact solution can be calculated by using the phasors in (6) and (7).

$$P^*_{in} = V^* I^* \cdot \cos\theta^* \tag{6}$$

$$P^*_{out} = R\left(I^*\right)^2 \tag{7}$$

In Table 1, the exact solution is compared with the approximate solutions with variances in step size. It shows that the smaller the step size, the smaller the error becomes. However, there is a great difference between the input and the output power. In particular, the error of input power is much bigger than that of output power in the *RL* circuit, when comparing it with the exact solution.

This fallacy can be analytically investigated, as follows: By reforming (3), the approximated current can be written as (8).

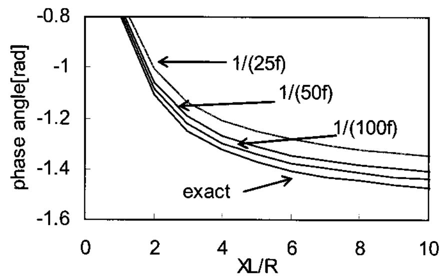
† Corresponding Author: Dept. of Electrical Engineering, Kunsan National University, Korea. (cancmh@kunsan.ac.kr)

* Dept. of Electrical Engineering, Kunsan National University, Korea. (btkim@kunsan.ac.kr)

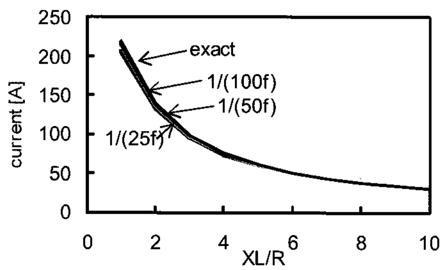
Received 12 November 2008; Accepted 3 April 2009

Table 1. Calculation Results for Various Step Sizes

		Exact solution	Solutions according to time step sizes		
			1/(25f)	1/(50f)	1/(100f)
P_{in}	[W]	479.2	1057.6	771.3	626.0
P_{out}	[W]		481.9	480.1	479.5
PF	[%]	9.9	22.2	16.1	13.0
I	[A]	21.9	21.7	21.8	21.8



(a) Phase angle



(b) Amplitude

Fig. 1. Comparison of approximate solutions in a linear *RL* circuit

The current *i(t)* obviously converges to a periodic function in steady state, so we can let *i(t)* in (8) as $I_m\sin(\omega t+\alpha)$ which is of course an approximate solution. Then, the equation (8) can be solved for I_m and α. These are the functions of L, v, R, and the step size Δt, given by (9) and (10).

$$i(t)=\frac{\Delta t\cdot v(t)+L\cdot i(t-\Delta t)}{\Delta t\cdot R+L} \tag{8}$$

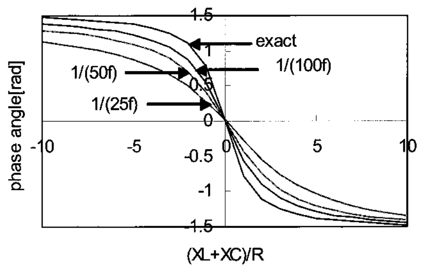
$$I_m=\left\{\sqrt{\left(R\cdot\Delta t+L-L\cdot\cos(\omega\Delta t)\right)^2+\left(L\cdot\sin(\omega\Delta t)\right)^2}\right\}^{-1}\Delta t\cdot V_m \tag{9}$$

$$\alpha=-\tan^{-1}\left(\frac{L\cdot\sin(\omega\Delta t)}{R\cdot\Delta t+L-L\cdot\cos(\omega\Delta t)}\right) \tag{10}$$

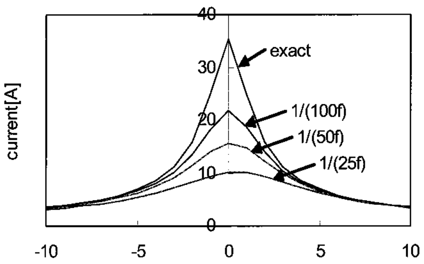
The approximate solutions for various step sizes are obtained by (9) and (10), and compared with the exact solution in Fig. 1. It shows that both the amplitude and phase angle include the computation errors, consequently causing a difference between the total input and output power. It can also be seen that as the ratio X_L/R increases, the phase error increases but the amplitude error decreases in the *RL* system.

In addition, using the same approach, the error in the *RLC* circuit can be investigated. The current in time-step form is written as (11) and (12). The amplitude and phase of an approximate current is obtained by substituting $I_m\sin(\omega t+\alpha)$ for *i(t)* in (11), written as (13) and (14), where the additional equations for symbols are given in the appendix.

For instance, the various approximate solutions of an *RLC* circuit with R=1 Ω, V=35volt-60Hz are obtained by using (13) and (14), and the results are compared in Fig. 2. It describes the fact that there are considerable errors in both amplitude and phase. In particular, the time-stepped result shows a serious deviation from the exact solution when the system approaches a resonant point.



(a) Phase angle



(b)Amplitude

Fig. 2. Comparison of approximate solutions in a linear RLC circuit

$$i(t)=\frac{\Delta t\cdot v(t)+L\cdot i(t-\Delta t)-(1/C)\cdot i_{sum}(t-\Delta t)\Delta t}{\Delta t\cdot R+L+(1/C)\cdot\Delta t^2} \tag{11}$$

$$i_{sum}(t-\Delta t)=\sum_{t=0}^{t-\Delta t}i(t)\Delta t \tag{12}$$

$$I_m=\frac{X_0}{X_1}\cdot V_m \tag{13}$$

$$\alpha=\tan^{-1}\frac{B_0}{A_0}-\tan^{-1}\frac{B_1}{A_1} \tag{14}$$

3. Correction Of Time-step Error In FE-Analysis

3.1 Linear System

In practice, it is nearly impossible to obtain the exact inductance of a magnetic system using an analytical method because of its complex geometry and nonlinearity. For that reason, the time-step FEM is widely used in analysis of systems. As a result of FE-analysis, we obtain the approximate current whose amplitude and phase angle is given in (9)-(10) or (13)-(14) for an *RL/RLC* system. It is important to note that inductances of the equations are the exact values which are as yet unknown. However, if I_m and α have already been obtained by time step analysis, the exact inductance L can be calculated by rearranging (9) and (13), as shown in (15), where the supplementary symbols can be referred to in the appendix.

$$L = \begin{cases} \dfrac{R\cdot\Delta t\cdot\tan(-\alpha)}{\sin(\omega\cdot\Delta t)-(1-\cos(\omega\cdot\Delta t))\cdot\tan(-\alpha)} & (\text{for } RL) \\[2ex] \dfrac{G(R\cdot X_0\cdot\cos(\gamma_0)+\frac{\Delta t}{C})-R\cdot X_0\cdot\sin(\gamma_0)}{\frac{X_0^{\,2}}{\Delta t}(\sin(2\gamma_0)-G\cdot\cos(2\gamma_0))} & (\text{for } RLC) \end{cases} \quad (15)$$

Since the exact L can be obtained from (15), the exact current in a transient or a steady state is also calculated using an analytic method. The validity of the correction is easily proven with the results in Table 1. Assume that the approximate currents in the table were obtained by time-step analysis. Using (15), the exact L and current can be obtained, so the power characteristics are corrected to the exact values as shown in Table 2.

Table 2. Power Comparison Before and After Error Correction In a Linear *RL* System

	In/output power	Solutions according to time step sizes		
		1/(25f)	1/(50f)	1/(100f)
Before correction	P_{in} [W]	1057.6	771.3	626.0
	P_{out} [W]	481.9	480.1	479.5
	PF [%]	22.1	16.1	13.0
After correction	P_{in} [W]		479.21	
	P_{out} [W]		479.21	
	PF [%]		9.9	

3.2 Non-linear System

Most electric machines have magnetic non-linearity caused by the saturation of a magnetic core. In a non-linear system, the voltage equation is given as (16). Since the relationship between flux linkage λ and current is non-linear, the equation (15) is not true for the system. In fact, even if the property of λ(i) is perfectly known, the non-linear differential equation (16) cannot be directly solved by an analytic approach [4].

$$v(t) = R\cdot i^*(t) + \frac{d\lambda\left(i^*(t)\right)}{dt} \quad (16)$$

Thus, another correction with an indirect approach is investigated for non-linear systems. First, suppose that we obtain the approximate current *i(t)* by the time-step FEM. It is actually a solution obtained by solving (17) coupled with Maxwell's field equations [5]. From (17), the λ in the *RL* system is written as (18) in which the first term in the right side is calculated using the known *i(t)*. Setting the initial value λ(0) (when there is no current) as zero, we are able to determine the non-linear λ-*i* function which has a piece-wise linear property.

$$v(t) = \begin{cases} R\cdot i(t) + \dfrac{\lambda(i(t))-\lambda(i(t-\Delta t))}{\Delta t} & (\text{for } RL) \\[2ex] R\cdot i(t) + \dfrac{\lambda(i(t))-\lambda(i(t-\Delta t))}{\Delta t} + \dfrac{\Delta t}{C}\sum_{t=0}^{t} i(t) & (\text{for } RLC) \end{cases} \quad (17)$$

$$\lambda(i(t)) = (v(t) - R\cdot i(t))\Delta t + \lambda(i(t-\Delta t)) \quad (\text{for } RL) \quad (18)$$

Once the function λ-*i* is known, we can again easily solve the equation (17) with an arbitrary step size. In other words, it is possible to calculate it with a very small step size in a short time as it is no longer a field-coupled analysis but is instead a simple circuit simulation. With such a small step size, the error due to time-step method becomes negligible and only the error due to the λ-*i* approximation remains. Therefore, the calculation accuracy gets much better.

In order to verify the proposed method, let's assume a simple magnetic system as shown in Fig. 3, which has a resistance of 20 Ω, and a lamination stack length of 30 mm. The number of turns is 750 and the core is a silicon steel generally used in electric machines. With a sinusoidal voltage of 200V-60Hz, the time-stepped non-linear FE-analysis is carried out with various time step sizes and the simulated currents are depicted in Fig. 4. With each current, the magnetic characteristics are calculated by (18). Fig. 5 shows the piece-wise linear λ-*i* characteristic obtained from a current with 1/(25f) step size. Using the λ-*i* characteristics, the equation (17) is solved with a step size of 1/(1000f), where the Newton-Raphson algorithm is used for convergence in non-linear analysis [6]. As shown in Fig. 6, regardless of step size, every waveform is moved to almost the same point. This means that the corrected solutions are nearly an exact solution, so the input power becomes identical with the output power.

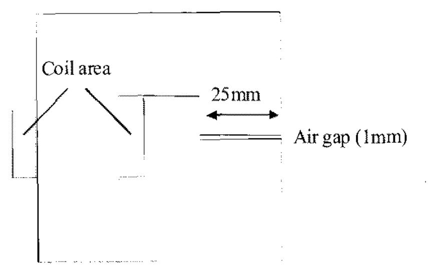


Fig. 3. Simple C-core inductor for simulation

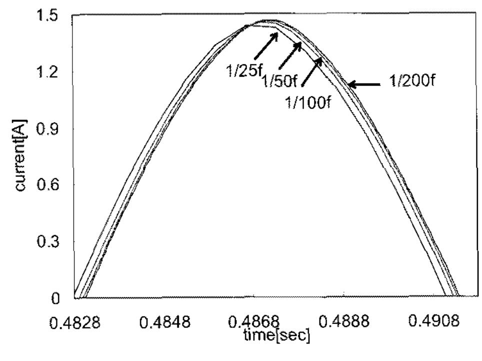


Fig. 4. Currents by time-step FEM in non-linear *RL* system (before correction)

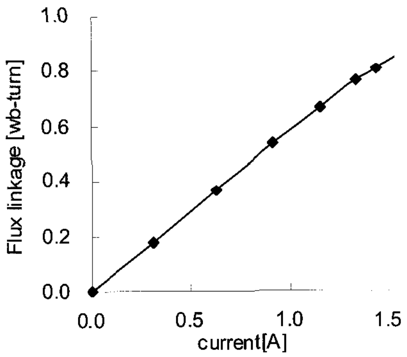


Fig. 5. Approximated λ-i characteristic of non-linear *RL* system

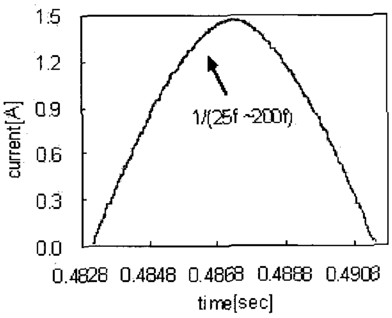


Fig. 6. Current correction of non-linear *RL* system

As an example for an *RLC* system, we assume the core in Fig. 3 where the coil has 170 turns and a capacitor having –j15 Ω is connected with a voltage source of 25V-60Hz. The currents obtained by FEA with variances in step sizes are shown in Fig. 7. It shows that the simulated currents are very different to one another, so it is very difficult to expect the exact solution from the system. In Fig. 8, the currents corrected by the proposed method are shown. Comparing them with the results in Fig. 7, they are dramatically moved to the same position, though showing a little difference near the peak due to the inaccuracy of the λ-*i* model.

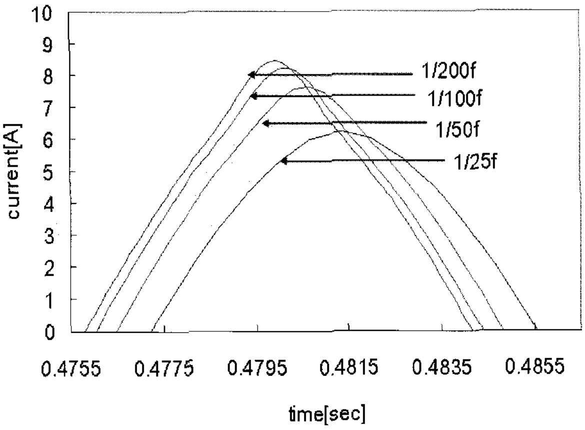


Fig. 7. Currents by time-step FEM in non-linear *RLC* system (before correction)

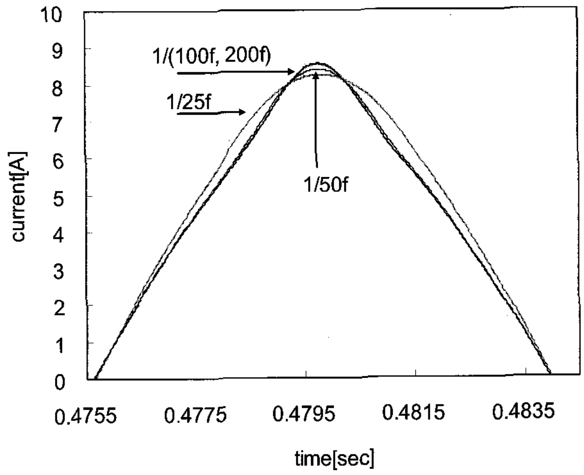


Fig. 8. Corrected currents of a non-linear *RLC* system

The power relationship between input and output is compared in Table 3. It shows that the power differences between the original results are greatly reduced and the accuracy of the power factor is also improved by the proposed method.

Table 3. Power Comparison Before and After Error Correction In a Linear *RL* System

	In/output power		Solutions according to time step sizes			
			1/(25f)	1/(50f)	1/(100f)	1/(200f)
Before correction	P_{in}	[W]	80.6	65.6	49.0	39.2
	P_{out}	[W]	18.8	25.1	27.2	28.0
	PF	[%]	74.3	52.3	37.5	29.6
After correction	P_{in}	[W]	34.0	31.1	30.1	29.4
	P_{out}	[W]	31.4	28.8	28.5	28.5
	PF	[%]	24.3	23.2	23.1	23.1

4. Conclusion

First, this paper deals with an estimation of the error produced by time-step approximation in FEA, and proposes new correction methods for linear and non-linear systems. It proves that the error in the linear case can be eliminated and the error in the non-linear case is dramatically reduced by the proposed methods. Therefore, these methods can be used efficiently to obtain accurate solutions for an otherwise time-consuming FE-analysis problem. In this study, application of the proposed method is limited to a sinusoidal voltage input, but it can be also used for a non-sinusoidal one.

Appendix

$X_0 = \sqrt{A_0^2 + B_0^2}$, $\quad X_1 = \sqrt{A_1^2 + B_1^2}$

$A_0 = 1 - \cos(\omega \cdot \Delta t)$, $\quad B_0 = \sin(\omega \Delta t)$

$$\gamma_0 = \tan^{-1}\left(\frac{B_0}{A_0}\right), \quad G = \tan\left(\tan^{-1}\left(\frac{B_0}{A_0}\right) - \alpha\right)$$

$$A_1 = R \cdot X_0 \cdot \cos(\gamma_0) + \frac{L}{\Delta t} \cdot X_0^2 \cdot \cos(2\gamma_0) + \frac{\Delta t}{C}$$

$$B_1 = R \cdot X_0 \cdot \sin(\gamma_0) + \frac{L}{\Delta t} \cdot X_0^2 \sin(2\gamma_0)$$

Acknowledgment

This work is the outcome of a Manpower Development Program for Energy and Resources supported by the Ministry of Knowledge and Economy (MKE) and a second stage BK21 project of the Korean Research Foundation.

References

[1] Jabbar, M. A., Phyu, Hla Nu, and Liu, Z. J., "Analysis of the Starting Process of a Disk Drive Spindle Motor By Time Stepping Finite Element Method", *IEEE Transactions on Magnetics,* vol. 50, pp. 3204-3206, July 2004.

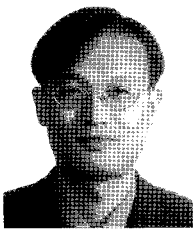
[2] Ho, S. L., and Fu, W. N., "Application of Automatic Choice of Step Size for Time Stepping Finite Element Method to Induction Motors", *IEEE Transactions on Magnetics,* vol. 33, no. 2, pp. 1370-1373, March 1997.

[3] Gear, C. William, *Numerical Initial Value Problem in Ordinary Differential Equations,* Prentice-Hall, Inc, Englewood Cliffs, New Jersey, 1971.

[4] Jordan, D. W. and Smith, P, *Nonlinear Ordinary Differential Equations,* Oxford Press, Oxford, 1999.

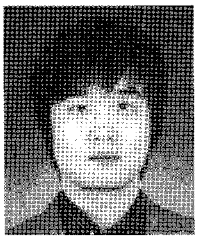
[5] Brauer, J. R., Vander Heiden, R. H., and Arkadan, A. A., "Nonliear transient finite element modeling of a capacitor discharge magnetizing fixture," *IEEE Transactions on Magnetics,* Vol. 29, pp. 2051-2054, March 1993.

[6] Burden, Richard L., and Douglas, J., *Faires, Numerical Analysis,* Thomson Brooks/Cole, Singapore, 2002.



Byung-Tack Kim received his B.S., M.S. and Ph.D. degrees in electrical engineering from Hanyang University. He has worked as a Senior Researcher in the CAD/CAM center at Samsung Electro-mechanics Inc. and the Digital Appliance Laboratory at LG Electronics Inc. He is currently a Professor at Kunsan National University. His research interests are the design of electric machines, numerical analysis and optimization.

Byoung-Hun Yu received his B.S. degree in electrical engineering from Kunsan National University. He is currently studying for his Master's degree at Kunsan National University. His research interests are the design of electric machines, numerical analysis and optimization.



Myoung-Hyun Choi received his B.S. degree in electrical engineering from Kunsan National University. He is currently studying for his Master's degree at Kunsan National University. His research interests are the design of electric machines, numerical analysis and optimization.

Ho-Hyun Kim received his B.S. degree in electrical engineering from Kunsan National University. He is currently studying for his Master's degree at Kunsan National University. His research interests are the design of electric machines, numerical analysis and optimization.