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# Measuring a Value of Contract Flexibility in the Third-Party Warehousing

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## ABSTRACT

This paper considers the value of warehousing contract under probabilistic demands. We consider a supply chain consisting of a supplier, a retailer and its third-party warehousing partner who provides the warehousing service to the retailer through an outsourcing contract. A typical contract is specified by initial space commitment and modification schedule. The retailer decides the order quantity for the supplier and space commitment for the outsourcing contract. Since there is close relationship between order quantity and space commitment to minimize the total cost including ordering cost, inventory carrying cost, shortage cost, and warehousing cost, we develop an analytical model under probabilistic demands, where the retailer can determine the optimal order size and space commitment level jointly. We found the closed-form optimum for a single-period case and the optimal conditions for a two-period case. To evaluate the value of contract flexibility for the two-period case, we compared the total cost under two policies; one with modification, under which the base commitment can be changed at the start of each period and the other without modification. From results of our numerical analysis, we showed that the modification policy is more cost-effective as the variability of demand increases.

Keywords: Outsourcing Contract, Contract Flexibility, Third-party Logistics, Supply Chain Management

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## 1. Introduction

Since the late 1980s logistics outsourcing has been recognized as a strategic weapon that can provide competitive advantages and help curtail distribution costs (see [2]). As part of logistics outsourcing, third-party warehousing is growing rapidly over a decade.

In this paper, we consider a supply chain with a single supplier, a retailer and its third-party warehousing partner (Figure 1). The retailer faces the dynamic market demands. To satisfy the demands, the retailer issues the purchase order for its supplier. For the purchase order, the supplier delivers the product to the retailer. Even though the physical inventory of the product is stocked at the third-party warehouse under the outsourcing contract, the ownership of the inventory belongs to the retailer. The product inventory is used to fulfill the market demands. Similar to the inventory movement from the supplier and the retailer, the physical product is transferred from the third-party warehouse to the end customer and the ownership of the product from the retailer to the customer. The decision problem faced by the retailer is to determine the order quantity for the supplier and the space commitment size for the warehousing service provider.

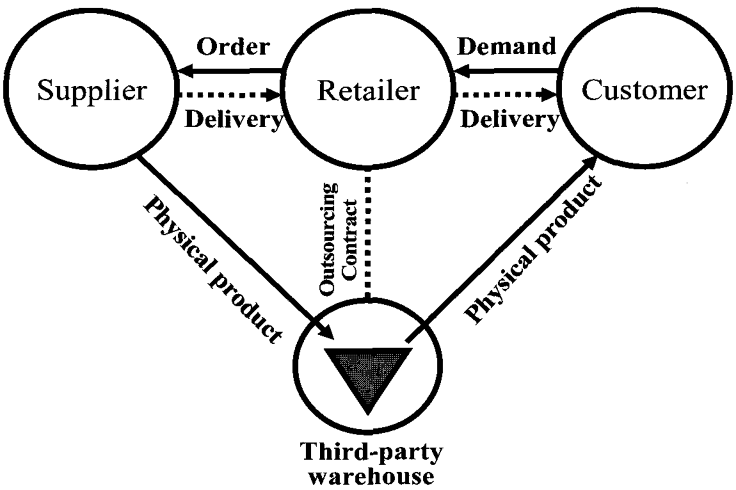


Figure 1. Third-party warehousing

The third-party warehousing process is based on the agreement between a retailer and a service provider through the outsourcing contract. The contract generally involves a user entering a contract with the warehousing provider for specific services at agreed-upon prices over a fixed contract period. One of the most important decisions is how large should be the space the user commits to the service provider. While the service provider wants to fix the commitment over the contract duration to ensure stable sales, the user does not want to fix the commitment size so as to deal with the uncertainty of the space requirement. For resolving this situation, most outsourcing contract contains the space commitment and certain flexibility by giving the user some modification opportunities. The more frequently the retailer can modify the space commitment, the more flexible the outsourcing contract. That is because if the retailer can change the commitment more frequently, she can cope with the variability of space requirement. Our research question is whether and how the contract flexibility in terms of modification frequency can give the benefit of logistical outsourcing.

We consider a supply chain in which a retailer procures a product from a supplier and sells the product to the market. The retailer faces variability of the market demand and tries to fulfill the demand through the appropriate inventory policy. This relationship between the demand pattern and the dependent space requirement caused by physical inventory affects the decision problem, that is, how large space the user should commit to the third-party warehouse firm. Based on this logic, we develop an analytical model so that the retailer's inventory policy and commitment level be determined jointly. In addition, we analyze the effect of contract flexibility when demand characteristics and cost parameters vary.

The rest of the paper is organized as follows. In Section 2, we give a brief literature review. Section 3 develops an analytical model to solve the decision problem for the retailer. In that section, we extend the single period model to a two-period model so that we could have a conjecture for the multi-period case. Section 4 gives the results of our numerical analysis, which show the value of contract flexibility as some of the key parameters vary. Finally, we draw conclusions and managerial implications from the results.

## 2. Literature Review

The literatures to address the benefits of third party logistics are increased in a decade (e.g. see [3, 5, 10, 11]). Lee [4] summarized the benefits as in Table 1.

Table 1. Benefits of third party logistics (Lee [4])

| | |
|---|---|
| Daugherty *et al.* [3] | Lower labor costs<br>Flexibility<br>Access to better information systems<br>Improved delivery and service |
| Lieb and Randall [5] | Reduction of transportation cost<br>Reduction of material management and storage cost<br>Decrease of stock level<br>Shortened response time to customers' requests<br>Improved control of distribution channels |
| Nemoto and Tezuka [10] | Economies of scale and scope<br>Savings on capital investment<br>Risk-sharing |
| Persson and Virum [11] | Better focus on the core businesses<br>Access to world-class processes, products, services or technology<br>Better capability of adjusting to changing environmental needs<br>Reducing the need for capital investments<br>Better cash-flow<br>Reducing operating costs<br>Exchanging fixed costs with variable costs<br>Access to resources not available in own organization |

For the supply chain contracts, Tsay *et al.* [12] gives a broad review which classifies the literatures according to the contract clauses. It contains quantity discounts and minimum commitments which are similar mechanism to our base commitment for the space requirement. Cachon [1] includes broader context than Tsay *et al.* [12]. It reviews the supply chain contracts as the variations on the newsboy problem under lots of different conditions.

Without considering the warehouse side, our problem extends the classical newsboy problem. Khouja [8] is one of the most recent reviews for the newsboy problem literatures. One of the relevant literatures is Nahmias and Schmidt [9] who developed an efficient heuristic solution for the multi-item newsboy problem subject to a single constraint of a specific form. More recently, Matsuyama [6] generalizes the newsboy problem to multi-period case which consists of multiple ordering cycles.

More relevant to our model, Chen *et al.* [2] analyzed three types of third-party warehousing contract. With relaxing the restriction on the range of commitment change and on modification schedules, they showed the effect of contract flexibility under dynamic demand patterns. However, they assumed that the space requirement was not serially correlated with that of the previous period and did not consider dynamic relationship between the market demand and the retailer's inventory policy. As another relevant literature, Lee [4] addressed the logistical outsourcing problem from which our research was originated. She used the similar cost structure with Chen *et al.* [2] with a continuous-review inventory model. In her model, she did not assume the trade-off relationship between the order quantity and the space requirements so that the total cost did not include the effect of order quantity change.

In our model, by chasing the root cause of variability of the space requirement, the order quantity for the supplier is determined to fulfill the dynamic market demand and the retailer decides the appropriate base commitment at the same time.

## 3. Formulation of Model

### 3.1 Assumptions

We consider the following inventory model. Items are purchased for a single period at a cost of $c$ dollars per item. We assume that the purchase leads to immediate delivery. The stock is replenished only once at the start of the period and demanded right away. Shortages are backordered but not at the end of planning horizon. Under third party warehousing contract, we consider only the opportunity cost of on-hand inventory because the other factors including the employee fee, space charges and warehouse maintenance cost are transferred to the third-party warehousing firm through the outsourcing contract. The opportunity cost of holding inventory means the income foregone by tying up money in inventory and not investing it elsewhere. This capital charge for on-hand inventory is given by $h$ dollars per unit item and charged as a function of excess stock over the amount demanded. The cost of unsatisfied demand is given by $s$ dollars per unit (assuming that $s > c$). It includes the cost of lost sales, loss of goodwill, customer dissatisfaction, and special administrative efforts resulting from the inability to meet demand. Denote by $Q$, the quantity purchased at the beginning of the period. We assume that demand during the period denoted by $D$ is not known in advance but is a random variable with probability density $f(D)$, and cumulative distribution function $F(D)$.

Through the outsourcing contract with third-party warehousing service provider, the retailer pays the warehousing cost which consists of fixed charge for the space commitment and variable charge for the overflow. The fixed charge $p_1B$ is paid if the commitment size is $B$ and the cost of per-unit committed space is $p_1$. Denote by $V$ the space requirement for inventory. The space requirement for on-hand inventory is assumed to be a linear function of on-hand inventory level. That is, we assume that space $\alpha I$ is required when on-hand inventory level is $I$ and space requirement per unit item is $\alpha$. Note that $V$ is a random variable because the demand is a random variable and, in turn, the inventory level is also a random variable. If the space requirement $V$ exceeds the commitment size, the retailer pays the premium charge, $p_2(V-B)$, for the overflow where $p_2$ is the premium charge per unit space and $V-B$ is the level of overflow when the commitment size is $B$. Therefore, expected warehousing cost occurred is $p_1B + p_2E[\max(0, V-B)]$ since the space requirement $V$ is a random variable. $E[\,]$ means the expected value of random variable.

### 3.2 Single-Period Model

In a single-period model, the order-up-to point $Q$ and base commitment level $B$ is to be determined. The objective is to minimize the total cost occurred during a single period which is composed of procurement costs, inventory carrying costs, shortage costs, and warehousing costs. Regarding the order-up-to point, a classical trade-off is needed between the risk of being short and thereby incurring the shortage costs and the risk of having an excess and thereby incurring wasted costs of ordering and capital charge for excess units. Under third party warehousing situation, there exists another trade-off between base commitment cost and premium charge for overflow. For the warehousing costs, there exist base commitment charge and premium charge for overflow. If the retailer commits too much, then she should bear the fixed commitment cost even when there is a little on-hand inventory. Conversely, if the retailer commits too less, then she can not have the benefit of discount by commitment. In addition, because there is close relationship between order quantity and inventory level, in turn between order quantity and space requirements, the two decision variables, $Q$ and $B$, are to be determined jointly.

#### *3.2.1 Without initial inventory*

In this section, we consider an optimal inventory policy having no initial inventory.

We assume that there is no disposal of the remaining inventory at the end of the period. Define $TC(Q, B)$ as the expected total cost under the policy $(Q, B)$. Then

$$TC(Q, B) = cQ + \int_0^Q h(Q-D)f(D)dD + \int_Q^\infty s(D-Q)f(D)dD + \left[p_1 B + \int_B^{\alpha Q} p_2(V-B)f(V)dV\right] \quad (1)$$

Total costs consist of five parts. The first part is the ordering cost. The second part is the inventory carrying costs when the demand is realized under the stock level, whereas the third is the shortage costs when the demand surpasses the inventory level. The fourth term is the basic charge for space commitment. The final part is the premium charge for space overflow when the space requirement is over the base commitment. The problem is to find the order quantity and the space commitment in order to minimize the total costs, $TC(Q, B)$.

**Proposition 1.**

The optimal solution to minimize the $TC(Q, B)$ is $Q^*$ and $B^*$ such that

$$F\left(Q^*\right) = \frac{s-c-\alpha p_1}{s+h} \quad (2)$$

$$F\left(Q^* - \frac{B^*}{\alpha}\right) = \frac{p_1}{p_2} \quad (3)$$

**Proof.**

See Appendix A. ■

We can find $Q^*$ from (2) at first, and then get $B^*$ from (3) after substituting the value of $Q^*$ into (3). Alternatively, we can state $Q^*$ and $B^*$ as follows.

$$Q^* = F^{-1}\left(\frac{s-c-\alpha p_1}{s+h}\right) \quad (2')$$

$$B^* = \max\left\{\alpha\left[Q^* - F^{-1}\left(\frac{p_1}{p_2}\right)\right], 0\right\} \quad (3')$$

**Proposition 2.**

$TC(Q, B)$ is convex with respect to $Q$ and $B$. Thus, $Q^*$ and $B^*$ guarantees the global optimum.

**Proof.**

See Appendix B. ■

In order for (2′) to be meaningful, $0 \le \frac{s-c-\alpha p_1}{s+h} \le 1$, or $s \ge c+\alpha p_1$ should be guaranteed. It makes sense in a single period case since the unsatisfied demand cost includes more than the lost revenue which is greater than the unit purchase cost plus unit warehousing cost for base commitment.

### 3.2.2 *With the initial inventory*

In the previous section 3.2.1, we assumed that there was no initial stock. To relax this assumption, we now consider the retailer has initial stock, $I$, at the starting point of the period. Then optimal ordering and commitment policy is described as follows.

$$\begin{cases} \text{do not order and set } B = B^I, & \text{if } I \ge Q^c \\ \text{order up to } Q^c \text{ (order } Q^c - I\text{) and set } B = B^c, & \text{if } I < Q^c \end{cases} \tag{4}$$

Where $Q^c = F^{-1}\left(\frac{s-c-\alpha p_1}{s+h}\right)$, $B^c = \max\left\{\alpha\left[Q^c - F^{-1}\left(\frac{p_1}{p_2}\right)\right], 0\right\}$

and $B^I = \max\left\{\alpha\left[I - F^{-1}\left(\frac{p_1}{p_2}\right)\right], 0\right\}$

$B^I$ is derived from (3′) simply by substituting $Q^*$ for $I$ since the initial stock level is given. The proof is straightforward from proposition 1 and 2. This optimal policy gives the possibility of extension to the second-period and multi-period model because the subsequent periods have positive inventory level or backlogs at the start of the period.

### 3.3 *Two-Period Model*

Based on the results of single period model, we extend it to the two-period case. To investigate the value of contract flexibility, we compare the optimum total costs under two policies. One is single base commitment policy which can not change the base commitment through two periods. The other is modification policy under which the retailer can change the base commitment level at the start of the second period. It is assumed that there is a sales chance at the end of period 2 with salvage cost $l$ per unit on-hand inventory. $l$ can have any sign. If $l<0$, then this means a salvage value per unit inventory. Therefore, it costs $h+l$ per remaining inventory through period 2. Through comparing the total costs of these two policies, we can evaluate the value of contract flexibility in terms of the modification frequency.

### *3.3.1 Modification of base commitment*

In this case, the retailer can change the base commitment. Therefore, the retailer should decide the commitment level for each period $\mathbf{B}=(B_1, B_2)$ and order quantity for each period $\mathbf{Q}=(Q_1, Q_2)$. We use the order-up-to policy for each period.

We can formulate the problem as a dynamic programming because the decision is made at start of each period without being affected by the decision of previous period. Define $g_t(I_{t-1})$ as the expected cost of following an optimum policy (minimum cost) from the beginning of period $t$ to final period when the net inventory at the end of period $t$ is $I_t$. In this section, the final period is period 2. Total costs over two periods can be denoted by $g_1(I_0)$ when the initial inventory is $I_0$. Therefore, the problem faced by the retailer in two period case is to minimize the $g_1(I_0)$. To obtain $g_1(I_0)$, it is necessary to first find $g_2(I_1)$. By the result of single period case, the optimal policy for period 2 is

$$\begin{cases} \text{do not order and set } B=B_2^l, & \text{if } I_1 \geq Q_2^c \\ \text{order } (Q_2^c - I_1) \text{ and set } B=B_2^c, & \text{if } I_1 < Q_2^c \end{cases} \tag{5}$$

where $Q_2^c = F^{-1}\left(\dfrac{s-c-\alpha p_1}{s+h+l}\right)$, $B_2^c = \max\left\{\alpha\left[Q_2^c - F^{-1}\left(\dfrac{p_1}{p_2}\right)\right], 0\right\}$, and

$$B_2^l = \max\left\{\alpha\left[I - F^{-1}\left(\frac{p_1}{p_2}\right)\right], 0\right\}$$

Note that the denominator of $Q_2^c$ is $s+h+l$ since it costs inventory carrying cost as well as salvage costs, $h+l$, per on-hand inventory of period 2. The cost of this optimum policy can be expressed as

$$g_2(I_1) = \begin{cases} L_2(I_1) + W_2(I_1, B_2^l), & \text{if } I_1 \geq Q_2^c \\ c(Q_2^c - I_1) + L_2(Q_2^c) + W_2(Q_2^c, B_2^c), & \text{if } I_1 < Q_2^c \end{cases} \tag{6}$$

$L_t(x)$ is defined by the expected holding plus shortage cost for a single period when there are $x$ units are available. $L_t(x)$ can be expressed as

$$L_t(x)=\begin{cases}\int_0^x h(x-D_t)f(D_t)dD_t+\int_x^\infty s(D_t-x)f(D_t)dD_t, & t<T\\ \int_0^x (h+l)(x-D_t)f(D_t)dD_t+\int_x^\infty s(D_t-x)f(D_t)dD_t, & t=T\end{cases} \tag{7}$$

Note that the salvage cost occurs only for the remaining inventory at the last period, $T=2$ in this section. $W_t(x, y)$ is the expected warehousing cost for a single period when base commitment level is $y$ and inventory level at the beginning of period $t$ is $x$. $W_t(x, y)$ can be expressed as

$$\begin{aligned}W_t(x, y)&=p_1y+\int_y^{\alpha x} p_2(V_t-y)f(V_t)dV_t\\ &=p_1y+\int_0^{x-\frac{y}{\alpha}} p_2(\alpha(x-D_t)-y)f(D_t)dD_t\end{aligned} \tag{8}$$

Note that $I_1$ is a random variable that depends upon the on-hand inventory level of at the end of period 1, that is, $I_1=Q_1-D_1$. Thus, (6) can be rearranged as follows.

$$g_2(I_1)=\begin{cases}L_2(Q_1-D_1)+W_2(Q_1-D_1,\ B_2^I), & \text{if } Q_1-D_1\geq Q_2^c\\ c(Q_2^c-Q_1+D_1)+L_2(Q_2^c)+W_2(Q_2^c,\ B_2^c), & \text{if } Q_1-D_1<Q_2^c\end{cases} \tag{9}$$

The expected value of $g_2(I_1)$ is given by

$$\begin{aligned}E[g_2(I_1)]&=\int_0^{Q_1-Q_2^c}\left[L_2(Q_1-D_1)+W_2(Q_1-D_1,\ B_2^I)\right]f(D_1)dD_1\\ &+\int_{Q_1-Q_2^c}^{\infty}\left[c(Q_2^c-Q_1+D_1)+L_2(Q_2^c)+W_2(Q_2^c,\ B_2^c)\right]f(D_1)dD_1\end{aligned} \tag{10}$$

The expected cost of the following the optimal policy for two periods is given by

$$g_1(I_0)=\min_{\substack{Q_1\geq 0\\ B_1\geq 0}}\left[cQ_1+L_1(Q_1)+W_1(Q_1,\ B_1)+E[g_2(I_1)]\right] \tag{11}$$

where $E[g_2(I_1)]$ is given by (10).

**Proposition 3.**

For period 1, the solution to minimize the total cost over two periods is $Q_1^*$ and $B_1^*$ such that

$$-s+(s+h)F(Q_1^*)+\alpha p_2 F\left(Q_1^*-\frac{B_1^*}{\alpha}\right)+(s+h+l)\int_0^{Q_1^*-Q_2^c} f(Q_1^*-D)F(D)dD=0 \tag{12}$$

$$F\left(Q_1^*-\frac{B_1^*}{\alpha}\right)=\frac{p_1}{p_2} \tag{13}$$

**Proof.**

See Appendix C. ■

We can find $Q_1^*$ and $B_1^*$ by solving jointly (12) and (13) after substituting $Q_2^c$ which can be obtained from (5).

**Proposition 4.**

$g_1(I_0)$ is convex with respect to $Q_1$ and $B_1$. Thus, $Q_1^*$ and $B_1^*$ guarantees the global optimum.

**Proof.**

See Appendix D. ■

Thus, the optimal policy for period 1 is

$$\begin{cases} \text{do not order and set } B=B_1^*, & \text{if } I_0 \ge Q_1^* \\ \text{order } (Q_1^*-I_0) \text{ and set } B=B_1^*, & \text{if } I_0 < Q_1^* \end{cases} \tag{14}$$

where $Q_1^*$ and $B_1^*$ satisfy (12) and (13).

**Proposition 5.**

Under modification policy, $Q_1^* \ge Q_2^c$ and $B_1^* \ge B_2^*$ if $s \ge c+\alpha p_1$.

**Proof.**

See Appendix E. ■

### *3.3.2 No modification of base commitment*

In this case, the retailer can not change the base commitment through two periods. Therefore, the retailer should decide a single $B$ rather than a schedule of base commitment $\mathbf{B} = (B_1, B_2)$. We use the order-up-to $\mathbf{Q} = (Q_1, Q_2)$ policy for each period.

Similar to the modification policy, the optimal policy for period 2 is

$$\begin{cases} \text{do not order,} & \text{if } I_1 \geq Q_2^c \\ \text{order } (Q_2^c - I_1), & \text{if } I_1 < Q_2^c \end{cases} \tag{15}$$

where $Q_2^c$ satisfies

$$(c-s)+(s+h+l)F(Q_2^c)+\alpha p_2 F\left(Q_2^c - \frac{B^*}{\alpha}\right) = 0 \tag{16}$$

Equation (16) is induced from the first order condition for period 2 with respect to $Q_2$. The cost of this optimum policy can be expressed as

$$g_2(I_1) = \begin{cases} L_2(I_1) + W_2(I_1, B), & \text{if } I_1 \geq Q_2^c \\ c(Q_2^c - I_1) + L_2(Q_2^c) + W_2(Q_2^c, B), & \text{if } I_1 < Q_2^c \end{cases} \tag{17}$$

The expected value of $g_2(I_1)$ is given by

$$\begin{aligned} E[g_2(I_1)] = & \int_0^{Q_1 - Q_2^c} \left[ L_2(Q_1 - D_1) + W_2(Q_1 - D_1, B) \right] f(D_1) dD_1 \\ & + \int_{Q_1 - Q_2^c}^{\infty} \left[ c(Q_2^c - Q_1 + D_1) + L_2(Q_2^c) + W_2(Q_2^c, B) \right] f(D_1) dD_1 \end{aligned} \tag{18}$$

The expected cost of the following the optimal policy for two periods is given by

$$g_1(I_0) = \min_{\substack{Q_1 \geq 0 \\ B \geq 0}} \left[ cQ_1 + L_1(Q_1) + W_1(Q_1, B) + E[g_2(I_1)] \right] \tag{19}$$

where $E[g_2(I_1)]$ is given by (18).

**Proposition 6.**

For period 1, the solution to minimize the total cost over two periods is $Q_1^*$ and $B^*$

such that

$$
\begin{aligned}
&-s+(s+h)F(Q_1^*)+\alpha p_2 F\left(Q_1^*-\frac{B^*}{\alpha}\right)-\alpha p_1 F(Q_1^*-Q_2^c) \\
&+(s+h+l)\int_0^{Q_1^*-Q_2^c} f(Q_1^*-D)F(D)dD \\
&+\alpha p_2\left[F\left(Q_2^c-\tfrac{B^*}{\alpha}\right)F(Q_1^*-Q_2^c)+\int_0^{Q_1^*-Q_2^c} f(Q_1^*-D-\tfrac{B^*}{\alpha})F(D)dD\right]=0
\end{aligned} \tag{20}
$$

$$
2p_1-p_2F\left(Q_1^*-\frac{B^*}{\alpha}\right)-p_2F\left(Q_2^c-\frac{B^*}{\alpha}\right)-p_2\int_0^{Q_1^*-Q_2^c} f(Q_1^*-D-\tfrac{B^*}{\alpha})F(D)dD=0 \tag{21}
$$

**Proof.**

See Appendix F. ■

We can find $Q_1^*$, $Q_2^c$ and $B^*$ numerically by solving jointly (16), (20) and (21).

**Proposition 7.**

$g_1(I_0)$ is convex with respect to $Q_1$ and $B$. Thus, $Q_1^*$ and $B^*$ guarantees the global optimum.

**Proof.**

See Appendix G. ■

Thus, the optimal policy for period 1 is

$$
\begin{cases}
\text{do not order and set } B=B^*, & \text{if } I_0 \ge Q_1^* \\
\text{order } (Q_1^*-I_0) \text{ and set } B=B^*, & \text{if } I_0 < Q_1^*
\end{cases} \tag{22}
$$

where $Q_1^*$ and $B^*$ satisfy (16), (20) and (21).

**Proposition 8.**

Under no modification policy, $Q_1^* \ge Q_2^c$ if $s \ge c+\alpha p_1$.

**Proof.**

See Appendix H. ■

### 3.3.3 *Value of contract flexibility and multi-period case*

Denote the expected cost of optimal policy for two periods under modification policy and no modification policy by $TC^F$ and $TC^0$ for each. The superscript of $F$ means *flexible*. $TC^F$ is expressed as (11) and $TC^0$ as (19). Each optimal quantities for $TC^F$ and $TC^0$ are obtained from proposition 3 and 6. A gap between $TC^F$ and $TC^0$ can be interpreted as the value of contract flexibility in terms of modification frequency. The induction of the gap becomes very cumbersome. Therefore, we will show the behavior of the gap through the numerical example in the next section.

We can extend the two-period model to the multi-period case. Although the proof can be more cumbersome, based on the proposition 5 and 8, we conjecture that the optimal ordering and commitment policy have the form as follows.

**(i) Under modification policy**

At the beginning of period $t$, $t = 1, 2, \cdots, T$,

$$\begin{cases} \text{do not order and set } B_t^* = B_t^I, & \text{if } I_{t-1} \ge Q_t^c \\ \text{order } (Q_t^c - I_{t-1}) \text{ and set } B_t^* = B_t^c, & \text{if } I_{t-1} < Q_t^c \end{cases} \tag{23}$$

Furthermore, $Q_T^c \le Q_{T-1}^c \le \cdots \le Q_2^c \le Q_1^c$ and $B_T^c \le B_{T-1}^c \le \cdots \le B_2^c \le B_1^c$

**(ii) Under no modification policy**

At the beginning of period $t$, $t = 1, 2, \cdots, T$,

$$\begin{cases} \text{do not order and set } B_t^* = B^*, & \text{if } I_{t-1} \ge Q_t^c \\ \text{order } (Q_t^c - I_{t-1}) \text{ and set } B_t^* = B^*, & \text{if } I_{t-1} < Q_t^c \end{cases} \tag{24}$$

Furthermore, $Q_T^c \le Q_{T-1}^c \le \cdots \le Q_2^c \le Q_1^c$

## 4. Numerical Example

Under two-period model, we compare the total costs between modification policy and no modification policy, i.e. $TC^F$ and $TC^0$. Basically, the demands for each period are identically normally distributed with mean $\mu$ and standard deviation $\sigma$. The value of basic parameters we used in the numerical analysis is in Table 2. The mean and standard deviation of demand are based on the refrigerator's one-year demand of Korean consumer electronic manufacturer. We arbitrarily chose the rest parameters to show the examples.

Table 2. Basic parameters

| $c$ | $h$ | $l$ | $s$ | $P_1$ | $P_2$ | $\alpha$ | *mean* | *std.dev.* | *C.V.* |
|---|---|---|---|---|---|---|---|---|---|
| 60 | 10 | 5 | 240 | 16 | 24 | 0.2 | 53,014 | 26,507 | 0.5 |

The value of flexible contract is measured by the gap between $E(TC^F)$ and $E(TC^0)$ over $E(TC^0)$. Formally, $Gap = E(TC^0) - E(TC^F)$. $Gap / E(TC^0)$ is used to measure the value of flexible contract. We firstly find the optimal solutions for each policy and then substitute them in the total cost function. Since the demand of period 1 is not realized, we used the expected total costs, that is, $E(TC^F)$ and $E(TC^0)$. Computations are implemented by MATLAB® 6.1 using Pentium® IV PC.

In general, modification policy result in saving in expected total costs. As the coefficients of variation (C.V.) of demand increases, the Gap increases to 0.18% (Figure 2). Capital charge and salvage cost for inventory have a monotonic effect on the Gap (Figure 3, 4). That is, the larger the capital charge is, the smaller the value of flexible contract is. Conversely, the larger the salvage cost is, the larger the value is. The change of shortage costs or ordering costs increases the Gap improvement to a certain point and decreases afterwards (Figure 5, 6). As the ratio of premium charge over basic charge for space requirement increases, the degree of Gap improvement increases (Figure 7). In addition, we analyzed the value of flexible contract with respect to every parameters based on the uniform distribution demand which has the same mean and standard deviation as the normal distribution demand. As a result, there are two parameters that have different patterns from the normal distribution. For the uniformly distributed demand, the value of flexibility increases as the capital charge for inventory increases (Figure 8). As the price ratio increases, the value of flexibility decreases so that it has a reverse pattern from the normally distributed demand (Figure 9). Therefore, we observed that the value of flexible contract with respect to parameters could have a different pattern over the various probability distributions of demand.

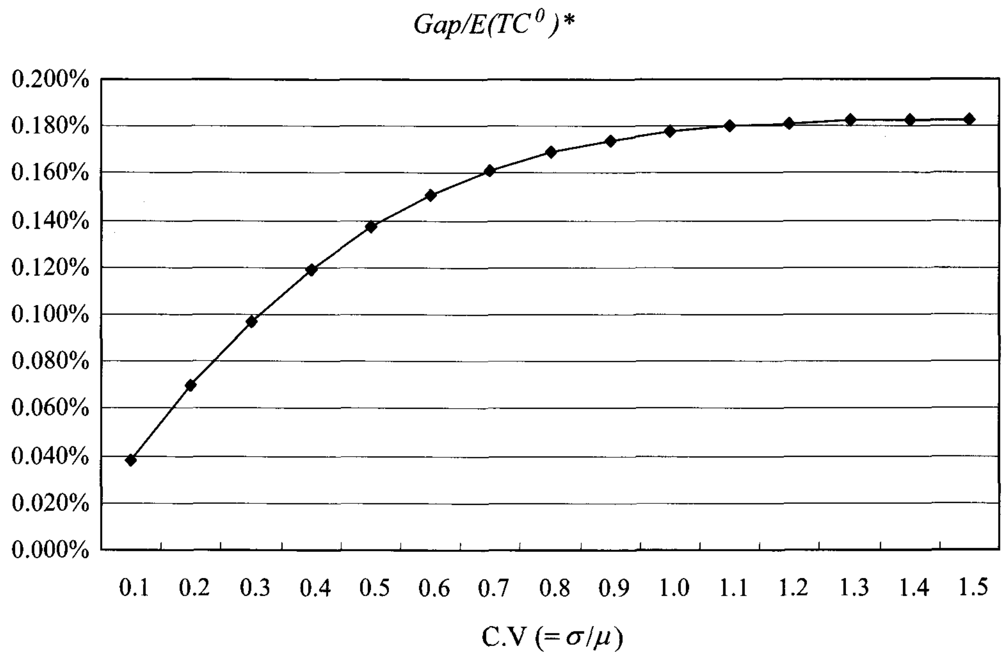


Figure 2. Coefficients of variation

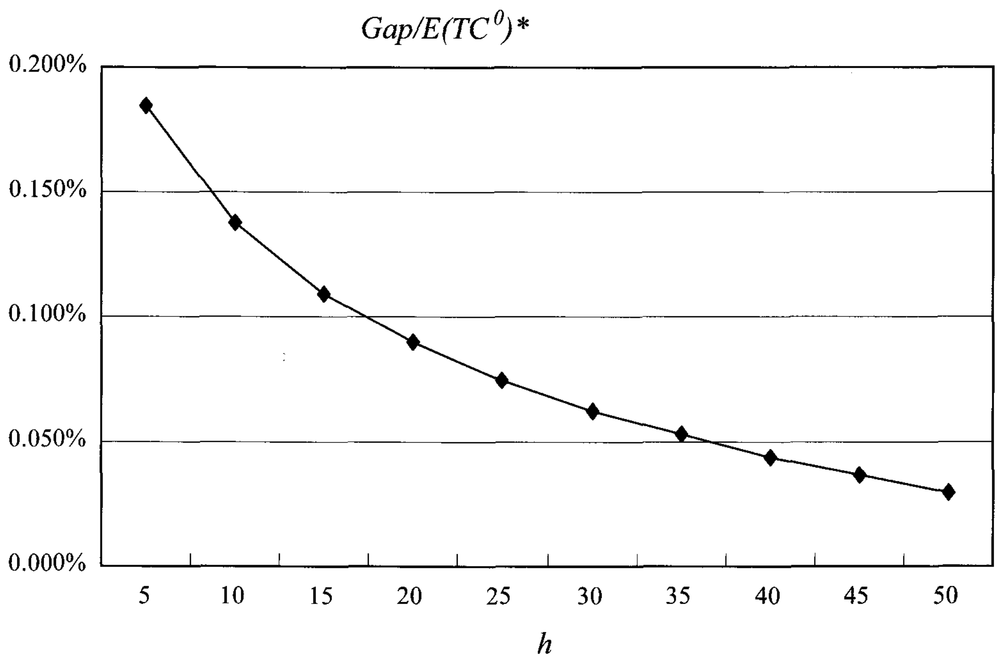


Figure 3. Capital charge for inventory

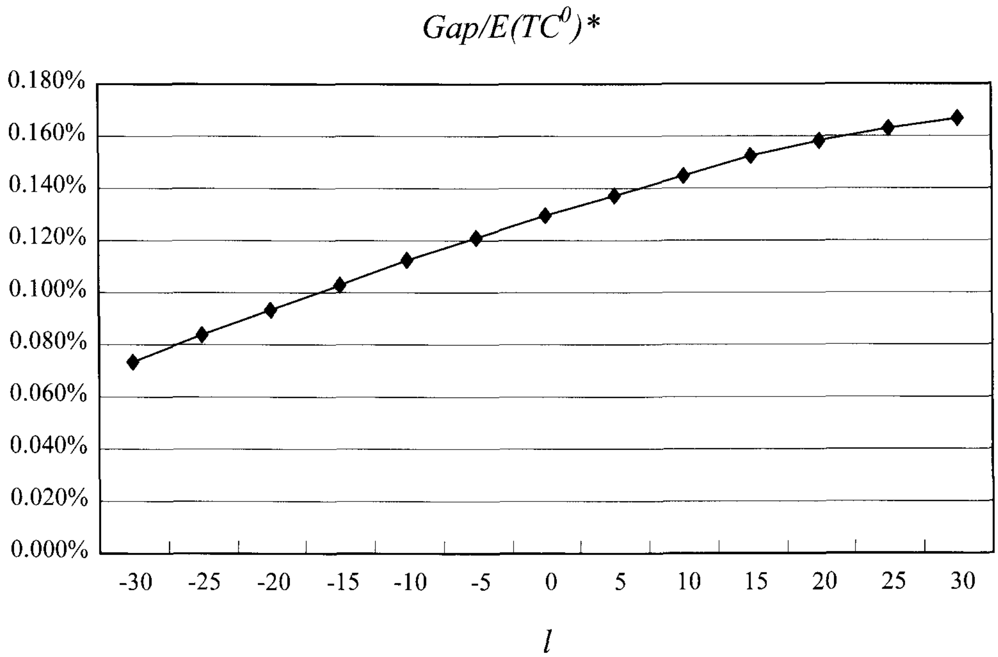


Figure 4. Salvage costs

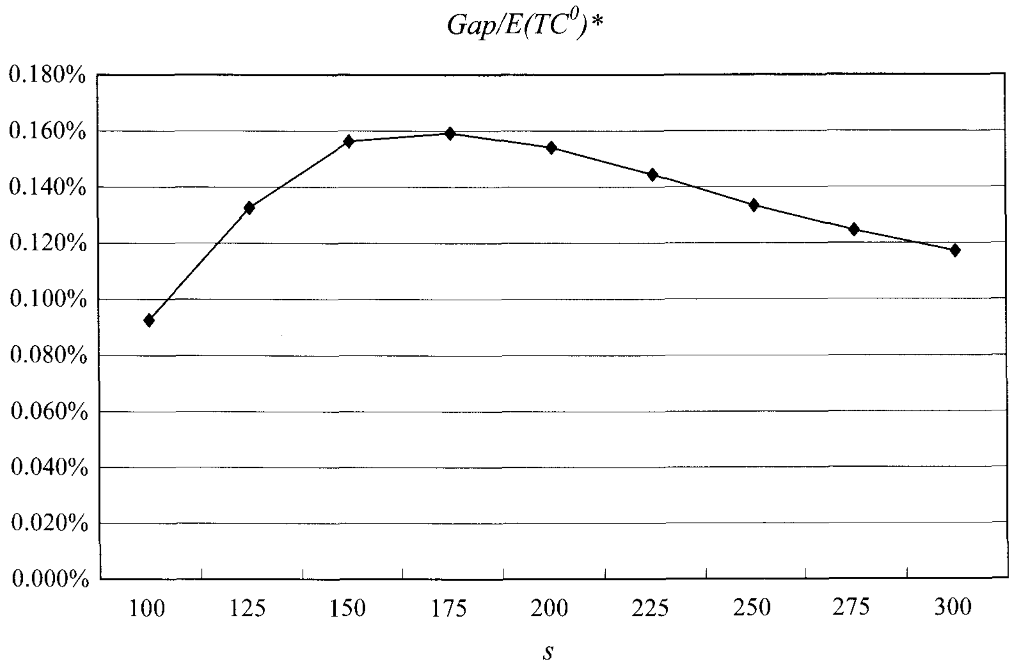


Figure 5. Shortage costs

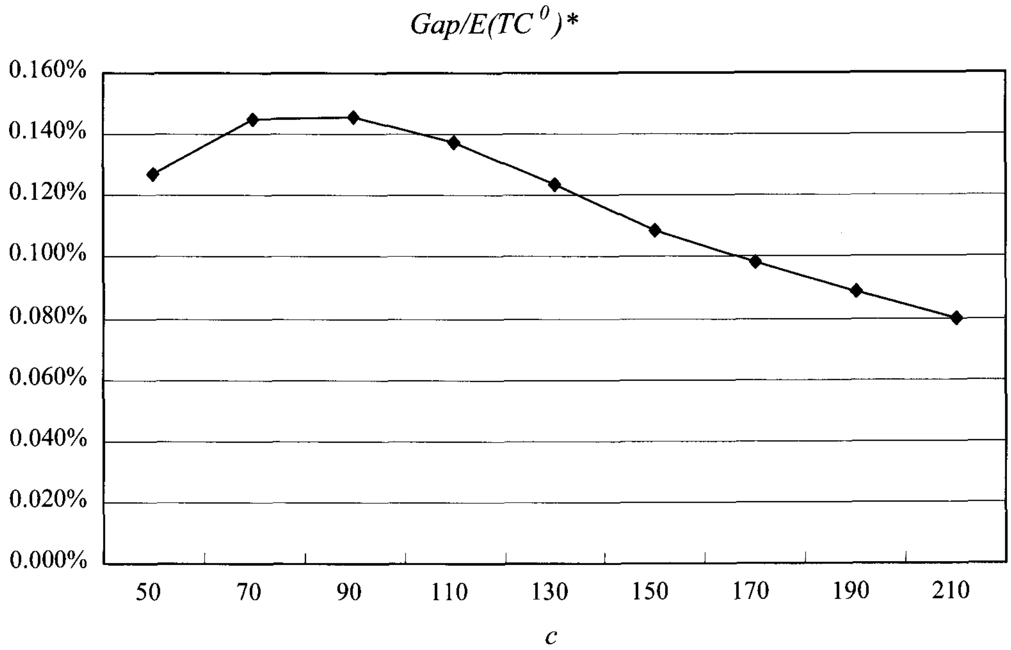


Figure 6. Ordering costs

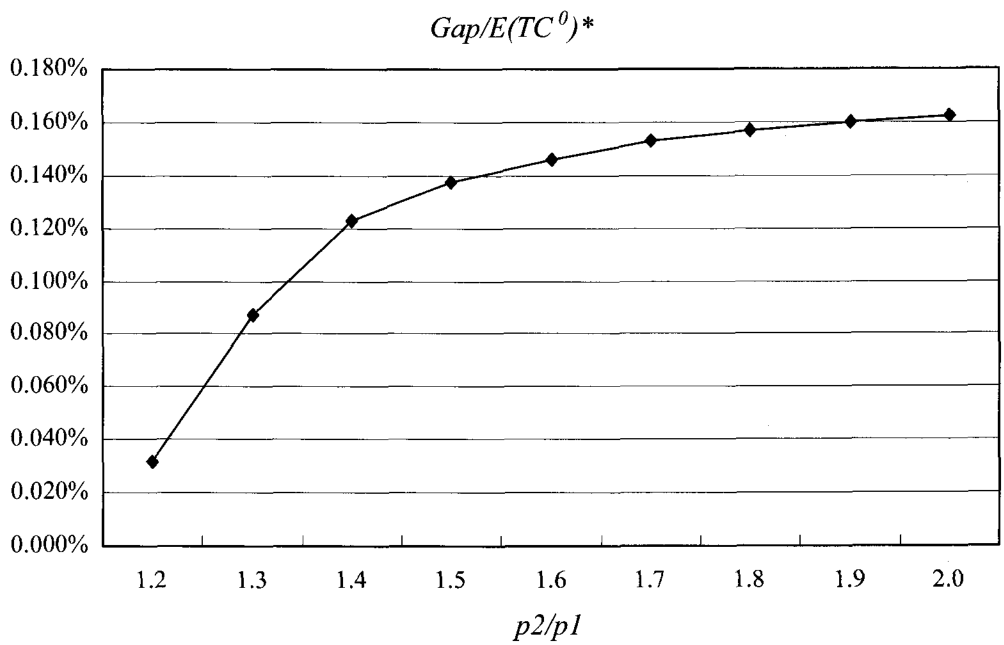


Figure 7. Premium charge over basic charge for warehousing service

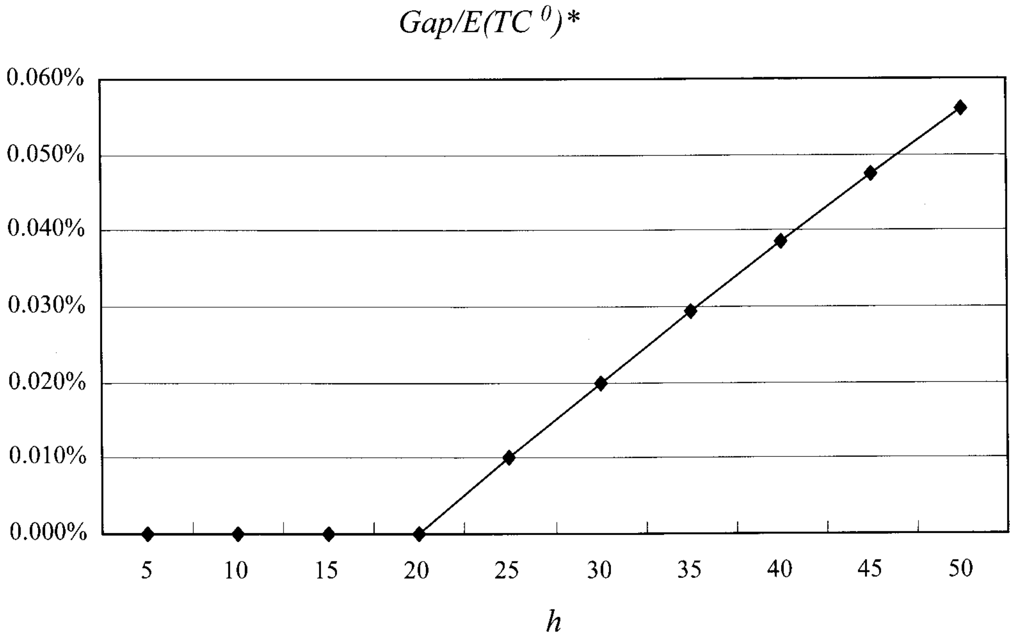


Figure 8. Capital charge for inventory (Uniformly distributed demand)

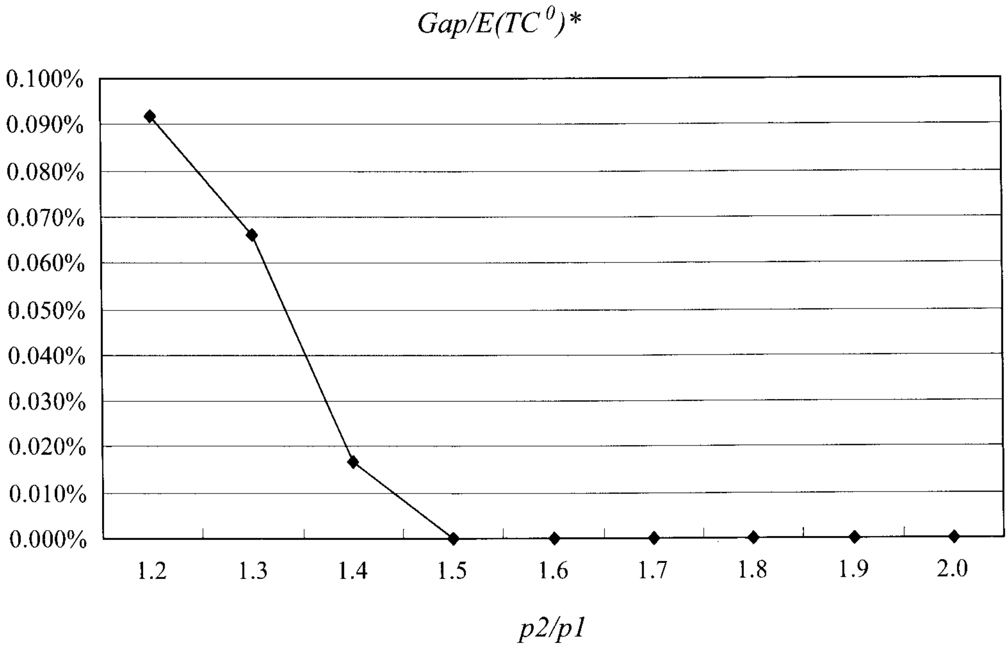


Figure 9. Premium charge over basic charge for warehousing service (Uniformly distributed demand)

## 5. Conclusion

This paper considers a warehousing contract under dynamic probabilistic demands. Considering a supply chain consisting of a supplier, a retailer and its third-party warehousing partner, we found there is a close relationship between the order quantity for the supplier and the space commitment for the warehousing partner. If the retailer increases the order quantity for the supplier, then the space requirement will be increased. When we need more warehouse space through a third-party warehousing contract which normally consists of a base charge for the predetermined commitment and a premium charge for the overflow, the larger the base commitment, the more cost-effective in terms of the unit warehousing cost because the unit charge for the base commitment is cheaper than the premium charge. In the retailer's decision making, there exist two different trade-offs. One is a classical trade-off between the shortage costs and the inventory capital charge. This trade-off mainly decides the order quantity for the supplier. The other is that between the fixed base commitment charge and the variable premium charge for the overflow of space requirement. This trade-off mainly decides the base commitment level. In addition, the decision about the optimal order quantity affects the space requirement. Therefore, we need to optimize the order quantity and space commitment level jointly. We develop an analytical model under probabilistic demands that the retailer can determine the optimal order size and space commitment level. We found the closed-form optimum solution for the single period and the optimal conditions for two-period case. To evaluate the value of contract flexibility for the two-period case, we compared the total costs under two policies; modification policy under which the base commitment can be changed at the start of each period and no modification policy. From the results of numerical analysis, we showed that the modification policy was more cost-effective as the variability of demand increased.

Even though this paper is a first step in exploring our research question, there are a few limitations. We have assumed instantaneous order fulfillments and delivery, which are unrealistic in some cases. To be more realistic, the continuous demand in a certain period should have been considered. To do this, this paper can be extended to the continuous review inventory policy. Space requirement types can be varied rather than linear increments. For example, the space requirements are increasing nonlinearly or stepwise. Furthermore, the cost functions can be assumed in other forms.

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# Appendices

## Appendix A: Proof of Proposition 1

The expected total cost under the policy $(Q, B)$ is as follows.

$$TC(Q, B) = cQ + \int_0^Q h(Q-D)f(D)dD + \int_Q^\infty s(D-Q)f(D)dD + \left(p_1 B + \int_B^{\alpha Q} p_2(V-B)f(V)dV\right) \quad \text{(A1)}$$

Since $V$ is the space requirement for inventory after satisfying the demand, $V$ is equal to 0 for $D \geq Q$, and $\alpha(Q-D)$ for $0 \leq D < Q$ which is the case the inventory remains after the demand is fulfilled. Note that over the range of $0 \leq D < Q$ the probability density functions of $V$, $f(V)$, can be transformed to $f(D)$ by virtue of the transformation method (see p. 280 of [7]) as follows.

$$f(V) = f(D)\left|\frac{\partial D}{\partial V}\right| = \frac{1}{\alpha} f(D) \quad \text{(A2)}$$

Also, since $dV = (-\alpha)dD$ for $0 \leq D < Q$, the final term of $TC(Q, B)$ can be restated as

$$\int_B^{\alpha Q} p_2(V-B)f(V)dV = \int_0^{Q-\frac{B}{\alpha}} p_2\left(\alpha(Q-D)-B\right)f(D)dD \quad \text{(A3)}$$

First order conditions for optimality are

$$\frac{\partial TC}{\partial Q} = (c-s) + (s+h)F(Q) + p_2 \alpha F\left(Q - \frac{B}{\alpha}\right) = 0 \quad \text{(A4)}$$

$$\frac{\partial TC}{\partial B} = p_1 - \int_0^{Q-\frac{B}{\alpha}} p_2 f(D)dD = p_1 - p_2 F\left(Q - \frac{B}{\alpha}\right) = 0 \quad \text{(A5)}$$

These give the optimal solution as follows.

$$F\left(Q^*\right) = \frac{s - c - \alpha p_1}{s+h} \quad \text{(A6)}$$

$$F\left(Q^* - \frac{B^*}{\alpha}\right) = \frac{p_1}{p_2} \quad \text{(A7)}$$

**Appendix B: Proof of Proposition 2**

The Hessian of expected total costs is as follows.

$$\begin{pmatrix} \dfrac{\partial^2 TC}{\partial Q_1^2} & \dfrac{\partial^2 TC}{\partial Q_1 \partial B_1} \\ \dfrac{\partial^2 TC}{\partial B_1 \partial Q_1} & \dfrac{\partial^2 TC}{\partial B_1^2} \end{pmatrix} = \begin{pmatrix} (h+s)f(Q)+\alpha p_2 f\left(Q-\dfrac{B}{\alpha}\right) & -p_2 f\left(Q-\dfrac{B}{\alpha}\right) \\ -p_2 f\left(Q-\dfrac{B}{\alpha}\right) & \dfrac{1}{\alpha} p_2 f\left(Q-\dfrac{B}{\alpha}\right) \end{pmatrix} \tag{B1}$$

The Hessian matrix is positive definite since $\dfrac{\partial^2 TC}{\partial Q_1^2} > 0$ and $\dfrac{\partial^2 TC}{\partial Q_1^2}\dfrac{\partial^2 TC}{\partial B_1^2} - \left(\dfrac{\partial^2 TC}{\partial B_1 \partial Q_1}\right)^2 > 0$.

Therefore, the expected total cost function is convex and $Q^*$ and $B^*$ guarantees the global optimum.

**Appendix C: Proof of Proposition 3**

Total costs over two periods with initial inventory $I_0$ is

$$g_1(I_0) = \min_{\substack{Q_1 \ge 0 \\ B_1 \ge 0}} \left[ cQ_1 + L_1(Q_1) + W_1(Q_1, B_1) + E[g_2(I_1)] \right] \tag{C1}$$

where

$$\begin{aligned} E[g_2(I_1)] &= \int_0^{Q_1 - Q_2^c} \left[ L_2(Q_1 - D_1) + W_2(Q_1 - D_1, B_2^I) \right] f(D_1) dD_1 \\ &+ \int_{Q_1 - Q_2^c}^{\infty} \left[ c(Q_2^c - Q_1 + D_1) + L_2(Q_2^c) + W_2(Q_2^c, B_2^c) \right] f(D_1) dD_1 \end{aligned} \tag{C2}$$

$E[g_2(I_1)]$ can be restated as follows.

$$\begin{aligned} E[g_2(I_1)] &= \underbrace{\int_0^{Q_1 - Q_2^c} L_2(Q_1 - D_1) f(D_1) dD_1}_{(A)} + \underbrace{\int_0^{Q_1 - Q_2^c} W_2(Q_1 - D_1, B_2^I) f(D_1) dD_1}_{(B)} \\ &+ \underbrace{\int_{Q_1 - Q_2^c}^{\infty} c(Q_2^c - Q_1 + D_1) f(D_1) dD_1}_{(C)} + \underbrace{\int_{Q_1 - Q_2^c}^{\infty} L_2(Q_2^c) f(D_1) dD_1}_{(D)} \\ &+ \underbrace{\int_{Q_1 - Q_2^c}^{\infty} W_2(Q_2^c, B_2^c) f(D_1) dD_1}_{(E)} \end{aligned} \tag{C3}$$

To obtain the first order derivative of $E[g_2(I_1)]$, we used the Liebnitz's rule for each term.

For part ($A$),

$$\frac{\partial(A)}{\partial Q_1} = -sF(Q_1 - Q_2^c) + (s+h+l)\int_0^{Q_1-Q_2^c} F(Q_1 - D_1)f(D_1)dD_1 + L_2(Q_2^c)f(Q_1 - Q_2^c) \quad \text{(C4)}$$

since

$$L_2(Q_1 - D_1) = \int_0^{Q_1-D_1}(h+l)(Q_1 - D_1 - D_2)f(D_2)dD_2 + \int_{Q_1-D_1}^{\infty} s(D_2 - Q_1 + D_1)f(D_2)dD_2 \quad \text{(C5)}$$

and

$$\frac{\partial L_2(Q_1-D_1)}{\partial Q_1} = -s + (s+h+l)F(Q_1 - D_1) \quad \text{(C6)}$$

For part ($B$),

If $I \geq F^{-1}\left(\frac{p_1}{p_2}\right)$ or $I \geq Q_2^c - \frac{B_2^c}{\alpha}$, then $B_2^I = \alpha\left(Q_1 - D_1 - Q_2^c + \frac{B_2^c}{\alpha}\right)$.

$$W_2(Q_1 - D_1, B_2^I) = \alpha p_1\left(Q_1 - D_1 - Q_2^c + \frac{B_2^c}{\alpha}\right) + \int_0^{Q_2^c - \frac{B_2^c}{\alpha}} \alpha p_2(Q_2^c - \tfrac{B_2^c}{\alpha} - D_2)f(D_2)dD_2$$

$$\frac{\partial(B)}{\partial Q_1} = \alpha p_1 F(Q_1 - Q_2^c) + (p_1 B_2^c + k)f(Q_1 - Q_2^c) \quad \text{(C8)}$$

where $k \equiv \int_0^{Q_2^c - \frac{B_2^c}{\alpha}} \alpha p_2(Q_2^c - \frac{B_2^c}{\alpha} - D_2)f(D_2)dD_2$ which is independent of $D_1$.

For part (C),

$$\frac{\partial(C)}{\partial Q_1} = \int_{Q_1-Q_2^c}^{\infty}(-c)f(D_1)dD_1 = cF(Q_1 - Q_2^c) - c \quad \text{(C9)}$$

For part ($D$),

$$\frac{\partial(D)}{\partial Q_1} = \int_{Q_1-Q_2^c}^{\infty} \frac{\partial L_2(Q_2^c)}{\partial Q_1} f(D_1)dD_1 - L_2(Q_2^c)f(Q_1 - Q_2^c) = -L_2(Q_2^c)f(Q_1 - Q_2^c) \quad \text{(C10)}$$

since $L_2(Q_2^c)$ is independent of $Q_1$ and thus $\frac{\partial L_2(Q_2^c)}{\partial Q_1}=0$.

For part ($E$),

$$\frac{\partial(E)}{\partial Q_1}=\int_{Q_1-Q_2^c}^{\infty}\frac{\partial W_2(Q_2^c,B_2^c)}{\partial Q_1}f(D_1)dD_1-W_2(Q_2^c,\,B_2^c)f(Q_1-Q_2^c)=-W_2(Q_2^c,\,B_2^c)f(Q_1-Q_2^c) \tag{C11}$$

since $W_2(Q_2^c,\,B_2^c)$ is independent of $Q_1$ and thus $\frac{\partial W_2(Q_2^c,\,B_2^c)}{\partial Q_1}=0$.

Summing up the each derivative from part ($A$) to ($E$),

$$\frac{\partial E[g_2(I_1)]}{\partial Q_1}=(c-s+\alpha p_1)F(Q_1-Q_2^c)+(s+h+l)\int_0^{Q_1-Q_2^c}F(Q_1-D_1)f(D_1)dD_1-c \tag{C12}$$

since $W_2(Q_2^c,\,B_2^c)=p_1B_2^c+k$.

Thus, the first order optimal condition of $g_1(I_0)$ with respect to $Q_1$ is

$$\frac{\partial g_1(I_0)}{\partial Q_1}=-s+(s+h)F(Q_1)+\alpha p_2F\left(Q_1-\frac{B_1}{\alpha}\right)+(s+h+l)\int_0^{Q_1-Q_2^c}f(Q_1-D_1)F(D_1)dD_1=0 \tag{C13}$$

since

$$\int_0^{Q_1-Q_2^c}F(Q_1-D_1)f(D_1)dD_1=\left(\frac{s-c-\alpha p_1}{s+h+l}\right)F(Q_1-Q_2^c)+\int_0^{Q_1-Q_2^c}f(Q_1-D_1)F(D_1)dD_1 \tag{C14}$$

This gives equation (14).

The first order optimal condition of $g_1(I_0)$ with respect to $B_1$ is

$$\frac{\partial g_1(I_0)}{\partial B_1}=p_1-\int_0^{Q_1-\frac{B_1}{\alpha}}p_2f(D_1)dD_1=p_1-p_2F\left(Q_1-\frac{B_1}{\alpha}\right)=0 \tag{C15}$$

since $g_2(I_1)$ is independent of $B_1$ and thus $\frac{\partial E[g_2(I_1)]}{\partial B_1}=0$.

This gives equation (15).

**Appendix D: Proof of Proposition 4**

The Hessian of $g_1(I_0)$ is

$$\frac{\partial^2 g_1(I_0)}{\partial Q_1^2} = (s+h)f(Q_1) + \alpha p_2 f\left(Q_1 - \frac{B_1}{\alpha}\right) + (s+h+l)\left[f(Q_2^c)F(Q_1 - Q_2^c) + \int_0^{Q_1 - Q_2^c} f'(Q_1 - D)F(D)dD\right] \quad \text{(D1)}$$

$$\frac{\partial^2 g_1(I_0)}{\partial Q_1 \partial B_1} = \frac{\partial^2 g_1(I_0)}{\partial B_1 \partial Q_1} = -p_2 f\left(Q_1 - \frac{B_1}{\alpha}\right) \quad \text{(D2)}$$

$$\frac{\partial^2 g_1(I_0)}{\partial B_1^2} = \frac{p_2}{\alpha} f\left(Q_1 - \frac{B_1}{\alpha}\right) \quad \text{(D3)}$$

Hessian is positive definite since

$$\frac{\partial^2 g_1(I_0)}{\partial Q_1^2} > 0 \text{ and } \frac{\partial^2 g_1(I_0)}{\partial Q_1^2}\frac{\partial^2 g_1(I_0)}{\partial B_1^2} - \left(\frac{\partial^2 g_1(I_0)}{\partial B_1 \partial Q_1}\right)^2 > 0.$$

Therefore, $g_1(I_0)$ is convex with respect to $Q_1$ and $B_1$ and thus $Q_1^*$ and $B_1^*$ guarantees the global optimum.

**Appendix E: Proof of Proposition 5**

The first order condition for optimality for period 1 is

$$\frac{\partial g_1(I_0)}{\partial Q_1} = -s + (s+h)F(Q_1) + \alpha p_2 F\left(Q_1 - \frac{B_1}{\alpha}\right) + (s+h+l)\int_0^{Q_1 - Q_2^c} f(Q_1 - D_1)F(D_1)dD_1 = 0 \quad \text{(E1)}$$

$$\frac{\partial g_1(I_0)}{\partial B_1} = p_1 - p_2 F\left(Q_1 - \frac{B_1}{\alpha}\right) = 0 \quad \text{(E2)}$$

Because of the second equality, the first condition can be restated as

$$\frac{\partial g_1(I_0)}{\partial Q_1} = (\alpha p_1 - s) + (s+h)F(Q_1) + (s+h+l)\int_0^{Q_1 - Q_2^c} f(Q_1 - D_1)F(D_1)dD_1 = 0 \quad \text{(E3)}$$

Since $\frac{\partial^2 g_1(I_0)}{\partial Q_1^2} > 0$, if $\frac{\partial g_1(I_0)}{\partial Q_1} < 0$ at a certain point of $Q_1$, the optimal solution

$Q_1^*$ is larger than the point.

$$\text{At } Q_1 = Q_2^c,\ \frac{\partial g_1(I_0)}{\partial Q_1} = \frac{-c(s+h)-(s-\alpha p_1)l}{s+h+l} \tag{E4}$$

since $Q_2^c = F^{-1}\left(\frac{s-c-\alpha p_1}{s+h+l}\right)$.

Assuming that $s-c-\alpha p_1 \geq 0$ to exist a positive $Q_2^c$, $\frac{\partial g_1(I_0)}{\partial Q_1} \leq 0$. Therefore, $Q_1^* \geq Q_2^c$.

Since $F\left(Q_1^* - \frac{B_1^*}{\alpha}\right) = F\left(Q_2^c - \frac{B_2^*}{\alpha}\right) = \frac{p_1}{p_2}$ and $Q_1^* \geq Q_2^c$, $B_1^*$ should be larger than $B_2^*$. That is, $B_1^* \geq B_2^*$.

**Appendix F: Proof of Proposition 6**

Total costs over two periods with initial inventory $I_0$ is

$$g_1(I_0) = \min_{\substack{Q_1 \geq 0 \\ B \geq 0}} \left[cQ_1 + L_1(Q_1) + W_1(Q_1, B) + E[g_2(I_1)]\right] \tag{F1}$$

where

$$\begin{aligned} E[g_2(I_1)] = &\int_0^{Q_1-Q_2^c} \left[L_2(Q_1-D_1) + W_2(Q_1-D_1, B)\right] f(D_1)dD_1 \\ &+ \int_{Q_1-Q_2^c}^{\infty} \left[c(Q_2^c - Q_1 + D_1) + L_2(Q_2^c) + W_2(Q_2^c, B)\right] f(D_1)dD_1 \end{aligned} \tag{F2}$$

$E[g_2(I_1)]$ can be restated as follows.

$$\begin{aligned} E[g_2(I_1)] = &\underbrace{\int_0^{Q_1-Q_2^c} L_2(Q_1-D_1)f(D_1)dD_1}_{(A)} + \underbrace{\int_0^{Q_1-Q_2^c} W_2(Q_1-D_1,\ B)f(D_1)dD_1}_{(B)} \\ &+ \underbrace{\int_{Q_1-Q_2^c}^{\infty} c(Q_2^c - Q_1 + D_1)f(D_1)dD_1}_{(C)} + \underbrace{\int_{Q_1-Q_2^c}^{\infty} L_2(Q_2^c)f(D_1)dD_1}_{(D)} \\ &+ \underbrace{\int_{Q_1-Q_2^c}^{\infty} W_2(Q_2^c,\ B)f(D_1)dD_1}_{(E)} \end{aligned} \tag{F3}$$

To obtain the first order derivative of $E[g_2(I_1)]$, we used the Liebnitz's rule for each term.

For part $(A)$,

$$\frac{\partial(A)}{\partial Q_1} = -sF(Q_1 - Q_2^c) + (s+h+l)\int_0^{Q_1 - Q_2^c} F(Q_1 - D_1) f(D_1) dD_1 + L_2(Q_2^c) f(Q_1 - Q_2^c) \quad \text{(F4)}$$

since

$$L_2(Q_1 - D_1) = \int_0^{Q_1 - D_1} (h+l)(Q_1 - D_1 - D_2) f(D_2) dD_2 + \int_{Q_1 - D_1}^{\infty} s(D_2 - Q_1 + D_1) f(D_2) dD_2 \quad \text{(F5)}$$

and

$$\tfrac{\partial L_2(Q_1 - D_1)}{\partial Q_1} = -s + (s+h+l)F(Q_1 - D_1) \quad \text{(F6)}$$

For part $(B)$,

$$W_2(Q_1 - D_1, B) = \alpha p_1 B + \int_0^{(Q_1 - D_1) - \frac{B}{\alpha}} p_2 \big(\alpha(Q_1 - D_1 - D_2) - B\big) f(D_2) dD_2 \quad \text{(F7)}$$

$$\frac{\partial W_2(Q_1 - D_1, B)}{\partial Q_1} = \int_0^{Q_1 - D_1 - \frac{B}{\alpha}} \alpha p_2 f(D_2) dD_2 = \alpha p_2 F(Q_1 - D_1 - \tfrac{B}{\alpha}) \quad \text{(F8)}$$

$$\frac{\partial(B)}{\partial Q_1} = \alpha p_2 \int_0^{Q_1 - Q_2^c} F(Q_1 - D_1 - \tfrac{B}{\alpha}) f(D_1) dD_1 + W_2(Q_2^c, B) f(Q_1 - Q_2^c) \quad \text{(F9)}$$

For part $(C)$,

$$\frac{\partial(C)}{\partial Q_1} = \int_{Q_1 - Q_2^c}^{\infty} (-c) f(D_1) dD_1 = cF(Q_1 - Q_2^c) - c \quad \text{(F10)}$$

For part $(D)$,

$$\frac{\partial(D)}{\partial Q_1} = \int_{Q_1 - Q_2^c}^{\infty} \tfrac{\partial L_2(Q_2^c)}{\partial Q_1} f(D_1) dD_1 - L_2(Q_2^c) f(Q_1 - Q_2^c) = -L_2(Q_2^c) f(Q_1 - Q_2^c) \quad \text{(F11)}$$

since $L_2(Q_2^c)$ is independent of $Q_1$ and thus $\tfrac{\partial L_2(Q_2^c)}{\partial Q_1} = 0$.

For part $(E)$,

$$\frac{\partial(E)}{\partial Q_1} = \int_{Q_1 - Q_2^c}^{\infty} \tfrac{\partial W_2(Q_2^c, B)}{\partial Q_1} f(D_1) dD_1 - W_2(Q_2^c, B) f(Q_1 - Q_2^c) = -W_2(Q_2^c, B) f(Q_1 - Q_2^c) \quad \text{(F12)}$$

since $W_2(Q_2^c, B)$ is independent of $Q_1$ and thus $\frac{\partial W_2(Q_2^c, B)}{\partial Q_1} = 0.$

Summing up the each derivative from part ($A$) to ($E$),

$$\begin{aligned}\frac{\partial E[g_2(I_1)]}{\partial Q_1} &= (c-s)F(Q_1 - Q_2^c) + (s+h+l)\int_0^{Q_1 - Q_2^c} F(Q_1 - D_1) f(D_1) dD_1 \\ &+ \alpha p_2 \int_0^{Q_1 - Q_2^c} F(Q_1 - D_1 - \tfrac{B}{\alpha}) f(D_1) dD_1 - c\end{aligned} \tag{F13}$$

Thus, the first order optimal condition of $g_1(I_0)$ with respect to $Q_1$ is

$$\begin{aligned}\frac{\partial g_1(I_0)}{\partial Q_1} &= (c-s) + (s+h)F(Q_1) + \alpha p_2 F\left(Q_1 - \frac{B}{\alpha}\right) \\ &+ (c-s)F(Q_1 - Q_2^c) + (s+h+l)\int_0^{Q_1 - Q_2^c} F(Q_1 - D_1) f(D_1) dD_1 \\ &+ \alpha p_2 \int_0^{Q_1 - Q_2^c} F(Q_1 - D_1 - \tfrac{B}{\alpha}) f(D_1) dD_1 - c = 0\end{aligned} \tag{F14}$$

By integrating by parts,

$$\int_0^{Q_1 - Q_2^c} F(Q_1 - D_1) f(D_1) dD_1 = \left(\frac{s - c - \alpha p_1}{s+h+l}\right) F(Q_1 - Q_2^c) + \int_0^{Q_1 - Q_2^c} f(Q_1 - D_1) F(D_1) dD_1 \tag{F15}$$

$$\int_0^{Q_1 - Q_2^c} F(Q_1 - D_1 - \tfrac{B}{\alpha}) f(D_1) dD_1 = F(Q_2^c - \tfrac{B}{\alpha}) F(Q_1 - Q_2^c) + \int_0^{Q_1 - Q_2^c} f(Q_1 - D_1 - \tfrac{B}{\alpha}) F(D_1) dD_1 \tag{F16}$$

Thus,

$$\begin{aligned}\frac{\partial g_1(I_0)}{\partial Q_1} &= -s + (s+h)F(Q_1) + \alpha p_2 F\left(Q_1 - \frac{B}{\alpha}\right) - \alpha p_1 F(Q_1 - Q_2^c) \\ &+ (s+h+l)\int_0^{Q_1 - Q_2^c} f(Q_1 - D_1) F(D_1) dD_1 \\ &+ \alpha p_2 \left[ F(Q_2^c - \tfrac{B}{\alpha}) F(Q_1 - Q_2^c) + \int_0^{Q_1 - Q_2^c} f(Q_1 - D_1 - \tfrac{B}{\alpha}) F(D_1) dD_1 \right] = 0\end{aligned} \tag{F17}$$

This gives equation (22).

Next, we find the first order condition for $B$. Different from the modification policy, we need to start from finding $\frac{\partial E[g_2(I_1)]}{\partial B}$ because $E[g_2(I_1)]$ is a function of $B$.

$$E[g_2(I_1)] = \int_0^{Q_1-Q_2^c} \left[ L_2(Q_1 - D_1) + W_2(Q_1 - D_1, B) \right] f(D_1)dD_1 + \int_{Q_1-Q_2^c}^{\infty} \left[ c(Q_2^c - Q_1 + D_1) + L_2(Q_2^c) + W_2(Q_2^c, B) \right] f(D_1)dD_1 \quad \text{(F18)}$$

$$\frac{\partial E[g_2(I_1)]}{\partial B} = \int_0^{Q_1-Q_2^c} \frac{\partial W_2(Q_1 - D_1, B)}{\partial B} f(D_1)dD_1 + \int_{Q_1-Q_2^c}^{\infty} \frac{\partial W_2(Q_2^c, B)}{\partial B} f(D_1)dD_1 \quad \text{(F19)}$$

$$W_2(Q_1 - D_1, B) = p_1 B + \int_0^{Q_1-D_1-\frac{B}{\alpha}} p_2(\alpha(Q_1 - D_1 - D_2) - B) f(D_2)dD_2 \quad \text{(F20)}$$

$$\frac{\partial W_2(Q_1 - D_1, B)}{\partial B} = p_1 - p_2 F\left(Q_1 - D_1 - \tfrac{B}{\alpha}\right) \quad \text{(F21)}$$

$$W_2(Q_2^c, B) = p_1 B + \int_0^{Q_2^c - \frac{B}{\alpha}} p_2(\alpha(Q_2^c - D_2) - B) f(D_2)dD_2 \quad \text{(F22)}$$

$$\frac{\partial W_2(Q_2^c, B)}{\partial B} = p_1 - p_2 F\left(Q_2^c - \tfrac{B}{\alpha}\right) \quad \text{(F23)}$$

$$\therefore \frac{\partial E[g_2(I_1)]}{\partial B} = p_1 - p_2 \int_0^{Q_1-Q_2^c} F\left(Q_1 - D_1 - \tfrac{B}{\alpha}\right) f(D_1)dD_1 - p_2 F\left(Q_2^c - \tfrac{B}{\alpha}\right)\left[1 - F\left(Q_1 - Q_2^c\right)\right] \quad \text{(F24)}$$

$$\frac{\partial g_1(I_0)}{\partial B} = 2p_1 - p_2 F\left(Q_1 - \tfrac{B}{\alpha}\right) - p_2 F\left(Q_2^c - \tfrac{B}{\alpha}\right) - p_2 \int_0^{Q_1-Q_2^c} f\left(Q_1 - D_1 - \tfrac{B}{\alpha}\right) F(D_1)dD_1 \quad \text{(F25)}$$

Since

$$\int_0^{Q_1-Q_2^c} F\left(Q_1 - D_1 - \tfrac{B}{\alpha}\right) f(D_1)dD_1 = F(Q_2^c - \tfrac{B}{\alpha})F(Q_1 - Q_2^c) + \int_0^{Q_1-Q_2^c} f\left(Q_1 - D_1 - \tfrac{B}{\alpha}\right) F(D_1)dD_1 \quad \text{(F26)}$$

This gives equation (23).

## Appendix G: Proof of Proposition 7

The Hessian of $g_1(0)$ is

$$\frac{\partial^2 g_1(I_0)}{\partial Q_1^2} = (s+h)f(Q_1) + (s+h+l)\left[ F(Q_2^c)f(Q_1 - Q_2^c) + \int_0^{Q_1-Q_2^c} f(Q_1 - D)f(D)dD \right] + \alpha p_2 \left[ F\left(Q_2^c - \frac{B}{\alpha}\right) f(Q_1 - Q_2^c) + \int_0^{Q_1-Q_2^c} f\left(Q_1 - D - \frac{B}{\alpha}\right) f(D)dD + f\left(Q_1 - \frac{B}{\alpha}\right)\right] + (c-s)f(Q_1 - Q_2^c) \quad \text{(G1)}$$

$$\frac{\partial^2 g_1(I_0)}{\partial Q_1 \partial B} = \frac{\partial^2 g_1(I_0)}{\partial B \partial Q_1} = -p_2 \left[ f\left(Q_1 - \frac{B}{\alpha}\right) + \int_0^{Q_1-Q_2^c} f\left(Q_1 - D - \frac{B}{\alpha}\right) f(D)dD \right] \quad \text{(G2)}$$

$$\frac{\partial^2 g_1(I_0)}{\partial B^2} = \frac{p_2}{\alpha}\left[ f\left(Q_1 - \frac{B}{\alpha}\right) + \int_0^{Q_1-Q_2^c} f\left(Q_1 - D - \frac{B}{\alpha}\right) f(D)dD + \int_{Q_1-Q_2^c}^{\infty} f\left(Q_2^c - \frac{B}{\alpha}\right) f(D)dD \right] \quad \text{(G3)}$$

Hessian is positive definite since

$$\frac{\partial^2 g_1(I_0)}{\partial Q_1^2} > 0 \text{ and } \frac{\partial^2 g_1(I_0)}{\partial Q_1^2}\frac{\partial^2 g_1(I_0)}{\partial B^2} - \left(\frac{\partial^2 g_1(I_0)}{\partial B \partial Q_1}\right)^2 > 0 .$$

Therefore, $g_1(I_0)$ is convex with respect to $Q_1$ and $B$ and thus $Q_1^*$ and $B^*$ guarantees the global optimum.

**Appendix H: Proof of Proposition 8**

The first order condition for optimality for period 1 is

$$\begin{aligned} \frac{\partial g_1(I_0)}{\partial Q_1} &= -s + (s+h)F(Q_1) + \alpha p_2 F\left(Q_1 - \frac{B}{\alpha}\right) - \alpha p_1 F(Q_1 - Q_2^c) \\ &+ (s+h+l)\int_0^{Q_1-Q_2^c} f(Q_1 - D)F(D)dD \qquad \text{(H1)} \\ &+ \alpha p_2 \left[ F\left(Q_2^c - \tfrac{B}{\alpha}\right)F(Q_1 - Q_2^c) + \int_0^{Q_1-Q_2^c} f(Q_1 - D - \tfrac{B}{\alpha})F(D)dD \right] = 0 \end{aligned}$$

$$\frac{\partial g_1(I_0)}{\partial B} = 2p_1 - p_2 F\left(Q_1 - \tfrac{B}{\alpha}\right) - p_2 F\left(Q_2^c - \tfrac{B}{\alpha}\right) - p_2 \int_0^{Q_1-Q_2^c} f\left(Q_1 - D_1 - \tfrac{B}{\alpha}\right)F(D_1)dD_1 = 0 \quad \text{(H2)}$$

Also, the first order condition for period 2 is

$$\frac{\partial g_2(I_1)}{\partial Q_2} = (c-s) + (s+h+l)F(Q_2^c) + \alpha p_2 F\left(Q_2^c - \frac{B}{\alpha}\right) = 0. \quad \text{(H3)}$$

Since $\frac{\partial^2 g_1(I_0)}{\partial Q_1^2} > 0$, if $\frac{\partial g_1(I_0)}{\partial Q_1} < 0$ at a certain point of $Q_1$, the optimal solution $Q_1^*$ is larger than the point.

$$\text{At } Q_1 = Q_2^c, \ \frac{\partial g_1(I_0)}{\partial Q_1} = -s + (s+h)F(Q_2^c) + \alpha p_2 F\left(Q_2^c - \frac{B}{\alpha}\right) = -c - lF(Q_2^c) < 0. \quad \text{(H4)}$$

Assuming that $s - c - \alpha p_1 \geq 0$ to exist a positive $Q_2^c$, $\frac{\partial g_1(I_0)}{\partial Q_1} \leq 0$ at $Q_1 = Q_2^c$.

Therefore, $Q_1^* \geq Q_2^c$.