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On Convergence for Sums of Rowwise Negatively Associated Random Variables

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Abstract

Let $\{X_{ni}|1 \le i \le n,\ n \ge 1\}$ be an array of rowwise negatively associated random variables. In this paper we discuss $n^{\alpha p-2}h(n)\max_{1\le k\le n}|\sum_{i=1}^{k} X_{ni}|/n^{\alpha} \to 0$ completely as $n \to \infty$ under not necessarily identically distributed with suitable conditions for $\alpha > 1/2$, $0 < p < 2$, $\alpha p \ge 1$ and a slowly varying function $h(x) > 0$ as $x \to \infty$. In addition, we obtain the complete convergence of moving average process based on negative association random variables which extends the result of Zhang (1996).

Keywords: Negatively associated random variables, slowly varying function, complete convergence, almost sure convergence.

1. Introduction

Hsu and Robbins (1947) introduced the concept of complete convergence of a sequence $\{X_n\}$ of random variables as follows. A sequence $\{X_n\}$ of random variables is said to converge completely to a constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \epsilon) < \infty, \quad \text{for every } \epsilon > 0.$$

Moreover, it was proved that the sequence of arithmetic means of independent identically distributed (*i.i.d.*) random variables converges completely to the expected value if the variance of the summands is finite by Hsu and Robbins (1947). This result has been generalized and extended in several directions and carefully studied by many authors (see, Pruitt, 1966; Rohatgi, 1971; Gut, 1992; Li *et al.*, 1992; Ghosal and Chandra, 1998; Hu *et al.*, 1986; Hu *et al.*, 1999; Kuczmaszewska and Szynal, 1994; Ahmed *et al.*, 2002; Wang *et al.*, 1993). Complete convergence for a sequence of random variables plays a central role in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel or Kolmogorov. Since then, serious attempts have been made to relax these strong conditions. For example, independence has been relaxed to pairwise independence or pairwise negative quadrant dependence or, even replaced by conditions of dependence such as mixing or martingale. In particular, many authors showed that many results could be obtained by replacing *i.i.d.* condition by uniformly bounded condition. We recall that an array $\{X_{ni}|1 \le i \le n,\ n \ge 1\}$ of random variables is said to be uniformly bounded by a random variable X if for all n and $x \ge 0$,

$$\sup_{i\ge1} P(|X_{ni}| > x) \le P(|X| > x).$$

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A finite sequence of random variables $\{X_i|1 \le i \le n\}$ is said to be negatively associated(NA) if for any two disjoint nonempty subsets A_1 and A_2 of $\{1,2,\ldots,n\}$ and f_1 and f_2 are any two coordinatewise nondecreasing functions,

$$\mathrm{Cov}\Big(f_1(X_i, i \in A_1),\ f_2(X_j, j \in A_2)\Big) \le 0,$$

whenever the covariance is finite. If for every $n \ge 2$, $X_1, X_2, \ldots, X_n$ are NA, then the sequence $\{X_i|i \in N\}$ is said to be NA. This definition is introduced by Alam and Saxena (1981). Many authors derived several important properties about NA sequences and also discussed some applications in the area of statistics, probability, reliability and multivariate analysis. Compared to positively associated random variables, the study of NA random variables has received less attention in the literature. Recently, some authors focussed on the problem of limiting behavior of partial sums of NA sequences. Su and Qin (1997) studied some limiting results for NA sequences, Liang (2000) and Baek *et al.* (2003) considered some complete convergence for negatively dependent random variables.

The main purpose of this paper, it is to discuss the complete convergence for sums of rowwise NA random variables under suitable conditions. As an application, we obtain the complete convergence of moving average processes based on NA random variables which extends the result of Zhang (1996) and we obtained some corollaries.

2. Preliminaries

This section will contain a background materials which will be used in obtaining the main results in the next sections and C will represent positive constants whose value may change from one place to another.

Lemma 1. (**Hu *et al.*, 1986**) *For any $r \ge 1$, $E|X|^r < \infty$ if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| > \epsilon n) < \infty, \quad \textit{for any } \epsilon > 0.$$

More precisely,

$$r2^{-r} \sum_{n=1}^{\infty} n^{r-1} P(|X| > n) \le E|X|^r \le 1 + r2^r \sum_{n=1}^{\infty} n^{r-1} P(|X| > n).$$

Lemma 2. (**Matular, 1992**) *Let $\{X_i|i \ge 1\}$ be a sequence of NA random variables with $EX_i = 0$ and $EX_i^2 < \infty$. Then there exists a positive C such that*

$$P(\max(|X_1|, \ldots, |X_1 + \cdots + X_n|) > \varepsilon) \le C \sum_{k=1}^{n} Var(X_k), \quad \textit{for all } \varepsilon > 0.$$

Lemma 3. (**Burton and Dehling, 1990**) *Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i$, $b = \sum_{i=-\infty}^{\infty} |a_i|$. Suppose $\Phi : [-b, b] \to R$ is a function satisfying the following conditions:*

(i) Φ is bounded and continuous at a.

(ii) There exist $\delta > 0$ and $C > 0$ such that for all $|x| \le \delta$, $|\Phi(x)| \le C|x|$.

Then $\lim_{n\to\infty} 1/n \sum_{i=-\infty}^{\infty} \Phi(\sum_{j=i+1}^{i+n} a_j) = \Phi(a)$.

Remark 1. Taking $\Phi(x) = |x|^q, q \geq 1$, from Lemma 3 we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^q = |a|^q.$$

3. Main Results

Theorem 1. *Let $\alpha > 1/2$, $0 < p < 2$ and $\alpha p \geq 1$. Let $\{X_{ni} | 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ for some $\alpha \leq 1$ and let $\sup_{i\geq 1} P(|X_{ni}| > x) \leq P(|X| > x)$ for all n and $x \geq 0$. Suppose that $h(x) > 0$ is a slowly varying function as $x \to \infty$ and let $h(x) \geq C > 0$ for $\alpha p = 1$. If $E|X|^p h(|X|^{1/\alpha}) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni} \right| \geq \epsilon n^\alpha \right) < \infty, \quad \textit{for every } \epsilon > 0.$$

Proof: Let $Y_i = n^\alpha I(X_{ni} > n^\alpha) + X_{ni} I(|X_{ni}| \leq n^\alpha) - n^\alpha I(X_{ni} < -n^\alpha)$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni} \right| \geq \epsilon n^\alpha \right) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni} \right| \geq \epsilon n^\alpha, |X_{ni}| \leq n^\alpha, 1 \leq i \leq n \right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni} \right| \geq \epsilon n^\alpha, \text{there exists } i \text{ such that } |X_{ni}| \geq n^\alpha \right) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq i\leq n} |X_{ni}| \geq n^\alpha \right) + \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} (Y_i - EY_i) \right| \geq \epsilon n^\alpha \right) \\
&= I_1 + I_2 \text{ (say).}
\end{aligned}$$

First, it is omitted, since we can easily prove that $I_1 < \infty$.
Next, in order to prove that $I_2 < \infty$, we first prove that

$$n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} EY_i \right| \to 0 \quad \text{as } n \to \infty.$$

So,

$$\begin{aligned}
n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} EY_i \right| &\leq n^{-\alpha} \max_{1\leq k\leq n} \left[\left| \sum_{i=1}^{k} EX_{ni} I(|X_{ni}| \leq n^\alpha) + \sum_{i=1}^{k} n^\alpha P(|X_{ni}| > n^\alpha) \right| \right] \\
&\leq n^{-\alpha} \sum_{i=1}^{n} |EX_{ni}| I(|X_{ni}| \leq n^\alpha) + nP(|X| > n^\alpha) \\
&= I_3 + I_4 \text{ (say).}
\end{aligned}$$

To prove that $I_3 \to 0$ as $n \to \infty$ and $I_4 \to 0$ as $n \to \infty$, we first need to prove that

$$\sum_{k=1}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha) < \infty. \tag{3.1}$$

Without loss of generality, since $h(x) \ge C > 0$ for $\alpha p = 1$, we obtain that

$$\begin{aligned}\sum_{k=1}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha) &\le C\sum_{k=1}^{\infty} kh(k)P(k^\alpha \le |X| < (k+1)^\alpha) \\ &\le C\sum_{k=1}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha)\, h\left(|X|^{\frac{1}{\alpha}}\right) \\ &\le CE|X|^p h\left(|X|^{\frac{1}{\alpha}}\right) < \infty, \quad \text{by Lemma 1.}\end{aligned} \tag{3.2}$$

When $\alpha p > 1$, since $h(x)$ is a slowly varying function, by choosing $N > 0$ such that $k \ge N$, we obtain that $k^{1-\alpha p}h^{-1}(k) < 1$. Thus, we have that

$$\begin{aligned}\sum_{k=1}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha) &\le \sum_{k=1}^{N-1} kP(k^\alpha \le |X| < (k+1)^\alpha) + \sum_{k=N}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha) \\ &\le C + \sum_{k=N}^{\infty} k^{\alpha p}h(k)P(k^\alpha \le |X| < (k+1)^\alpha) \\ &\le CE|X|^p \left(|X|^{\frac{1}{\alpha}}\right) \to 0 \quad \text{as } n \to \infty,\end{aligned} \tag{3.3}$$

which, together with (3.2) and (3.3), yields (3.1).

When $1/2 < \alpha \le 1$, since $EX_{ni} = 0$, we obtain that

$$\begin{aligned}I_3 &\le n^{1-\alpha}E|X|I(|X| > n^\alpha) \\ &\le n^{1-\alpha}\sum_{k=n}^{\infty} k^\alpha P(k^\alpha \le |X| < (k+1)^\alpha) \\ &\le C\sum_{k=n}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

When $\alpha > 1$, by (3.1) and Kronecker Lemma, we obtain that

$$\begin{aligned}I_3 &= n^{1-\alpha}E|X|I(|X| \le n^\alpha) \\ &\le Cn^{1-\alpha}\sum_{k=0}^{n} k^\alpha P(k^\alpha \le |X| < (k+1)^\alpha) \longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

Now, by (3.1), we get

$$\begin{aligned}I_4 &= nP(|X| > n^\alpha) \\ &\le Cn\sum_{k=n}^{\infty} P(k^\alpha \le |X| < (k+1)^\alpha) \\ &\le C\sum_{k=n}^{\infty} kP(k^\alpha \le |X| < (k+1)^\alpha) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, we have that

$$n^{-\alpha} \max_{1\le k\le n} \left| \sum_{i=1}^{k} EY_i \right| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Next, note that $\{(Y_i - EY_i)|i \ge 1\}$ is still a rowwise NA random variables by the definition of NA random variables. Thus, by using Lemma 2, we get

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} (Y_i - EY_i) \right| \ge \epsilon n^{\alpha} \right) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} h(n) \sum_{i=1}^{n} E|Y_i|^2 \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} h(n) \sum_{i=1}^{n} \left[E|X_{ni}|^2 I(|X_{ni}| \le n^{\alpha}) + n^{2\alpha} P(|X_{ni}| > n^{\alpha}) \right] \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} h(n) E|X|^2 I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1} h(n) P(|X| > n^{\alpha}) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} h(n) E|X|^2 I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1} h(n) P(|X| > n^{\alpha}) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} h(n) \sum_{k=1}^{n} n^{2\alpha} P(k^{\alpha} \le |X| < (k+1)^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1} h(n) \sum_{k=n}^{\infty} P(k^{\alpha} \le |X| < (k+1)^{\alpha}) \\
&\le C \sum_{k=1}^{\infty} k^{2\alpha} P(k^{\alpha} \le |X| < (k+1)^{\alpha}) \sum_{n=k}^{\infty} n^{\alpha p-1-2\alpha} h(n) + \sum_{k=1}^{\infty} P(k^{\alpha} \le |X| < (k+1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha p-1} h(n) \\
&\le C \sum_{k=1}^{\infty} k^{\alpha p} h\left(| X |^{\frac{1}{\alpha}} \right) P(k^{\alpha} \le |X| < (k+1)^{\alpha}) \\
&\le CE|X|^p h\left(|X|^{\frac{1}{\alpha}} \right) < \infty.
\end{aligned}$$

□

Taking $X_{ni} = X_i$ for $1 \le i \le n$ and $h(x) = \log^{-2} n$ in Theorem 1, we can immediately obtain the following corollary.

Corollary 1. *Let $\alpha > 1/2$, $0 < p < 2$, $\alpha p \ge 1$ and let $\{X_i|i \ge 1\}$ be an identically distributed NA random variables with $EX_1 = 0$ for some $\alpha \le 1$. If $E|X_1|^p < \infty$, then we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \log^{-2} n P\left(\sum_{i=1}^{n} |X_i| \ge \varepsilon n^{\alpha} \right) < \infty, \quad \textit{for all } \varepsilon > 0.$$

Remark 2. The condition of identical distribution can be weakened slightly to be uniformly bounded in probability. When $\{X_i|i \ge 1\}$ is a sequence of *i.i.d.* random variables, if we take $p = 1/\alpha$ for some $0 < p < 1$ and $\log^{-2} n = 1$, then Corollary 1 becomes the result of Bai and Su (1985).

Corollary 2. *Under the conditions of Theorem 1, we have that*

$$\frac{1}{n^{\alpha}}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \ \ a.s. \ \ as \ \ n \longrightarrow \infty. \tag{3.4}$$

Proof:

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| \ge \epsilon n^{\alpha}\right)\\
&= \sum_{l=0}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} n^{\alpha p-2}h(n)P\left(\frac{1}{n^{\alpha}}\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| \ge \epsilon\right)\\
&\ge \sum_{l=1}^{\infty} P\left(\frac{1}{l^{\alpha}}\max_{1\le k\le 2^l}\left|\sum_{i=1}^{k} X_{ni}\right| \ge \epsilon\right).
\end{aligned} \tag{3.5}$$

By Borel-Cantelli Lemma and (3.5), we have

$$P\left(\frac{1}{l^{\alpha}}\max_{1\le k\le 2^l}\left|\sum_{i=1}^{k} X_{ni}\right| \ge \epsilon, \ \text{i.o.}\right) = 0,$$

and hence

$$\frac{1}{l^{\alpha}}\max_{1\le k\le 2^l}\left|\sum_{i=1}^{k} X_{ni}\right| \longrightarrow 0 \quad \text{a.s. as } l \to \infty. \tag{3.6}$$

From (3.6) and the fact that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{1}{n^{\alpha}}\sum_{i=1}^{n} X_{ni}\right| &\le \lim_{l\to\infty}\frac{1}{l^{\alpha}}\max_{2^{l-1}\le n\le 2^l}\left|\sum_{i=1}^{n} X_{ni}\right|\\
&\le \lim_{l\to\infty}\frac{1}{l^{\alpha}}\max_{1\le k\le 2^l}\left|\sum_{i=1}^{k} X_{ni}\right|
\end{aligned}$$

the desired result (3.4) follows and the proof is completed. □

4. Application

In this section, we present one result about the complete convergence of linear processes which follows from Theorem 1. We give a general version of Zhang (1996) from the identically distributed and ϕ-mixing case to the *NA* random variables.

Let $\{X_i, i \in Z\}$, where $Z_+ = \{1, 2, 3, \cdots\}$ denote a sequence of random variables and $\{a_i | i \in Z_+\}$ a sequence of real numbers with $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Define a linear process of the form

$$Y_k = \sum_{i=-\infty}^{\infty} a_{i+k}X_i, \quad k \in Z_+, \qquad \text{where } Z_+ = \{1, 2, 3, \ldots\}. \tag{4.1}$$

Theorem 2. *Assume that* $\{X_i| -\infty < i < \infty\}$ *is a sequence of NA random variables with* $EX_i = 0$. *Let* $h(x) > 0$ *be a slowly varying function as* $x \to \infty$ *and* $r \geq 1, 1 \leq t < 2$, $h(x)$ *is increasing function for* $r = 1$ *and* $\{Y_i|i \geq 1\}$ *be defined as in (4.1) of this section. If for all n and* $x \geq 0$

$$\sup_{i\geq 1} P(|X_{ni}| > x) \leq P(|X| > x) \quad and \quad E|X|^{rt}h(|X|^t) < \infty,$$

then we have

$$\sum_{n=1}^{\infty} n^{r-2}h(n)P\left(\left|\sum_{i=1}^{n} Y_i\right| \geq \varepsilon n^{\frac{1}{t}}\right) < \infty, \quad for\ all\ \varepsilon > 0.$$

Proof: Let $X_{ni} = a_{ni}X_i$ and $a_{ni} = 1/n^{1/t}\sum_{k=1}^{n} a_{i+k}$ and note that

$$\left(\frac{1}{n}\right)^{\frac{1}{t}} \sum_{k=1}^{n} Y_k = \sum_{i=-\infty}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{t}} \sum_{k=1}^{n} a_{i+k}X_i = \sum_{i=-\infty}^{\infty} a_{ni}X_i.$$

By taking $p = rt, t = 1/\alpha$ and $\alpha p = r$ in Theorem 1, by Lemma 3 and Remark 1, similarly to proof of Theorem 1, we can obtain the result of Theorem 2, the proof is completed. □

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