Communications of the Korean Statistical Society
2009, Vol. 16, No. 3, 529–540

Weak Laws of Large Numbers for Weighted Sums of Fuzzy Random Variables

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Abstract

In this paper, we present some results on weak laws of large numbers for weighted sums of fuzzy random variables taking values in the space of fuzzy numbers of the real line R. We first give improvements of WLLN for weighted sums of convex-compactly uniformly integrable fuzzy random variables obtained by Joo and Hyun (2005). And then,we consider the case that the averages of expectations of fuzzy random variables converges. As results, WLLN for weighted sums of convexly tight or identically distributed case is obtained.

Keywords: Fuzzy random variables, fuzzy numbers, weak laws of large numbers, tightness.

1. Introduction

Strong laws of large numbers for sums of independent fuzzy random variables have been studied by several researchers. Klement *et al.* (1986) introduced L^1-metric d_1 and uniform metric d_∞ on the space of fuzzy sets (*cf.* Section 2) and proved SLLN for *i.i.d.* fuzzy random variables in the sense of d_1. Inoue (1991) extended their result (Klement *et al.*, 1986) to the case of independent and level-wise tight fuzzy random variables. Colubi *et al.* (1999) obtained SLLN for *i.i.d.* fuzzy random variables with respect to d_∞ by approximation method. Molchanov (1999) gave a short proof of SLLN for *i.i.d.* fuzzy random variables in the sense of d_∞. This was also proved by Joo and Kim (2001), independently. Recently, Joo (2002) obtained a SLLN for independent and convexly tight fuzzy random variables by using the Skorokhod metric d_s which was introduced by Joo and Kim (2000). Beside that, there are many other things such as Uemura (1993), Colubi *et al.* (2001), Feng (2002), Proske and Puri (2002), which studied SLLN for Banach space-valued fuzzy random variables. On the other hand, it is known the results of Taylor *et al.* (2001), Joo (2004) for weak laws of large numbers for fuzzy random variables.

It is one of significant problems how we can generalize laws of large numbers for sums of fuzzy random variables to the case of weighted sums. Related to this problem, Guan and Li (2004) obtained some results under restrictive condition and Joo and Hyun (2005) gave weak law of large numbers for weighted sums of convex-compactly uniformly integrable fuzzy random variables. Also, Joo *et al.* (2006) established strong convergence for weighted sums of fuzzy random sets.

The purpose of this paper is to obtain improvements of the above weak laws of large numbers for weighted sums of integrable fuzzy random variables. Section 2 is devoted to describe some preliminary results which will be used in later section. Finally, in Section 3, we establish some results on weak laws of large numbers for weighted sums of fuzzy random variables.

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2. Preliminaries

In this section, we describe some preliminary results for fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0, 1]$ with the following properties;

(1) $\tilde{u}$ is normal, *i.e.*, there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semicontinuous.

(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is convex, *i.e.* $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0, 1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \operatorname{supp} \tilde{u}, & \text{if } \alpha = 0, \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0, 1]$. The following theorem shows that a fuzzy number $\tilde{u}$ is completely determined by the closed intervals $L_\alpha \tilde{u} = [u^l_\alpha, u^r_\alpha]$.

Theorem 1. *For $\tilde{u} \in F(R)$, we denote $L_\alpha \tilde{u} = [u^l_\alpha, u^r_\alpha]$ and consider u^l, u^r as functions of α. Then the followings hold;*

(1) u^l is a bounded increasing function on $[0, 1]$.

(2) u^r is a bounded decreasing function on $[0, 1]$.

(3) $u^l_1 \leq u^r_1$.

(4) u^l and u^r are left continuous on $[0, 1]$ *and right continuous at 0.*

Furthermore, if v^l and v^r satisfy above (1)~(4), then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^l_\alpha, v^r_\alpha]$.

Proof: See Goetschel and Voxman (1986). □

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the family of closed intervals $\{[u^l_\alpha, u^r_\alpha] : 0 \leq \alpha \leq 1\}$, where u^l and u^r satisfy (1)~(4) of Theorem 1. We denote the right-limit of u^l (resp. u^r) at α by $u^l_{\alpha^+}$ (resp. $u^r_{\alpha^+}$).

The linear structure on $F(R)$ is defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}\left(\dfrac{z}{\lambda}\right), & \text{if } \lambda \neq 0, \\ \tilde{0}(z), & \text{if } \lambda = 0, \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$.

Then it is well-known that if

$$\tilde{u} = \left\{\left[u^l_\alpha, u^r_\alpha\right] : 0 \leq \alpha \leq 1\right\}$$

and

$$\tilde{v} = \left\{ \left[v_\alpha^l, v_\alpha^r \right] : 0 \le \alpha \le 1 \right\},$$

then

$$\tilde{u} \oplus \tilde{v} = \left\{ \left[u_\alpha^l + v_\alpha^l, u_\alpha^r + v_\alpha^r \right] : 0 \le \alpha \le 1 \right\}$$

and

$$\lambda \tilde{u} = \begin{cases} \left\{ \left[\lambda u_\alpha^l, \lambda u_\alpha^r \right] : 0 \le \alpha \le 1 \right\}, & \lambda \ge 0, \\ \left\{ \left[\lambda u_\alpha^r, \lambda u_\alpha^l \right] : 0 \le \alpha \le 1 \right\}, & \lambda < 0. \end{cases}$$

We can define L^1-metric d_1 and uniform metric d_∞ on $F(R)$ as follows;

$$\begin{aligned} d_1(\tilde{u}, \tilde{v}) &= \int_0^1 \max\left(\left| u_\alpha^l - v_\alpha^l \right|, \left| u_\alpha^r - v_\alpha^r \right| \right) d\alpha \\ d_\infty(\tilde{u}, \tilde{v}) &= \sup_{0 \le \alpha \le 1} \max\left(\left| u_\alpha^l - v_\alpha^l \right|, \left| u_\alpha^r - v_\alpha^r \right| \right) \\ &= \max\left(\sup_{0 \le \alpha \le 1} \left| u_\alpha^l - v_\alpha^l \right|, \sup_{0 \le \alpha \le 1} \left| u_\alpha^r - v_\alpha^r \right| \right). \end{aligned}$$

The norm of $\tilde{u} \in F(R)$ is defined by

$$\|\tilde{u}\| = d_\infty\left(\tilde{u}, \tilde{0} \right) = \max\left(\left| u_0^l \right|, \left| u_0^r \right| \right).$$

It is well known that $(F(R), d_1)$ is separable but is not complete, and that $(F(R), d_\infty)$ is complete but is not separable (For details, see Klement *et al.* (1986)). Joo and Kim (2000) introduced the Skorokod metric d_s on $F(R)$ which makes it a separable metric space as follows:

Definition 1. *Let T denote the class of strictly increasing, continuous mapping of* $[0, 1]$ *onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define*

$$d_s(\tilde{u}, \tilde{v}) = \inf \left\{ \epsilon > 0 : \text{there exists a } t \in T \text{ such that } \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon \right\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

It follows immediately that d_s is a metric on $F(R)$ and $d_s(\tilde{u}, \tilde{v}) \le d_\infty(\tilde{u}, \tilde{v})$. The metric d_s will be called the Skorokhod metric. It is well-known that $(F(R), d_s)$ is separable and topologically complete. Also, d_∞-convergence implies d_s-convergence and d_s-convergence implies d_1-convergence. But the converses are not true (For details, see Joo and Kim (2000)).

3. Main Results

Throughout this paper, let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A fuzzy number valued function

$$\tilde{X} : \Omega \to F(R), \quad \tilde{X} = \left\{ \left[X_\alpha^l, X_\alpha^r \right] : 0 \le \alpha \le 1 \right\}$$

is called a fuzzy random variable if for each $\alpha \in [0,1]$, X^l_α and X^r_α are random variable in the usual sense. It is well-known that $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable (See Kim, 2002). So we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated.

A fuzzy random variable $\tilde{X}$ is called integrable if $E\|\tilde{X}\| < \infty$. The expectation of integrable fuzzy random variable $\tilde{X}$ is a fuzzy number defined by

$$E\left(\tilde{X}\right) = \left\{\left[EX^l_\alpha, EX^r_\alpha\right] : 0 \le \alpha \le 1\right\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variable and $\{\lambda_{ni}\}$ be a double sequence of real numbers that not necessarily Toeplitz but satisfying

$$\sum_{i=1}^{\infty} |\lambda_{ni}| \le C, \qquad \text{for each } n,$$

where $C > 0$ is constant not depending on n.

In this section, we establish sufficient conditions for

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\right) \to 0 \text{ in probability as } n \to \infty.$$

This problem was solved by Joo and Hyun (2005) for the case of convex-compactly uniformly integrable fuzzy random variables. First, we give a generalization of Joo and Hyun (2005) by assuming the following condition.

For each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,

$$\max\left(E \max_{1\le k\le m} \left|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}\right|,\ E \max_{1\le k\le m} \left|X^r_{n,\alpha^+_{k-1}} - X^r_{n,\alpha_k}\right|\right) < \epsilon, \tag{3.1}$$

where $\tilde{X}_n = \{[X^l_{n,\alpha}, X^r_{n,\alpha}] : 0 \le \alpha \le 1\}$ and $X^l_{n,\alpha^+_{k-1}}$ (resp. $X^r_{n,\alpha^+_{k-1}}$) denotes the right limit of X^l_n (resp. X^r_n) at α.

Theorem 2. *Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variables satisfying* (3.1). *If for each $\alpha \in [0,1]$ and $j = l, r$,*

$$\sum_{i=1}^{n} \lambda_{ni}\left(X^j_{i,\alpha} - EX^j_{i,\alpha}\right) \to 0 \ \textit{in probability as } n \to \infty,$$

then

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^n \lambda_{ni} E\left(\tilde{X}_i\right)\right) \to 0 \ \textit{in probability as } n \to \infty.$$

Proof: We first show that if for each $\alpha \in [0,1]$,

$$\sum_{i=1}^{n} \lambda_{ni}\left(X^l_{i,\alpha} - EX^l_{i,\alpha}\right) \to 0 \text{ in probability as } n \to \infty,$$

then

$$\sup_{0\le\alpha\le1}\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^l_{i,\alpha} - EX^l_{i,\alpha}\right)\right| \to 0 \text{ in probability as } n \to \infty.$$

Let $\epsilon > 0$ and $0 < \delta < 1$ be given. By (3.1), there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0, 1]$ such that for all n,

$$E \max_{1\le k\le m} \left| X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k} \right| < \frac{\epsilon\delta}{3C}. \tag{3.2}$$

Then it is trivial that for all n,

$$\max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(EX^l_{i,\alpha^+_{k-1}} - EX^l_{i,\alpha_k} \right) \right| \le \sum_{i=1}^n |\lambda_{ni}| \, E \max_{1\le k\le m} \left| X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k} \right| < \frac{\epsilon}{3}. \tag{3.3}$$

We note that

$$\begin{aligned}
&\sup_{0\le\alpha\le 1} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha} - EX^l_{i,\alpha} \right) \right| \\
&= \max_{1\le k\le m} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha} - EX^l_{i,\alpha} \right) \right| \\
&\le \max_{1\le k\le m} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha} - X^l_{i,\alpha_k} \right) \right| + \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k} \right) \right| \\
&\quad + \max_{1\le k\le m} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} \left| \sum_{i=1}^n \lambda_{ni} \left(EX^l_{i,\alpha_k} - EX^l_{i,\alpha} \right) \right| \\
&\le \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k} \right) \right| + \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k} \right) \right| + \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(EX^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}} \right) \right| \\
&\le \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k} \right) \right| + \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k} \right) \right| + \frac{\epsilon}{3} \quad \text{by (3.3).}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&P\left(\sup_{0\le\alpha\le 1} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha} - EX^l_{i,\alpha} \right) \right| > \epsilon \right) \\
&\le P\left(\max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k} \right) \right| > \frac{\epsilon}{3} \right) + P\left(\max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k} \right) \right| > \frac{\epsilon}{3} \right) \\
&= (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

Now for (I), by (3.2),

$$\begin{aligned}
(\mathrm{I}) &\le \frac{3}{\epsilon} E \max_{1\le k\le m} \left| \sum_{i=1}^n \lambda_{ni} \left(X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k} \right) \right| \\
&\le \frac{3}{\epsilon} \sum_{i=1}^n |\lambda_{ni}| E \max_{1\le k\le m} \left| X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k} \right| \\
&\le \frac{3}{\epsilon} \frac{\epsilon\delta}{3} = \delta.
\end{aligned}$$

For (II), by assumption,

$$\begin{aligned}\text{(II)} &\le \sum_{k=1}^{m} P\left(\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^{l}_{i,\alpha_k} - EX^{l}_{i,\alpha_k}\right)\right| > \frac{\epsilon}{3}\right)\\ &\le \delta, \quad \text{for sufficiently large } n.\end{aligned}$$

This implies that

$$P\left(\sup_{0\le\alpha\le1}\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^{l}_{i,\alpha} - EX^{l}_{i,\alpha}\right)\right| > \epsilon\right) \le 2\delta, \quad \text{for sufficiently large } n,$$

which completes the proof.

It can be proved by similar arguments that if for each $\alpha \in [0, 1]$,

$$\sum_{i=1}^{n} \lambda_{ni}\left(X^{r}_{i,\alpha} - EX^{r}_{i,\alpha}\right) \to 0 \text{ in probability as } n \to \infty,$$

then

$$\sup_{0\le\alpha\le1}\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^{r}_{i,\alpha} - EX^{r}_{i,\alpha}\right)\right| \to 0 \text{ in probability as } n \to \infty .$$

Therefore, the desired result follows from the fact that

$$\begin{aligned}&P\left\{d_\infty\left(\oplus_{i=1}^{n}\lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^{n}\lambda_{ni}E\tilde{X}_i\right) > \epsilon\right\}\\ &\le P\left\{\sup_{0\le\alpha\le1}\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^{l}_{i,\alpha} - EX^{l}_{i,\alpha}\right)\right| > \epsilon\right\} + P\left\{\sup_{0\le\alpha\le1}\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^{r}_{i,\alpha} - EX^{r}_{i,\alpha}\right)\right| > \epsilon\right\}.\end{aligned}$$

□

Corollary 1. *Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variables satisfying* (3.1). *If for each $j = l, r$,*

$$Cov\left(X^{j}_{k,\alpha}, X^{j}_{m,\alpha}\right) = 0, \quad \text{for each } k \ne m \text{ and each } \alpha \in [0, 1], \tag{3.4}$$

and

$$\sum_{i=1}^{n} Var\left(X^{j}_{i,\alpha}\right) = o\left(\left(\max_{1\le i\le n}\lambda^2_{ni}\right)^{-1}\right), \quad \text{for each } \alpha \in [0, 1], \tag{3.5}$$

then

$$d_\infty\left(\oplus_{i=1}^{n}\lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^{n}\lambda_{ni}E\tilde{X}_i\right) \to 0 \text{ in probability as } n \to \infty.$$

Proof: For each $j = l, r$ and $\alpha \in [0, 1]$, $\{X^{j}_{n,\alpha}\}$ is a sequence of uncorrelated random variables by (3.4). Thus, for each $\epsilon > 0$,

$$\begin{aligned}P\left\{\left|\sum_{i=1}^{n} \lambda_{ni}\left(X^{j}_{i,\alpha} - EX^{j}_{i,\alpha}\right)\right| > \epsilon\right\} &\le \frac{1}{\epsilon^2}\text{Var}\left(\sum_{i=1}^{n}\lambda_{ni}X^{j}_{i,\alpha}\right) = \frac{1}{\epsilon^2}\sum_{i=1}^{n}\lambda^2_{ni}\text{Var}\left(X^{j}_{i,\alpha}\right)\\ &\le \frac{1}{\epsilon^2}\left(\max_{1\le i\le n}\lambda^2_{ni}\right)\sum_{i=1}^{n}\text{Var}\left(X^{j}_{i,\alpha}\right) \to 0 \quad \text{by (3.5).}\end{aligned}$$

□

We recall that the space $F(R)$ of fuzzy numbers can be embedded into a proper Banach space. Thus we can define the concept of convexity in $F(R)$. That is, $K \subset F(R)$ is said to be convex if $\lambda\tilde{u} \oplus (1-\lambda)\tilde{v} \in K$ whenever $\tilde{u}, \tilde{v} \in K$ and $\lambda \in [0,1]$.

With adopting this concept of convexity in $F(R)$, we can introduce the notions of convex tightness and convex-compactly uniform integrability of fuzzy random variables.

Definition 2. *Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables.*

(1) $\{\tilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$, there exists a compact subset K of $(F(R), d_s)$ such that

$$P\left(\tilde{X}_n \notin K\right) < \epsilon, \quad \text{for all } n.$$

If we can choose K to be convex and compact, then $\{\tilde{X}_n\}$ is said to be convexly tight.

(2) $\{\tilde{X}_n\}$ is said to be compactly uniformly integrable if for each $\epsilon > 0$ there exists a compact subset K of $(F(R), d_s)$ such that

$$\int_{\{\tilde{X}_n \notin K\}} \left\|\tilde{X}_n\right\| \, dP < \epsilon, \quad \text{for all } n.$$

If we can choose K to be convex and compact, then $\{\tilde{X}_n\}$ is said to be convex-compactly uniformly integrable.

Corollary 2. *Let $\{\tilde{X}_n\}$ be a sequence of convex-compactly uniformly integrable fuzzy random variables. If for each $\alpha \in [0,1]$ and $j = l, r$,*

$$\sum_{i=1}^{n} \lambda_{ni}\left(X^j_{i,\alpha} - EX^j_{i,\alpha}\right) \to 0 \text{ in probability as } n \to \infty,$$

then

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i\right) \to 0 \text{ in probability as } n \to \infty.$$

Proof: It suffices to prove that (3.1) is satisfied. Let $\epsilon > 0$ be given. By convex-compact uniform integrability of $\{\tilde{X}_n\}$, we can choose a compact convex subset K of $(F(R),\ d_s)$ such that

$$E\left(I_{\{\tilde{X}_n \notin K\}} \left\|\tilde{X}_n\right\|\right) < \frac{\epsilon}{4}.$$

Now by Theorem 3.11 of Kim (2004), there exists a finite partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$, such that

$$\sup_{\tilde{u}\in K} \max\left(\max_{1\le k\le m}\left|u^l_{\alpha^+_{k-1}} - u^l_{\alpha_k}\right|,\ \max_{1\le k\le m}\left|u^r_{\alpha^+_{k-1}} - u^r_{\alpha_k}\right|\right) < \frac{\epsilon}{2}.$$

Thus, we have for $j = l, r$ and for all n,

$$\begin{aligned} E\left(\max_{1\le k\le m}\left|X^j_{n,\alpha^+_{k-1}} - X^j_{n,\alpha_k}\right|\right) &= E\left(I_{\{\tilde{X}_n\in K\}}\max_{1\le k\le m}\left|X^j_{n,\alpha^+_{k-1}} - X^j_{n,\alpha_k}\right|\right) + E\left(I_{\{\tilde{X}_n\notin K\}}\max_{1\le k\le m}\left|X^j_{n,\alpha^+_{k-1}} - X^j_{n,\alpha_k}\right|\right) \\ &< \frac{\epsilon}{2} + 2EI_{\{\tilde{X}_n\notin K\}}\left\|\tilde{X}_n\right\| < \epsilon, \end{aligned}$$

which completes the proof. □

Corollary 3. *Let $\{\tilde{X}_n\}$ be a sequence of convexly tight fuzzy random sets such that*

$$\sup_n \|\tilde{X}_n\|^p = M < \infty, \quad \text{for some } p > 1.$$

If for each $\alpha \in [0,1]$ and $j = l, r$,

$$\sum_{i=1}^{n} \lambda_{ni}\left(X^j_{i,\alpha} - EX^j_{i,\alpha}\right) \to 0 \quad \text{in probability as } n \to \infty,$$

then

$$d_\infty\left(\oplus_{i=1}^{n}\lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^{n}\lambda_{ni}E\tilde{X}_i\right) \to 0 \quad \text{in probability as } n \to \infty.$$

Remark 1. We note that the results of this section have no restrictions on weights λ_{ni} except that

$$\sum_{i=1}^{\infty} |\lambda_{ni}| \leq C, \quad \text{for each } n.$$

So the above results include weak laws of large numbers for sum of fuzzy random variables.

Unfortunately, the next example shows that (3.1) need not be satisfied for a single fuzzy random variable,

Example 1. For $\lambda \in [0,1]$, let $\tilde{u}_\lambda$ be a fuzzy number defined by

$$\tilde{u}_\lambda(x) = \begin{cases} \lambda, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then it follows that for each $\alpha \in [0,1]$, $u^r_{\lambda,\alpha} = 1$ and

$$u^l_{\lambda,\alpha} = \begin{cases} 0, & \text{if } \alpha \leq \lambda, \\ 1, & \text{if } \alpha > \lambda. \end{cases}$$

Now let $\tilde{X}$ be a fuzzy random variable with

$$P\left(\tilde{X} \in \{\tilde{u}_\lambda : \lambda_1 \leq \lambda \leq \lambda_2\}\right) = \lambda_2 - \lambda_1.$$

Then it is obvious that $E\|\tilde{X}\| = 1$ and that since $P(X^l_\alpha = 0) = 1 - \alpha, P(X^l_\alpha = 1) = \alpha$,

$$E\left(\tilde{X}\right) = \{[\alpha, 1] : 0 \leq \alpha \leq 1\}.$$

But for any partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$, we have that for all except a finite number of points,

$$\max_{1\leq k\leq m} \left|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}\right| = 1.$$

Therefore, $E\max_{1\le k\le m}\left|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}\right| = 1$ and so (3.1) can not be satisfied for $\tilde{X}$.

This example shows that Theorem 2 cannot be valid for a sequence of identically distributed fuzzy random variables. But if we replace (3.1) by assumption that $\{\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\}$ is convergent, we can also obtain WLLN. The next theorem is a generalization of SLLN of Guan and Li (2004) which is obtained for weighted sums of independent fuzzy random variables by assuming that $\{1/n \oplus_{i=1}^n \mathrm{co}(E\tilde{X}_i)\}$ is convergent with respect to d_∞.

Theorem 3. *Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variables such that for some $\tilde{u} \in F(R)$,*

$$\lim_{n\to 0} d_s\left(\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i,\ \tilde{u}\right) = 0.$$

If for each $\alpha \in [0,1]$ and $j = l, r$,

$$\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha} - EX^j_{i,\alpha}\right) \to 0 \quad \text{in probability as } n \to \infty$$

and

$$\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha^+} - EX^j_{i,\alpha^+}\right) \to 0 \quad \text{in probability as } n \to \infty,$$

then

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^n \lambda_{ni} E\left(\tilde{X}_i\right)\right) \to 0 \quad \text{in probability as } n \to \infty.$$

Proof: Let $\epsilon > 0$. By assumption, there exists a $t \in T$ such that for large n,

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i,\ t(\tilde{u})\right) < \frac{\epsilon}{18}.$$

By applying Lemma 3.3 of Joo and Kim (2001) to $\tilde{v} = t(\tilde{u})$, we can find a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ be a partition of $[0,1]$ such that

$$\max\left(\max_{1\le k\le m}\left|v^l_{\alpha^+_{k-1}} - v^l_{\alpha_k}\right|,\ \max_{1\le k\le m}\left|v^r_{\alpha^+_{k-1}} - v^r_{\alpha_k}\right|\right) < \frac{\epsilon}{18}.$$

Then we have that for $j = l, r$, and for large n,

$$\begin{aligned}\left|\sum_{i=1}^n \lambda_{ni}\left(EX^j_{i,\alpha_k} - EX^j_{i,\alpha^+_{k-1}}\right)\right| &\le \left|\sum_{i=1}^n \lambda_{ni} EX^j_{i,\alpha_k} - v^j_{\alpha_k}\right| + \left|\sum_{i=1}^n \lambda_{ni} EX^j_{i,\alpha^+_{k-1}} - v^j_{\alpha^+_{k-1}}\right| + \left|v^j_{\alpha^+_{k-1}} - v^j_{\alpha_k}\right| \\ &< \frac{\epsilon}{6}.\end{aligned}$$

We note that for $\alpha_{k-1} < \alpha \le \alpha_k$,

$$\begin{aligned}&\left|\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha} - EX^j_{i,\alpha}\right)\right| \\ &\le \left|\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha_k} - EX^j_{i,\alpha^+_{k-1}}\right)\right| + \left|\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha^+_{k-1}} - EX^j_{i,\alpha_k}\right)\right| \\ &\le \left|\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha_k} - EX^j_{i,\alpha_k}\right)\right| + \left|\sum_{i=1}^n \lambda_{ni}\left(X^j_{i,\alpha^+_{k-1}} - EX^j_{i,\alpha^+_{k-1}}\right)\right| + 2\left|\sum_{i=1}^n \lambda_{ni}\left(EX^j_{i,\alpha_k} - EX^j_{i,\alpha^+_{k-1}}\right)\right|,\end{aligned}$$

and so,

$$\sup_{\alpha_{k-1}<\alpha\leq\alpha_k}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha}^{j}-EX_{i,\alpha}^{j}\right)\right| \leq \left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_k}^{j}-EX_{i,\alpha_k}^{j}\right)\right| + \left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_{k-1}^{+}}^{j}-EX_{i,\alpha_{k-1}^{+}}^{j}\right)\right| + \frac{\epsilon}{3}.$$

Then

$$\begin{aligned}\sup_{0\leq\alpha\leq 1}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha}^{j}-EX_{i,\alpha}^{j}\right)\right| &= \max_{1\leq k\leq m}\sup_{\alpha_{k-1}<\alpha\leq\alpha_k}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha}^{j}-EX_{i,\alpha}^{j}\right)\right| \\ &\leq \max_{1\leq k\leq m}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_k}^{j}-EX_{i,\alpha_k}^{j}\right)\right| + \max_{1\leq k\leq m}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_{k-1}^{+}}^{j}-EX_{i,\alpha_{k-1}^{+}}^{j}\right)\right| + \frac{\epsilon}{3}.\end{aligned}$$

Therefore, by assumption we have

$$\begin{aligned}&P\left(\sup_{0\leq\alpha\leq 1}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha}^{j}-EX_{i,\alpha}^{j}\right)\right|>\epsilon\right) \\ &\leq P\left(\max_{1\leq k\leq m}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_k}^{j}-EX_{i,\alpha_k}^{j}\right)\right|>\frac{\epsilon}{3}\right)+P\left(\max_{1\leq k\leq m}\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_{k-1}^{+}}^{j}-EX_{i,\alpha_{k-1}^{+}}^{j}\right)\right|>\frac{\epsilon}{3}\right) \\ &\leq \sum_{k=1}^{m}P\left(\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_k}^{j}-EX_{i,\alpha_k}^{j}\right)\right|>\frac{\epsilon}{3}\right)+\sum_{k=1}^{m}P\left(\left|\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha_{k-1}^{+}}^{j}-EX_{i,\alpha_{k-1}^{+}}^{j}\right)\right|>\frac{\epsilon}{3}\right) \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

This completes the proof. □

Corollary 4. *Let $\{\tilde{X}_n\}$ be a sequence of identically distributed fuzzy random variables with $E\|\tilde{X}_1\| < \infty$, and $\{\lambda_{ni}\}$ be a double sequence of real numbers satisfying*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\lambda_{ni}=\lambda, \quad \textit{for some } \lambda\in R.$$

If for each $\alpha\in[0,1]$ and $j=l,r$,

$$\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha}^{j}-EX_{1,\alpha}^{j}\right)\to 0 \quad \textit{in probability as } n\to\infty$$

and

$$\sum_{i=1}^{n}\lambda_{ni}\left(X_{i,\alpha^{+}}^{j}-EX_{1,\alpha^{+}}^{j}\right)\to 0 \quad \textit{in probability as } n\to\infty,$$

then

$$d_\infty\left(\oplus_{i=1}^{n}\lambda_{ni}\tilde{X}_i,\ \lambda E\left(\tilde{X}_1\right)\right)\to 0 \quad \textit{in probability as } n\to\infty.$$

Proof: We note that

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i,\ \lambda E\tilde{X}_1\right) = \left|\sum_{i=1}^n \lambda_{ni} - \lambda\right| \|E\tilde{X}_1\| \to 0$$

as $n \to \infty$, which implies

$$\lim_{n\to 0} d_s\left(\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i,\ \lambda E\tilde{X}_1\right) = 0.$$

Since

$$d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i,\ \lambda E\left(\tilde{X}_1\right)\right) \le d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i,\ \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\right) + d_\infty\left(\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i,\ \lambda E\left(\tilde{X}_1\right)\right)$$

the desired result follows immediately from Theorem 3. □

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Received February 2009; Accepted April 2009