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LOWER AND UPPER BOUNDS FOR INVERSE ELEMENTS OF STRICTLY DIAGONALLY DOMINANT SEVENTH-DIAGONAL MATRICES

ZHUOHONG HUANG* AND TING-ZHU HUANG

ABSTRACT. In this paper,we give the lower and upper bounds for inverse elements of strictly diagonally dominant seventh-diagonal matrices, and improve the bounds on [SIAM. J. matrix Anal.Appl.20(1999)820-837].

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1. Introduction

Seventh-diagonal matrices arise in many topics of numerical analysis including boundary value problems approached by finite difference methods, interpolation by cubic splines, three-term difference equations and so on. For many problems, it is helpful to have upper and lower bounds for the entries (or the absolute values of the entries) of the inverse of a matrix. Shivakumar and Ji [1] gave the upper and lower bounds for diagonally dominant tridiagonal matrices. Nabben [3] established decay rates for the entries of inverses of certain banded matrices. Nabben [2], Peluso and Politi [4] improved the upper and lower bounds of Shivakumar and Ji have established. Later, Liu, Huang and Fu [5] obtained new upper and lower bounds on the inverse elements of strictly diagonally dominant tridiagonal matrices, they improved the related results in [1-4].

Let $A = (a_{ij})_{n\times n} \in C^{n\times n}$. For any positive integer number p, if $a_{ij} = 0$ and $|i-j| > p$, then we call A be a $2p+1$ banded matrix. In this paper, we only consider the case of $p = 3$. Let A be a seven-diagonal matrix with order n, $n \geq 7$ and $a_{ii} \neq 0$, for $i = 1, 2, \dots, n$. Let $X = A^{-1} = (x_{ij})_{n\times n}$ be the inverse of A. Let $x_j = (x_{1,j}, x_{2,j}, \cdots, x_{n,j})^T$ $(j = 1, 2, \dots, n)$ be the jth column of X. It is $Ax_j = e_j$, where e_j is the jth fundamental vector of R^n. Writing the first $j-1$

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equations, with $j \geq 2$, we have

$$\sum_{k=i-3}^{i+3} a_{i,k}x_{k,j} = 0,\ i = 1,2,\dots,n-1, j = i+1,\dots,n, \tag{1}$$

where, $a_{i,k} = 0,\ x_{k,j} = 0, k \leq 0$ or $k > n$. From (1) we have

$$\begin{aligned}
x_{1,j} =& -\frac{a_{1,2}}{a_{1,1}}x_{2,j} - \frac{a_{1,3}}{a_{1,1}}x_{3,j} - \frac{a_{1,4}}{a_{1,1}}x_{4,j} \\
\triangleq& -\alpha_1 x_{2,j} - \beta_1 x_{3,j} - \gamma_1 x_{4,j}, \\
x_{2,j} =& -\frac{a_{2,3} - a_{2,1}\beta_1}{a_{2,2} - a_{2,1}\alpha_1}x_{3,j} - \frac{a_{2,4} - a_{2,1}\gamma_1}{a_{2,2} - a_{2,1}\alpha_1}x_{4,j} - \frac{a_{2,5}}{a_{2,2} - a_{2,1}\alpha_1}x_{5,j} \\
\triangleq& -\alpha_2 x_{3,j} - \beta_2 x_{4,j} - \gamma_2 x_{5,j}, \\
x_{3,j} =& -\frac{a_{3,4} - (a_{3,2} - a_{3,1}\alpha_1)\beta_2 - a_{3,1}\gamma_1}{a_{3,3} - (a_{3,2} - a_{3,1}\alpha_1)\alpha_2 - a_{3,1}\beta_1}x_{4,j} \\
& - \frac{a_{3,5} - (a_{3,2} - a_{3,1}\alpha_1)\gamma_2}{a_{3,3} - (a_{3,2} - a_{3,1}\alpha_1)\alpha_2 - a_{3,1}\beta_1}x_{5,j} \\
& - \frac{a_{3,6}}{a_{3,3} - (a_{3,2} - a_{3,1}\alpha_1)\alpha_2 - a_{3,1}\beta_1}x_{6,j} \\
\triangleq& -\alpha_3 x_{4,j} - \beta_3 x_{5,j} - \gamma_3 x_{6,j}, \\
& \cdots \\
x_{i,j} =& -\frac{\zeta_i}{P_i}x_{i+1,j} - \frac{\eta_i}{P_i}x_{i+2,j} - \frac{a_{i,i+3}}{P_i}x_{i+3,j} \\
\triangleq& -\alpha_i x_{i+1,j} - \beta_i x_{i+2,j} - \gamma_i x_{i+3,j},
\end{aligned} \tag{2}$$

where,

$$\begin{aligned}
P_i =& a_{i,i} - [a_{i,i-1} - (a_{i,i-2} - a_{i,i-3}\alpha_{i-3})\alpha_{i-2} - a_{i,i-3}\beta_{i-3}]\alpha_{i-1} \\
& - (a_{i,i-2} - a_{i,i-3}\alpha_{i-3})\beta_{i-2} - a_{i,i-3}\gamma_{i-3}, \\
\zeta_i =& a_{i,i+1} - [a_{i,i-1} - a_{i,i-3}\beta_{i-3} - (a_{i,i-2} - a_{i,i-3}\alpha_{i-3})\alpha_{i-2}]\beta_{i-1} \\
& - (a_{i,i-2} - a_{i,i-3}\alpha_{i-1})\gamma_{i-2}, \\
\eta_i =& a_{i,i+2} - [a_{i,i-1} - a_{i,i-3}\beta_{i-3} - (a_{i,i-2} - a_{i,i-3}\alpha_{i-3})\alpha_{i-2}]\gamma_{i-1}.
\end{aligned}$$

Now we can repeat the same procedure (2) from the system $Ax_j = e_j$, with $j \leq n-1$:

$$\sum_{k=i-3}^{i+3} a_{i,k}x_{k,j} = 0,\ i = n,\dots,2,\ j = i-1,\dots,1, \tag{3}$$

where, $a_{i,k} = 0,\ x_{k,j} = 0, k \leq 0,$ or $k > n$. From (3) we have

$$x_{n,j} = -\frac{a_{n,n-3}}{a_{n,n}}x_{n-3,j} - \frac{a_{n,n-2}}{a_{n,n}}x_{n-2,j} - \frac{a_{n,n-1}}{a_{n,n}}x_{n-1,j}$$

$$
\begin{aligned}
&\stackrel{\triangle}{=} -\mu_n x_{n-3,j} - \nu_n x_{n-2,j} - \omega_n x_{n-1,j},\\
x_{n-1,j} &= -\frac{a_{n-1,n-4}}{a_{n-1,n-1} - a_{n-1,n}\omega_n} x_{n-4,j} - \frac{a_{n-1,n-3} - a_{n-1,n}\mu_n}{a_{n-1,n-1} - a_{n-1,n}\omega_n} x_{n-3,j}\\
&\quad - \frac{a_{n-1,n-2} - a_{n-1,n}\nu_n}{a_{n-1,n-1} - a_{n-1,n}\omega_n} x_{n-2,j}\\
&\stackrel{\triangle}{=} -\mu_{n-1} x_{n-4,j} - \nu_{n-1} x_{n-3,j} - \omega_{n-1} x_{n-2,j},\\
x_{n-2,j} &= -\frac{a_{n-2,n-5}}{\psi_{n-2}} x_{n-5,j} - \frac{a_{n-2,n-4} - \tau_{n-2}\mu_{n-1}}{\psi_{n-2}} x_{n-4,j}\\
&\quad - \frac{a_{n-2,n-3} - a_{n-2,n}\mu_n - \tau_{n-2}\nu_{n-1}}{\psi_{n-2}} x_{n-3,j}\\
&\stackrel{\triangle}{=} -\mu_{n-2} x_{n-5,j} - \nu_{n-2} x_{n-4,j} - \omega_{n-2} x_{n-3,j},\\
x_{i,j} &= -\frac{a_{i,i-3}}{\psi_i} x_{i-3,j} - \frac{a_{i,i-2} - \tau_i\mu_{i+1}}{\psi_i} x_{i-2,j}\\
&\quad - \frac{a_{i,i-1} - \xi_i\mu_{i+2} - \tau_i\nu_{i+1}}{\psi_i} x_{i-1,j}\\
&\stackrel{\triangle}{=} -\mu_i x_{i-3,j} - \nu_i x_{i-2,j} - \omega_i x_{i-1,j},
\end{aligned}
\tag{4}
$$

where,

$$
\begin{aligned}
\xi_i &= a_{i,i+2} - a_{i,i+3}\omega_{i+3},\ \tau_i = a_{i,i+1} - a_{i,i+3}\nu_{i+3} - \xi_i\omega_{i+2},\\
\psi_i &= a_{i,i} - \tau_i\omega_{i+1} - a_{i,i+3}\mu_{i+3} - \xi_i\nu_{i+2}, i = n, n-1\ldots, 2.
\end{aligned}
$$

According to (2) and (4), we obtain, for $j = 3, \ldots, n-2$,

$$
\begin{aligned}
x_{j-2,j} &= -\alpha_{j-2}x_{j-1,j} - \beta_{j-2}x_{j,j} - \gamma_{j-2}x_{j+1,j},\\
x_{j-1,j} &= -\alpha_{j-1}x_{j,j} - \beta_{j-1}x_{j+1,j} - \gamma_{j-1}x_{j+2,j},\\
x_{j+1,j} &= -\mu_{j+1}x_{j-2,j} - \nu_{j+1}x_{j-1,j} - \omega_{j+1}x_{j,j},\\
x_{j+2,j} &= -\mu_{j+2}x_{j-1,j} - \nu_{j+2}x_{j,j} - \omega_{j+2}x_{j+1,j}.
\end{aligned}
\tag{5}
$$

Further,

$$
\begin{aligned}
x_{j-2,j} &= \Psi_{j-2}x_{j,j},\ x_{j-1,j} = \Psi_{j-1}x_{j,j},\\
x_{j+1,j} &= \Phi_{j+1}x_{j,j},\ x_{j+2,j} = \Phi_{j+2}x_{j,j},
\end{aligned}
\tag{6}
$$

where,

$$
\begin{aligned}
\kappa_{j+1} &= \frac{\mu_{j+1}\beta_{j-2} - \omega_{j+1}}{1 - \mu_{j+1}\gamma_{j-2}},\ \iota_{j+1} = \frac{\mu_{j+1}\alpha_{j-2} - \nu_{j+1}}{1 - \mu_{j+1}\gamma_{j-2}},\\
\rho_{j-1} &= \frac{\gamma_{j-1}\nu_{j+2} - \alpha_{j-1}}{1 - \mu_{j+2}\gamma_{j-1}},\ \varsigma_{j-1} = \frac{\gamma_{j-1}\omega_{j+2} - \beta_{j-1}}{1 - \mu_{j+2}\gamma_{j-1}},\\
\Psi_{j-1} &= \frac{\rho_{j-1} + \varsigma_{j-1}\kappa_{j+1}}{1 - \varsigma_{j-1}\iota_{j+1}},\ \Phi_{j+1} = \frac{\kappa_{j+1} + \iota_{j+1}\rho_{j-1}}{1 - \iota_{j+1}\varsigma_{j-1}},\\
\Phi_{j+2} &= -(\mu_{j+2}\Psi_{j-1} + \nu_{j+2} + \omega_{j+2}\Phi_{j+1}),
\end{aligned}
$$

$$\Psi_{j-2} = -(\alpha_{j-2}\Psi_{j-1} + \beta_{j-2} + \gamma_{j-2}\Phi_{j+1}).$$

2. Lower and upper bounds

Let A be a strictly diagonally dominant seven-diagonal matrix. Then the elements of A satisfy the following conditions:

$$|a_{ii}| > \sum_{j=i-3}^{i+3} |a_{ij}|, i = 1, 2, \ldots, n \ (|a_{ij}| = 0, \text{ if } j \leq 0 \text{ or } j > n).$$

First of all, we define

$$\begin{aligned}
\tilde{P}_i &= |a_{i,i}| - \{[|a_{i,i-1}| + (|a_{i,i-2}| + |a_{i,i-3}||\tilde{\alpha}_{i-3}|)|\tilde{\alpha}_{i-2}| + |a_{i,i-3}||\tilde{\beta}_{i-3}|]\tilde{\alpha}_{i-1} \\
\tilde{\zeta}_i &= |a_{i,i+1}| + [|a_{i,i-1}| + |a_{i,i-3}\tilde{\beta}_{i-3}| + (|a_{i,i-2}| + |a_{i,i-3}\tilde{\alpha}_{i-3}|)|\tilde{\alpha}_{i-2}|]\tilde{\beta}_{i-1} \\
&\quad + (|a_{i,i-2}| + |a_{i,i-3}\tilde{\alpha}_{i-1}|)|\tilde{\gamma}_{i-2}|, \\
\tilde{\eta}_i &= |a_{i,i+2}| + [|a_{i,i-1}| + |a_{i,i-3}\tilde{\beta}_{i-3}| + (|a_{i,i-2}| + |a_{i,i-3}\tilde{\alpha}_{i-3}|)|\tilde{\alpha}_{i-2}|]\tilde{\gamma}_{i-1}, \\
\tilde{\alpha}_i &= \frac{\tilde{\zeta}_i}{\tilde{P}_i}, \ \tilde{\beta}_i = \frac{\tilde{\eta}_i}{\tilde{P}_i}, \ \tilde{\gamma}_i = \frac{|a_{i,i+3}|}{\tilde{P}_i}, \ i = 1, 2, \ldots, n-1, \\
\tilde{\tau}_i &= |a_{i,i+1}| + |a_{i,i+3}\tilde{\nu}_{i+3}| + |\tilde{\xi}_i\tilde{\omega}_{i+2}|, \ \tilde{\xi}_i = |a_{i,i+2}| + |a_{i,i+3}\tilde{\omega}_{i+3}|, \\
&\quad + (|a_{i,i-2}| + |a_{i,i-3}|\tilde{\alpha}_{i-3})\tilde{\beta}_{i-2} + |a_{i,i-3}|\tilde{\gamma}_{i-3}\}, \\
\tilde{\psi}_i &= |a_{i,i}| + |\tilde{\tau}_i\tilde{\omega}_{i+1}| + |a_{i,i+3}\tilde{\mu}_{i+3}| + |\tilde{\xi}_i\tilde{\nu}_{i+2}|, \\
\tilde{\mu}_i &= \frac{|a_{i,i-3}|}{\tilde{\psi}_i}, \tilde{\nu}_i = \frac{|a_{i,i-2}| + |\tilde{\tau}_i\tilde{\mu}_{i+1}|}{\tilde{\psi}_i}, \\
\tilde{\omega}_i &= \frac{|a_{i,i-1}| + |\tilde{\xi}_i\tilde{\mu}_{i+2}| + |\tilde{\tau}_i\tilde{\nu}_{i+1}|}{\tilde{\psi}_i}, \ i = n, n-1, \ldots, 2, \\
\tilde{\kappa}_{j+1} &= \frac{\tilde{\omega}_{j+1}\tilde{\beta}_{j-2} + \tilde{\mu}_{j+1}}{1 - \tilde{\omega}_{j+1}\tilde{\gamma}_{j-2}}, \ \tilde{\iota}_{j+1} = \frac{\tilde{\omega}_{j+1}\tilde{\alpha}_{j-2} + \tilde{\nu}_{j+1}}{1 - \tilde{\omega}_{j+1}\tilde{\gamma}_{j-2}}, \\
\tilde{\rho}_{j-1} &= \frac{\tilde{\gamma}_{j-1}\tilde{\nu}_{j+2} + \tilde{\alpha}_{j-1}}{1 - \tilde{\omega}_{j+2}\tilde{\gamma}_{j-1}}, \ \tilde{\varsigma}_{j-1} = \frac{\tilde{\gamma}_{j-1}\mu_{j+2} + \tilde{\beta}_{j-1}}{1 - \tilde{\omega}_{j+2}\tilde{\gamma}_{j-1}}, \\
\tilde{\Phi}_{j+1} &= \frac{\tilde{\kappa}_{j+1} + \tilde{\iota}_{j+1}\tilde{\rho}_{j-1}}{1 - \tilde{\iota}_{j+1}\tilde{\varsigma}_{j-1}}, \ \tilde{\Phi}_{j+2} = \tilde{\mu}_{j+2}\tilde{\Phi}_{j+1} + \tilde{\nu}_{j+2} + \tilde{\omega}_{j+2}, \\
\tilde{\Psi}_{j-2} &= \tilde{\alpha}_{j-2}\tilde{\Psi}_{j+2} + \tilde{\beta}_{j-2} + \tilde{\gamma}_{j-2}, \ \tilde{\Psi}_{j-1} = \tilde{\rho}_{j-1} + \tilde{\varsigma}_{j-1}\tilde{\Phi}_{j+1}, \\
\tilde{\Phi}_i &= \tilde{\mu}_i\tilde{\Phi}_{i-3} + \tilde{\nu}_i\tilde{\Phi}_{i-2} + \tilde{\omega}_i\tilde{\Phi}_{i-1}, \ \tilde{\Psi}_i = \tilde{\alpha}_i\tilde{\Psi}_{i+1} + \tilde{\beta}_i\tilde{\Psi}_{i+2} + \tilde{\gamma}_i\tilde{\Psi}_{i+3}.
\end{aligned}$$

Theorem 2.1. *Let A be a strictly diagonally dominant seventh-diagonal matrix, and $A^{-1} = X = (x_{ij})_{n\times n}$. Then for the elements of matrix X,*

$$|x_{i,j}| \geq (||\alpha_i\Psi_{i+1}| - |\beta_i\Psi_{i+2} + \gamma_i\Psi_{i+3}||)|x_{j,j}|,$$

and

$$|x_{i,j}| \leq (\tilde{\alpha}_i \tilde{\Psi}_{i+1} + \tilde{\beta}_i \tilde{\Psi}_{i+2} + \tilde{\gamma}_i \tilde{\Psi}_{i+3})|x_{j,j}|. \tag{7}$$

Proof. According to (6), we obtain $x_{j-2,j} = \Psi_{j-2}x_{j,j}, x_{j-1,j} = \Psi_{j-1}x_{j,j}$. Further, by (2), we have

$$\begin{aligned} x_{j-3,j} &= -\alpha_{j-3}x_{j-2,j} - \beta_{j-3}x_{j-1,j} - \gamma_{j-3}x_{j,j} \\ &= -(\alpha_{j-3}\Psi_{j-2} + \beta_{j-3}\Psi_{j-1} + \gamma_{j-3})x_{j,j} = \Psi_{j-3}x_{j,j}, \\ x_{j-4,j} &= -\alpha_{j-4}x_{j-3,j} - \beta_{j-4}x_{j-2,j} - \gamma_{j-4}x_{j-1,j} \\ &= -(\alpha_{j-4}\Psi_{j-3} + \beta_{j-4}\Psi_{j-2} + \gamma_{j-4}\Psi_{j-1})x_{j,j} = \Psi_{j-4}x_{j,j}, \\ x_{i,j} &= -\alpha_i x_{i+1,j} - \beta_i x_{i+2,j} - \gamma_i x_{i+3,j} \\ &= -(\alpha_i\Psi_{i+1} + \beta_i\Psi_{i+2} + \gamma_i\Psi_{i+3})x_{j,j} = \Psi_i x_{j,j}. \end{aligned} \tag{8}$$

Thus, we obtain the inequality (7). □

Theorem 2.2. *Let A be a strictly diagonally dominant seventh-diagonal matrix, and $A^{-1} = X = (x_{ij})_{n\times n}$. Then for the elements of matrix X,*

$$|x_{i,j}| \geq (||\mu_i\Phi_{i-3}| - |\nu_i\Phi_{i-2} + \omega_i\Phi_{i-1}||)|x_{j,j}|,$$

and

$$|x_{i,j}| \leq (\tilde{\mu}_i\tilde{\Phi}_{i-3} + \tilde{\nu}_i\tilde{\Phi}_{i-2} + \tilde{\omega}_i\tilde{\Phi}_{i-1})|x_{j,j}|. \tag{9}$$

Proof. According to (6), we obtain $x_{j+1,j} = \Phi_{j+1}x_{j,j}, x_{j+2,j} = \Phi_{j+2}x_{j,j}$. Further, by (4), we have

$$\begin{aligned} x_{j+3,j} &= -\mu_{j+3}x_{j,j} - \nu_{j+3}x_{j+1,j} - \omega_{j+3}x_{j+2,j} \\ &= -(\mu_{j+3} + \nu_{j+3}\Phi_{j+1} + \omega_{j+3}\Phi_{j+2})x_{j,j} = \Phi_{j+3}x_{j,j}, \\ x_{j+4,j} &= -\mu_{j+4}x_{j+1,j} - \nu_{j+4}x_{j+2,j} - \omega_{j+4}x_{j+3,j} \\ &= -(\mu_{j+4}\Phi_{j+1} + \nu_{j+4}\Phi_{j+2} + \omega_{j+4}\Phi_{j+3})x_{j,j} = \Phi_{j+4}x_{j,j}, \\ x_{i,j} &= -\mu_i x_{i-3,j} - \nu_i x_{i-2,j} - \omega_i x_{i-1,j} \\ &= -(\mu_i\Phi_{i-3} + \nu_i\Phi_{i-2} + \omega_i\Phi_{i-1})x_{j,j} = \Phi_i x_{j,j}. \end{aligned} \tag{10}$$

Thus, we obtain the inequality (9). □

Theorem 2.3. *Let A be a strictly diagonally dominant seventh-diagonal matrix, and $A^{-1} = X = (x_{ij})_{n\times n}$. Then for the elements of matrix X,*

$$\frac{1}{a_{j,j} + \tilde{h}_j} \leq |x_{j,j}| \leq \frac{1}{a_{j,j} - \tilde{h}_j}, \tag{11}$$

where,

$$\tilde{h}_j = \sum_{k=j-3}^{k=j-1} |a_{j,k}\tilde{\Psi}_k| + \sum_{k=j+1}^{k=j+3} |a_{j,k}\tilde{\Phi}_k|.$$

Proof. According to $AX = I$, we obtain the diagonal elements of X satisfy the following relations

$$\sum_{k=j-3}^{k=j+3} a_{j,k}x_{k,j} = 1. \tag{12}$$

Further, by (6),(8) and (10), we have

$$\begin{aligned} |1-a_{j,j}x_{j,j}| &= |\sum_{k=j-3,k\neq j}^{k=j+3} a_{j,k}x_{k,j}| \\ &\leq \sum_{k=j-3}^{k=j-1} |a_{j,k}\Psi_k x_{j,j}| + \sum_{k=j+1}^{k=j+3} |a_{j,k}\Phi_k x_{j,j}| \\ &\leq \sum_{k=j-3}^{k=j-1} |a_{j,k}\tilde{\Psi}_k x_{j,j}| + \sum_{k=j+1}^{k=j+3} |a_{j,k}\tilde{\Phi}_k x_{j,j}|. \end{aligned}$$

Thus, we obtain the inequality (11). □

Theorem 2.4. *Let A be a nonsingular diagonally dominant seventh-diagonal matrix. Then the upper bounds established in Theorem 3.1, 3.2 and 3.3 give the exact inverse of the comparison matrix $\mu(A) = (m_{ij})$ of A, which is given by*

$$m_{ii} = |a_{ii}|,\ m_{ij} = -|a_{ij}|, \text{for } i \neq j.$$

Proof. Since $\mu(A)$ be an nonsingular diagonally dominant M-matrix, by (3), we have

$$\begin{aligned} &-1 \leq \alpha_1 = \frac{-|a_{1,2}|}{|a_{1,1}|} = -\tilde{\alpha}_1 \leq 0, -1 \leq \beta_1 = \frac{-|a_{1,3}|}{|a_{1,1}|} = -\tilde{\beta}_i \leq 0, \\ &-1 \leq \gamma_1 = \frac{-|a_{1,4}|}{|a_{1,1}|} = -\tilde{\gamma}_1 \leq 0,\ -1 \leq \alpha_1+\beta_1+\gamma_1 \leq 0. \\ &-1 \leq \alpha_2 = \frac{-|a_{2,3}|-(-|a_{2,1}|)\beta_1}{|a_{2,2}|-(-|a_{2,1}|)\alpha_1} = -\frac{|a_{2,3}|+|a_{2,1}|\tilde{\beta}_1}{|a_{2,2}|-|a_{2,1}|\tilde{\alpha}_1} = -\tilde{\alpha}_2 \leq 0, \\ &-1 \leq \beta_2 = \frac{-|a_{2,4}|-(-|a_{2,1}|)\gamma_1}{|a_{2,2}|-(-|a_{2,1}|)\alpha_1} = -\frac{|a_{2,4}|+|a_{2,1}|\tilde{\gamma}_1}{|a_{2,2}|-|a_{2,1}|\tilde{\alpha}_1} = -\tilde{\beta}_2 \leq 0, \\ &-1 \leq \gamma_2 = \frac{-|a_{2,5}|}{|a_{2,2}|-(-|a_{2,1}|)\alpha_1} = -\frac{|a_{2,5}|}{|a_{2,2}|-|a_{2,1}|\tilde{\alpha}_1} = -\tilde{\gamma}_2 \leq 0, \\ &-1 \leq \alpha_2+\beta_2+\gamma_2 \leq 0. \end{aligned}$$

Since

$$(|\alpha_2|+|\beta_2|)|\alpha_1| \leq |\alpha_1|,\ (|\alpha_2|+|\beta_2|+|\gamma_2|)|\alpha_1| \leq |\alpha_1|,$$

then

$$\begin{aligned} -1 \leq \alpha_3 &= \frac{-|a_{3,4}|-[-|a_{3,2}|-(-|a_{3,1}|)\alpha_1]\beta_2-(-|a_{3,1}|)\gamma_1}{|a_{3,3}|-[-|a_{3,2}|-(-|a_{3,1}|)\alpha_1]\alpha_2-(-|a_{3,1}|)\beta_1} \\ &= -\frac{|a_{3,4}|+[|a_{3,2}|+|a_{3,1}|\tilde{\alpha}_1]\tilde{\beta}_2+|a_{3,1}|\tilde{\gamma}_1}{|a_{3,3}|-[(|a_{3,2}|+|a_{3,1}|\tilde{\alpha}_1)\tilde{\alpha}_2+|a_{3,1}|]\tilde{\beta}_1} = -\tilde{\alpha}_3 \leq 0, \end{aligned}$$

$$
\begin{aligned}
-1 \leq \beta_3 &= \frac{-|a_{3,5}| - [-|a_{3,2}| - (-|a_{3,1}|)\alpha_1]\gamma_2}{|a_{3,3}| - [-|a_{3,2}| - (-|a_{3,1}|)\alpha_1]\alpha_2 - (-|a_{3,1}|)\beta_1} \\
&= -\frac{|a_{3,5}| + (|a_{3,2}| + |a_{3,1}|\tilde{\alpha}_1)\tilde{\gamma}_2}{|a_{3,3}| - [(|a_{3,2}| + |a_{3,1}|\tilde{\alpha}_1)\tilde{\alpha}_2 + |a_{3,1}|\tilde{\beta}_1]} = -\tilde{\beta}_3 \leq 0, \\
-1 \leq \gamma_3 &= \frac{-|a_{3,6}|}{|a_{3,3}| - [-|a_{3,2}| - (-|a_{3,1}|)\alpha_1]\alpha_2 - (-|a_{3,1}|)\beta_1} \\
&= -\frac{|a_{3,6}|}{|a_{3,3}| - (|a_{3,2}| + |a_{3,1}|\tilde{\alpha}_1)\tilde{\alpha}_2 + |a_{3,1}|\tilde{\beta}_1} = -\tilde{\gamma}_3 \leq 0, \\
-1 &\leq \alpha_3 + \beta_3 + \gamma_3 \leq 0.
\end{aligned}
$$

and

$$
\begin{aligned}
P_i =& |a_{i,i}| - [-|a_{i,i-2}| - (-|a_{i,i-3}|)\alpha_{i-3}]\beta_{i-2} - (-|a_{i,i-3}|)\gamma_{i-3} \\
& - \{-|a_{i,i-1}| - [-|a_{i,i-2}| - (-|a_{i,i-3}|)\alpha_{i-3}]\alpha_{i-2} - (-|a_{i,i-3}|)\beta_{i-3}\}\alpha_{i-1} \\
=& |a_{i,i}| - \{[|a_{i,i-1}| + (|a_{i,i-2}| + |a_{i,i-3}|\tilde{\alpha}_{i-3})\tilde{\alpha}_{i-2} + |a_{i,i-3}|\tilde{\beta}_{i-3}]\tilde{\alpha}_{i-1} \\
& + (|a_{i,i-2}| + |a_{i,i-3}|\tilde{\alpha}_{i-3})\tilde{\beta}_{i-2} + |a_{i,i-3}|\tilde{\gamma}_{i-3}\} \\
=& \tilde{P}_i, \\
\zeta_i =& - |a_{i,i+1}| - (-|a_{i,i-2}| - (-|a_{i,i-3}|)\alpha_{i-1})\gamma_{i-2} \\
& - \{-|a_{i,i-1}| - (-|a_{i,i-3}|)\beta_{i-3} - [-|a_{i,i-2}| - (-|a_{i,i-3}|)\alpha_{i-3}]\alpha_{i-2}\}\beta_{i-1} \\
=& - [|a_{i,i+1}| + [|a_{i,i-1}| + |a_{i,i-3}|\tilde{\beta}_{i-3} + (|a_{i,i-2}| + |a_{i,i-3}|\tilde{\alpha}_{i-3})\tilde{\alpha}_{i-2}]\tilde{\beta}_{i-1} \\
& + (|a_{i,i-2}| + |a_{i,i-3}|\tilde{\alpha}_{i-1})\tilde{\gamma}_{i-2}] \\
=& - \tilde{\zeta}_i, \\
\eta_i =& - |a_{i,i+2}| - \{-|a_{i,i-1}| - (-|a_{i,i-3}|)\beta_{i-3} \\
& - [-|a_{i,i-2}| - (-|a_{i,i-3}|)\alpha_{i-3}]\alpha_{i-2}\}\gamma_{i-1} \\
=& - \{|a_{i,i+2}| + [|a_{i,i-1}| + |a_{i,i-3}|\tilde{\beta}_{i-3} + (|a_{i,i-2}| + |a_{i,i-3}|\tilde{\alpha}_{i-3})\tilde{\alpha}_{i-2}]\tilde{\gamma}_{i-1}\} \\
=& - \tilde{\eta}_i.
\end{aligned}
$$

According to induction inference, we obtain

$$
-1 \leq \alpha_i = \frac{\zeta_i}{P_i} = -\frac{\tilde{\zeta}_i}{\tilde{P}_i} = -\tilde{\alpha}_i \leq 0, \ -1 \leq \beta_i = \frac{\eta_i}{P_i} = -\frac{\tilde{\eta}_i}{\tilde{P}_i} = -\tilde{\beta}_i \leq 0,
$$

$$
-1 \leq \gamma_i = \frac{-|a_{i,i+3}|}{P_i} = -\frac{|a_{i,i+3}|}{\tilde{P}_i} = -\tilde{\gamma}_i \leq 0, \ -1 \leq \alpha_i + \beta_i + \gamma_i \leq 0.
$$

By the similar way to the above proof, we have

$$
\begin{aligned}
&-1 \leq \mu_i = -\tilde{\mu}_i \leq 0, \ i = 3, \ldots, n-1, \\
&-1 \leq \nu_i = -\tilde{\nu}_i \leq 0, \ i = 3, \ldots, n-2, \\
&-1 \leq \omega_i = -\tilde{\omega}_i \leq 0, \ i = 3, \ldots, n-2, \\
&-1 \leq \mu_i + \nu_i + \omega_i \leq 0, \ i = 3, \ldots, n-2.
\end{aligned}
$$

Further,

$$\kappa_{j+1} = \frac{\mu_{j+1}\beta_{j-2} - \omega_{j+1}}{1 - \mu_{j+1}\gamma_{j-2}} = \frac{\tilde{\mu}_{j+1}\tilde{\beta}_{j-2} + \tilde{\omega}_{j+1}}{1 - \tilde{\mu}_{j+1}\tilde{\gamma}_{j-2}} = \tilde{\kappa}_{j+1},$$
$$\iota_{j+1} = \frac{\mu_{j+1}\alpha_{j-2} - \nu_{j+1}}{1 - \mu_{j+1}\gamma_{j-2}} = \frac{\tilde{\mu}_{j+1}\tilde{\alpha}_{j-2} + \tilde{\nu}_{j+1}}{1 - \tilde{\mu}_{j+1}\tilde{\gamma}_{j-2}} = \tilde{\iota}_{j+1},$$

$$\rho_{j-1} = \frac{\gamma_{j-1}\nu_{j+2} - \alpha_{j-1}}{1 - \mu_{j+2}\gamma_{j-1}} = \frac{\tilde{\gamma}_{j-1}\tilde{\nu}_{j+2} + \tilde{\alpha}_{j-1}}{1 - \tilde{\mu}_{j+2}\tilde{\gamma}_{j-1}} = \tilde{\rho}_{j-1},$$
$$\varsigma_{j-1} = \frac{\gamma_{j-1}\omega_{j+2} - \beta_{j-1}}{1 - \mu_{j+2}\gamma_{j-1}} = \frac{\tilde{\gamma}_{j-1}\tilde{\omega}_{j+2} + \tilde{\beta}_{j-1}}{1 - \tilde{\mu}_{j+2}\tilde{\gamma}_{j-1}} = \tilde{\varsigma}_{j-1},$$
$$\Phi_{j+1} = \frac{\kappa_{j+1} + \iota_{j+1}\rho_{j-1}}{1 - \iota_{j+1}\varsigma_{j-1}} = \frac{\tilde{\kappa}_{j+1} + \tilde{\iota}_{j+1}\tilde{\rho}_{j-1}}{1 - \tilde{\iota}_{j+1}\tilde{\varsigma}_{j-1}} = \tilde{\Phi}_{j+1},$$
$$\begin{aligned}\Phi_{j+2} &= -(\mu_{j+2}\Psi_{j-1} + \nu_{j+2} + \omega_{j+2}\Phi_{j+1}) \\ &= \tilde{\mu}_{j+2}\tilde{\Psi}_{j-1} + \tilde{\nu}_{j+2} + \tilde{\omega}_{j+2}\tilde{\Phi}_{j+1} = \tilde{\Phi}_{j+2},\end{aligned}$$
$$\Psi_{j-1} = \frac{\rho_{j-1} + \varsigma_{j-1}\kappa_{j+1}}{1 - \varsigma_{j-1}\iota_{j+1}} = \frac{\tilde{\rho}_{j-1} + \tilde{\varsigma}_{j-1}\tilde{\kappa}_{j+1}}{1 - \tilde{\varsigma}_{j-1}\tilde{\iota}_{j+1}},$$
$$\begin{aligned}\Psi_{j-2} &= -(\alpha_{j-2}\Psi_{j-1} + \beta_{j-2} + \gamma_{j-2}\Phi_{j+1}) \\ &= \tilde{\alpha}_{j-2}\tilde{\Psi}_{j-1} + \tilde{\beta}_{j-2} + \tilde{\gamma}_{j-2}\tilde{\Phi}_{j+1} = \tilde{\Psi}_{j-2},\end{aligned}$$
$$\Psi_i = \alpha_i\Psi_{i+1} + \beta_i\Psi_{i+2} + \gamma_i\Psi_{i+3} = \tilde{\alpha}_i\tilde{\Psi}_{i+1} + \tilde{\beta}_i\tilde{\Psi}_{i+2} + \tilde{\gamma}_i\tilde{\Psi}_{i+3} = \tilde{\Psi}_i,$$
$$\Phi_i = \mu_i\Phi_{i-3} + \nu_i\Phi_{i-2} + \omega_i\Phi_{i-1} = \tilde{\mu}_i\tilde{\Phi}_{i-3} + \tilde{\nu}_i\tilde{\Phi}_{i-2} + \tilde{\omega}_i\tilde{\Phi}_{i-1} = \tilde{\Phi}_i,$$
$$\begin{aligned}h_j &= \sum_{k=j-3}^{k=j-1} |a_{j,k}|\Psi_k + \sum_{k=j+1}^{k=j+3} |a_{j,k}|\Phi_k \\ &= \sum_{k=j-3}^{k=j-1} |a_{j,k}|\tilde{\Psi}_k + \sum_{k=j+1}^{k=j+3} |a_{j,k}|\tilde{\Phi}_k = \tilde{h}_j.\end{aligned}$$

Thus,

$$x_{i,j} = \Phi_i x_{jj} = \tilde{\Phi}_i x_{jj},\ i = 1,\dots,n-1,\ j = i+1,\dots,n,$$
$$x_{i,j} = \Psi_i x_{jj} = \tilde{\Psi}_i x_{jj},\ i = 2,\dots,n,\ \ j = 1,\dots,i-1,$$
$$x_{j,j} = \frac{1}{|a_j| - h_j} = \frac{1}{|a_j| - \tilde{h}_j},\ j = 1,2,\dots,n.$$

Then we obtain the exact inverse of the comparison matrix $\mu(A)$ of A. □

Theorem 2.5([3,Theorem 3.12]). *Let A be a $2p+1$ banded M-matrix, and $A^{-1} = X = [x_{st}]$. Then for any s,t with $s \in \{(i-1)p+2,\dots,ip+1\}$ and $t \in \{(j-1)p+2,\dots,jp+1\}$ ($i = 1$ if $s = 1, j = 1$ if $t = 1$) with $i \neq j$,*

$$\theta_s^{-1} x_{st}\theta_t \le \rho^{|i-j|} x_{tt}, \qquad \theta_s^{-1} x_{st}\theta_t \le \rho^{|i-j|} x_{ss},$$

where $\rho = \rho(D^{-1}N)$ *is the spectral radius of* $D^{-1}N$ *with* $D = diag(A)$ *and* $N = D - A$, $\theta = \{\theta_1, \dots, \theta_n\}$ *denote the eigenvector corresponded with* ρ.

Since we obtain the exact inverse in Theorem 3.4, and the bounds in Theorem 3.5 [Theorem 3.12,3] are relate to the spectral radius ρ and eigenvector θ which corresponded with ρ of nonnegative matrices $D^{-1}N$, it is very difficult to calculate spectral radius and eigenvector, the rate of calculating the bounds be extremely slow. Then, when A be a nonsingular diagonally dominant seven-diagonal matrix, the bounds in Theorem 3.4 are better than the bounds in Theorem 3.5 [Theorem 3.12,3].

On the other hand, by the following numerical examples, we can illustrate the upper bounds is the exact inverse of A in Theorem 3.4, and better than the bounds that Nabben obtained in Theorem 3.5 [Theorem 3.12,3]:

Example 1. we consider the bounds for nonsingular banded diagonally dominant diagonal M-matrix with $p = 3$ and order $n = 7$, where $a_{i,i} = 30$ $(i = 1, \dots, n)$, $a_{i,i+1} = -2$, $a_{i+1,i} = -4$ $(i = 1, \dots, n-1)$, $a_{i,i+2} = -3$, $a_{i+2,i} = -6$ $(i = 1, \dots, n-2)$, $a_{i,i+3} = -4$, $a_{i+3,i} = -7$, $(i = 1, \dots, n-3)$.

We denote the matrix of upper bounds in [3], upper bounds and lower bounds in this paper of A^{-1} by $\hat{V} = (\hat{v}_{i,j})_{n\times n}$, $\tilde{V} = (\tilde{v}_{i,j})_{n\times n}$, $V = (v_{i,j})_{n\times n}$, respectively. Use the Matlab to program calculation on the microcomputer(the precision is $1.0e-7$), we obtain

$$\tilde{V} = \begin{pmatrix} 0.0367 & 0.0051 & 0.0057 & 0.0067 & 0.0022 & 0.0017 & 0.0012 \\ 0.0079 & 0.0376 & 0.0061 & 0.0070 & 0.0067 & 0.0022 & 0.0017 \\ 0.0108 & 0.0091 & 0.0390 & 0.0077 & 0.0070 & 0.0067 & 0.0022 \\ 0.0134 & 0.0124 & 0.0108 & 0.0411 & 0.0077 & 0.0070 & 0.0067 \\ 0.0067 & 0.0133 & 0.0122 & 0.0108 & 0.0390 & 0.0061 & 0.0057 \\ 0.0065 & 0.0068 & 0.0133 & 0.0124 & 0.0091 & 0.0376 & 0.0051 \\ 0.0053 & 0.0065 & 0.0067 & 0.0134 & 0.0108 & 0.0079 & 0.0367 \end{pmatrix} = A^{-1},$$

$$\hat{V} = \begin{pmatrix} 0.0367 & 0.0277 & 0.0214 & 0.0101 & 0.0100 & 0.0058 & 0.0058 \\ 0.0498 & 0.0376 & 0.0291 & 0.0137 & 0.0135 & 0.0078 & 0.0078 \\ 0.0668 & 0.0505 & 0.0390 & 0.0184 & 0.0181 & 0.0105 & 0.0105 \\ 0.0526 & 0.0397 & 0.0307 & 0.0411 & 0.0405 & 0.0234 & 0.0234 \\ 0.0507 & 0.0383 & 0.0296 & 0.0396 & 0.0390 & 0.0226 & 0.0226 \\ 0.0298 & 0.0225 & 0.0174 & 0.0233 & 0.0229 & 0.0376 & 0.0376 \\ 0.0291 & 0.0220 & 0.0170 & 0.0227 & 0.0224 & 0.0367 & 0.0367 \end{pmatrix}$$

Obviously, $\tilde{V} = A^{-1} < \hat{V}$.

3. Numerical examples

In this section, we consider some examples for different matrices A and compare the entries of A^{-1} with our bounds. Denote $\tilde{V}$, V as in example 1, and denote $\tilde{\delta} = \max\{\tilde{v}_{i,j} - |x_{i,j}|\}$, $\delta = \max\{|x_{i,j}| - v_{i,j}\}$.

Example 3.1 In this case, we consider the bounds for a strictly diagonally dominant seven-diagonal matrix with $a_{i,i} = 27(i = 1, \dots, n)$, $a_{i,i+1} = 2$, $a_{i+1,i}$

$= 4$ $(i = 1, \ldots, n-1)$, $a_{i,i+2} = -3$, $a_{i+2,i} = 6$ $(i = 1, \ldots, n-2)$, $a_{i,i+3} = -4$, $a_{i+3,i} = -7$, $(i = 1, \ldots, n-3)$.
The lower and upper bounds as follows:

n	7	70	700
$\tilde{\delta}$	1.75×10^{-2}	5.86×10^{-2}	5.88×10^{-2}
δ	1.05×10^{-2}	1.49×10^{-2}	1.49×10^{-2}

Example 3.2 In this case, we consider the bounds for a strictly diagonally dominant seven-diagonal matrix with $a_{i,i} = 40$ $(i = 1, \ldots, n)$,$a_{i,i+1} = -1$, $a_{i+1,i} = -1(i = 1, \ldots, n-1)$, $a_{i,i+2} = 2$, $a_{i+2,i} = 2$ $(i = 1, \ldots, n-2)$, $a_{i,i+3} = 3$, $a_{i+3,i} = 3$, $(i = 1, \ldots, n-3)$.
The lower and upper bounds as follows:

n	7	70	700
$\tilde{\delta}$	1.61260×10^{-4}	1.7580×10^{-4}	1.7580×10^{-4}
δ	9.4046×10^{-4}	9.5411×10^{-4}	9.5411×10^{-4}

Example 3.3 In this case, we consider the bounds for a strictly diagonally dominant seven-diagonal matrix with $a_{i,i} = 12$ $(i = 1, \ldots, n)$, $a_{i,i+1} = 1$, $a_{i+1,i} = 1$ $(i = 1, \ldots, n-1)$, $a_{i,i+2} = 1$, $a_{i+2,i} = 1$ $(i = 1, \ldots, n-2)$, $a_{i,i+3} = 1$, $a_{i+3,i} = 1$, $(i = 1, \ldots, n-3)$.
The lower and upper bounds as follows:

n	7	70	700
$\tilde{\delta}$	5.4×10^{-3}	5.8×10^{-3}	5.8×10^{-3}
δ	7.3×10^{-3}	7.6×10^{-3}	7.6×10^{-3}

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Zhuohong Huang received his BS and MS from Xiangtan Normal University and Xiangtan University in 2002 and 2007, respectively. Since 2007 he has been a DR at the University of Electronic Science and Technology of China. His research interests focus on the theory and computation of matrix and computational electromagnetics.

Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, PR China
e-mail: zhuohonghuang@yahoo.cn

Ting-Zhu Huang received his Ph.D degree at Xi'an Jiaotong University in 2001. Prefessor of computational and applied mathematics.

Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, PR China
e-mail: tingzhuhuang@126.com