J. Appl. Math. & Informatics Vol. 27(2009), No. 3 - 4, pp. 909 - 913
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LOCAL AND MEAN k-RAMSEY NUMBERS FOR THE FAMILY OF GRAPHS

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ABSTRACT. For a family of graphs $\mathcal{H}$ and an integer k, we denote by $R^k(\mathcal{H})$ the corresponding k-Ramsey number, which is defined to be the smallest integer n such that every k-coloring of the edges of K_n contains a monochromatic copy of a graph in $\mathcal{H}$. The local k-Ramsey number $R^k_{loc}(\mathcal{H})$ and the mean k-Ramsey number $R^k_{mean}(\mathcal{H})$ are defined analogously. Let $\mathcal{G}$ be the family of non-bipartite graphs and $\mathcal{T}_n$ be the family of all trees on n vertices. In this paper we prove that $R^k_{loc}(\mathcal{G}) = R^k_{mean}(\mathcal{G})$, and $R^2(\mathcal{T}_n) < R^2_{loc}(\mathcal{T}_n) = R^2_{mean}(\mathcal{T}_n)$ for all $n \geq 3$.

AMS Mathematical Subject Classification: 05C10, 05C15.
Keywords and phrases: local k-coloring, mean k-coloring, local k-Ramsey number, mean k-Ramsey number.

For a family of graphs $\mathcal{H}$ and an integer k, the *k-Ramsey number* $R^k(\mathcal{H})$ is defined to be the smallest integer n such that every k-coloring of the edges of K_n contains a monochromatic copy of a graph in $\mathcal{H}$. Over the past years the number of different colors has no longer been restricted to k, but a restriction is placed on the number of different colored edges incident to the vertices. To be precise, let G be a fixed graph on n vertices and c be a coloring of the edges of G. For every vertex $v \in V(G)$ define $k_c(v)$ as the number of distinct colors that appear on edges of G incident to v. The coloring c is called a *local k-coloring* if $k_c(v) \leq k$ for all v in G, while c is called a *mean k-coloring* if $\frac{1}{n}\sum_v k_c(v) \leq k$. Further the *local k-Ramsey number* $R^k_{loc}(\mathcal{H})$ (or *mean k-Ramsey number* $R^k_{mean}(\mathcal{H})$) is defined as the smallest positive integer n such that every local k-coloring (or mean k-coloring) of K_n contains a monochromatic copy of a graph in $\mathcal{H}$.

For these Ramsey numbers $R^k(\mathcal{H})$, $R^k_{loc}(\mathcal{H})$, and $R^k_{mean}(\mathcal{H})$, if $\mathcal{H}$ contains only a single graph H, then we denote them by $R^k(H)$, $R^k_{loc}(H)$, and $R^k_{mean}(H)$, respectively. Since every k-coloring of the edges of K_n is a local k-coloring and every local k-coloring is a mean k-coloring, it is clear that $R^k(\mathcal{H}) \leq R^k_{loc}(\mathcal{H}) \leq$

Received December 21, 2007. Revised April 23, 2008. Accepted May 12, 2008. *Corresponding author. This research was supported by National Natural Science Foundation of China (10571042), NSF of Hebei (A2005000144) and the SF of Hebei Normal University (L2004B03)

$R^k_{mean}(\mathcal{H})$ for every family of graphs $\mathcal{H}$ and $R^k(H) \leq R^k_{loc}(H) \leq R^k_{mean}(H)$ for every graph H. About the second of these inequalities we have known the following results.

Theorem A.[2] *For all $m \geq 3$, $R^2_{loc}(K_m) = R^2_{mean}(K_m)$.*

In [2] Caro and Tuza have raised a main open problem. Is it true that for every integer $k \geq 2$ and every graph G, $R^k_{loc}(G) = R^k_{mean}(G)$?

Theorem B.[3] *$R^k_{loc}(K_m) = R^k_{mean}(K_m)$ for each $m \geq 3$ and $k \geq 2$.*

Theorem C.[1] *For every ES-tree T on d vertices and for every sufficiently large k such that $\frac{k(k-1)}{d-1}$ is a integer, $R^k_{loc}(T) = R^k_{mean}(T) = (d-2)k+2$.*

Theorem B and Theorem C partly answered the above open problem.

In this paper we consider the inequality $R^k_{loc}(\mathcal{H}) \leq R^k_{mean}(\mathcal{H})$ and obtain the following result.

Theorem 1. *Let $\mathcal{G}$ be the family of non-bipartite graphs, then the equality $R^k_{loc}(\mathcal{G}) = R^k_{mean}(\mathcal{G})$ holds.*

Proof. Since every local k-coloring of the edges of K_n is a mean k-coloring, it is clear that $R^k_{loc}(\mathcal{G}) \leq R^k_{mean}(\mathcal{G})$. We need only to prove the reverse inequality.

Set $n = R^k_{mean}(\mathcal{G})-1$ and color K_n with a mean k-coloring such that it contains no monochromatic copy of any graph in $\mathcal{G}$. Let P represent this colored K_n, and let V_i be the set of vertices of P whose incident edges are colored by exactly i different colors, $1 \leq i \leq k$, and W be the set of vertices of P whose incident edges are colored by at least $k+1$ different colors. Then $n = \sum_{i=1}^k |V_i| + |W|$. For convenience let $|V_i| = x_i$ and $|W| = y$, then $n = \sum_{i=1}^k x_i + y$.

From the definition of a mean k-coloring, we know that $\sum_v k_f(v) \leq kn$, then $1 \cdot x_1 + 2 \cdot x_2 + \cdots + k \cdot x_k + (k+1) \cdot y \leq kn$.

However $kn = k(\sum_{i=1}^k x_i + y) = kx_1 + kx_2 + \cdots + kx_k + ky$, so $1 \cdot x_1 + 2 \cdot x_2 + \cdots + k \cdot x_k + (k+1) \cdot y \leq kx_1 + kx_2 + \cdots + kx_k + ky$, hence $\sum_{i=1}^{k-1}(k-i)x_i \geq y$.

Let $s = \sum_{i=1}^k (k-i+1)x_i$, then from the above inequality we know that $s \geq n$. Let $V(K_s) = \bigcup_{i=1}^k (k-i+1)V_i$. Color the edges of K_s in the following fashion: For each edge in any copy of V_i, $1 \leq i \leq k$, color it as it was colored in P; for each $u \in V_i$ (any copy) and $v \in V_j$ (any copy), $1 \leq i < j \leq k$, color the edge uv as it was colored in P. Then the only edges of K_s not yet colored are those joining a pair of vertices in two different copies of some V_i.

In order to color the edges between two different copies in $(k-i+1)V_i$, we regard each set V_i as a *vertex* of a new complete graph. Set $t = k-i+1$, then color the edges of the complete graph K_t by $\frac{t(t-1)}{2}$ distinct colors (which were not used in former). Clearly for every vertex in $V(K_t)$, the number of different colors that appear on edges of K_t incident to this vertex is equal to $t-1 = (k-i+1)-1 = k-i$. Then we color the edges joining a pair of vertices in two different copies of V_i by the corresponding colors in K_t. It is easy to check that for every vertex $v \in V(K_s)$ the number of different colors that appear on the edges of K_s incident to this vertex is equal to $i + (k-i) = k$. Therefore the coloring just described is a local k-coloring of K_s.

Since the edges between any pair of copies of V_i or the edges between V_i (any copy) and V_j (any copy) $(i \neq j)$ generate bipartite graphs, we conclude that if the locally k-colored K_s just described contains a monochromatic copy of a graph G in $\mathcal{G}$ then there must exist a monochromatic copy of $G_0 \subseteq G$ in $\mathcal{G}$ such that G_0 with at least one odd cycle appears in P, which is contrary to the coloring of P. Therefore in the above local k-coloring of edges of K_s there is no monochromatic copy of any graph in $\mathcal{G}$. Since $n \leq s$, $R^k_{mean}(\mathcal{G})-1 = n \leq s \leq R^k_{loc}(\mathcal{G})-1$, that is, $R^k_{mean}(\mathcal{G}) \leq R^k_{loc}(\mathcal{G})$. The proof is complete. □

The following problem is a natural generalization of the open problem raised by Caro and Tuza.

Problem 1. *Is it true that for every integer $k \geq 2$ and for every family of graphs $\mathcal{G}$, $R^k_{mean}(\mathcal{G}) = R^k_{loc}(\mathcal{G})$?*

Let $\mathcal{T}_n$ be the family of all trees on n vertices. In the following we prove that $R^2(\mathcal{T}_n) < R^2_{loc}(\mathcal{T}_n) = R^2_{mean}(\mathcal{T}_n)$ for all $n \geq 3$.

Lemma 1. $R^2(\mathcal{T}_n) = n$.

Proof. The proof of this lemma is clear, but because of its simplicity we present it here. It is clear that we need only to prove the inequality $R^2(\mathcal{T}_n) \leq n$. Let c be an arbitrary 2-coloring of edges of K_n and T be a monochromatic tree with the maximum number of vertices in this 2-colored K_n. Suppose that T is colored by the color 1. If T is not a spanning tree of K_n, then there exists a vertex $v \in V(K_n)$ such that $v \notin V(T)$. For any vertex $u \in V(T)$, by the maximality of T we know that the color of the edge uv is not the color 1, hence it must be the other color, say, the color 2. Therefore we obtain a star whose center is v and whose end vertices are all the vertices of T being colored by the color 2. But this star has one more vertex than T, contradicting the maximality of T, hence $R^2(\mathcal{T}_n) \leq n$, and the proof is complete. □

Lemma 2. *If $n = 2j$, then $R^2_{loc}(\mathcal{T}_n) = g(n) = 3j - 1$; If $n = 2j + 1$, then $R^2_{loc}(\mathcal{T}_n) = g(n) = 3j + 1$.*

Proof. We construct a local 2-coloring of the edges of $K_{g(n)-1}$ such that it contains no monochromatic copy of any tree in $\mathcal{T}_n$. First partition $V(K_{g(n)-1})$ into three parts H_1, H_2 and H_3 as follows. If $g(n)-1 = 3j-2$, then $|H_1| = |H_2| = j-1$ and $|H_3| = j$, and if $g(n) - 1 = 3j$, then $|H_1| = |H_2| = |H_3| = j$. Next color the edges of the complete graphs $G[H_1]$, $G[H_2]$, and $G[H_3]$ by three distinct colors and the edges of the complete bipartite graphs $G[H_1, H_2]$, $G[H_2, H_3]$ and $G[H_3, H_1]$ by the same three colors, respectively. It is clear that the coloring just described is a local 2-coloring of $K_{g(n)-1}$, which contains no monochromatic copy of any tree in $\mathcal{T}_n$, hence $R^2_{loc}(\mathcal{T}_n) \geq g(n)$.

In order to prove the reverse inequality we consider a local 2-coloring of the edges of $K_{g(n)}$. If there is a color which induces a connected subgraph H of $K_{g(n)}$ with at least n vertices, then H contains a spanning tree T with at least n vertices, and hence $K_{g(n)}$ contains a monochromatic copy of some tree in $\mathcal{T}_n$. Otherwise, we conclude that the maximum number of vertices of all connected subgraphs of $K_{g(n)}$ induced by any color is less than n. Let H be a connected subgraph of $K_{g(n)}$ induced by one of the colors and assume that $|V(H)| = i < n$. By the definition of a local 2-coloring it follows that the edges that join $V(H)$ and $V(K_{g(n)})\backslash V(H)$ are colored by only two colors, moreover, if $v \in V(H)$, then all the edges jointing v to all vertices in $V(K_{g(n)})\backslash V(H)$ have the same color. Hence there are at least $\lceil \frac{i}{2} \rceil$ vertices in H that are jointed to the vertices in $V(K_{g(n)})\backslash V(H)$ by the same color, which results in a complete bipartite graph with at least $\lceil \frac{i}{2} \rceil + (g(n) - i) = g(n) - \lfloor \frac{i}{2} \rfloor$ vertices. Since $i < n$, it follows that $g(n) - \lfloor \frac{i}{2} \rfloor \geq n$, so in this case $K_{g(n)}$ also contains a monochromatic copy of some tree in $\mathcal{T}_n$. Then we know that $R^2_{loc}(\mathcal{T}_n) \leq g(n)$ and the proof is complete. □

Lemma 3. $R^2_{loc}(\mathcal{T}_n) = R^2_{mean}(\mathcal{T}_n)$ *for all* $n \geq 3$.

Proof. It is clear that $R^2_{loc}(\mathcal{T}_n) \leq R^2_{mean}(\mathcal{T}_n)$, we need only to prove the reverse inequality $R^2_{loc}(\mathcal{T}_n) \geq R^2_{mean}(\mathcal{T}_n)$. Let $s = R^2_{mean}(\mathcal{T}_n) - 1$ and color the edges of K_s with a mean 2-coloring such that it contains no monochromatic copy of any tree in $\mathcal{T}_n$. Let V_1 be the set of vertices of K_s whose incident edges are colored by a single color, V_2 be the set of vertices whose incident edges are colored by exactly 2 different colors, and $V_{\geq 3}$ be the set of vertices whose incident edges are colored by at least 3 different colors. By Lemma 1 and Lemma 2 we know that $s \geq n$. If $V_1 \neq \phi$, then there must exist one vertex $v \in V(K_s)$ such that v is adjacent to any vertex in $V(K_s)\backslash\{v\}$. Since $s \geq n$, we obtain a monochromatic star with at leat n vertices, contrary to the coloring of K_s. If $V_1 = \phi$, then by the definition of a mean 2-coloring we know that $V_{\geq 3} = \phi$, hence the mean 2-coloring is a local 2-coloring and $R^2_{mean}(\mathcal{T}_n) - 1 \leq R^2_{loc}(\mathcal{T}_n) - 1$, that is, $R^2_{mean}(\mathcal{T}_n) \leq R^2_{loc}(\mathcal{T}_n)$. The proof is complete. □

Immediately combining Lemmas 1-3 we get our main result as follows.

Theorem 2. $R^2(\mathcal{T}_n) < R^2_{loc}(\mathcal{T}_n) = R^2_{mean}(\mathcal{T}_n)$ *for all* $n \geq 3$.

The following problem is a natural continuation of our theorem.

Problem 2. *Is it true for* $k \geq 2$ *and* $n \geq 3$, $R^k_{loc}(\mathcal{T}_n) = R^k_{mean}(\mathcal{T}_n)$*?*

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