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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR SINGULAR THREE-POINT BOUNDARY VALUE PROBLEMS

CHUNMEI MIAO* AND WEIGAO GE

ABSTRACT. In this paper, the singular three-point boundary value problem

$$\begin{cases} u''(t) + f(t,u) = 0, \quad t \in (0,1), \\ u(0) = 0, \ u(1) = \alpha u(\eta), \end{cases}$$

is studied, where $0 < \eta < 1$, $\alpha > 0$, $f(t,u)$ may be singular at $u = 0$. By mixed monotone method, the existence and uniqueness are established for the above singular three-point boundary value problems. The theorems obtained are very general and complement previous know results.

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1. Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Then Gupta [3] studied three-point boundary value problem for nonlinear ordinary differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by many authors, we refer the reader to [5-8] for some existence results of nonlinear multipoint boundary value problems. But so far, for the singular multi-point boundary value problems, to the author's knowledge, few papers have been seen in the literature.

P. K. Singh [9], the existence of a positive solution is obtained for the second-order three-point boundary value problem

$$y'' + f(x,y) = 0, \ 0 < x \leq 1,$$
$$y(0) = 0, \ y(1) = y(p),$$

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*Corresponding author.

where $p \in (0,1)$ is fixed and $f(x,y)$ is singular at $x=0,\ y=0$, and possibly at $y=\infty$. The method applies a fixed-point theorem (Gatica, Oliker and Waltman [4]) for mappings that are decreasing with respect to a cone.

Xu [10] considered the singular three-point boundary value problem

$$\begin{cases} y''(t)+f(t,y)=0, & t\in(0,1),\\ y(0)=0,\ y(1)=\alpha y(\eta), \end{cases}$$

where $0<\eta,\ \alpha\eta<1$, and $f(t,u)$ may be singular at $u=0$. By using the method of fixed point index, obtained the multiplicity results for positive solutions.

In paper [11], Ma by using some new existence principles to get positive solutions of the following nonlinear three-point singular boundary value problem

$$(\phi_p(u'))'+q(t)f(t,u)=0,\quad t\in(0,1),$$

subject to

$$u(0)-g(u'(0))=0,\ u(1)-\beta u(\eta)=0,$$

or

$$u(0)-\alpha u(\eta)=0,\ u(1)-g(u'(1))=0,$$

where $f(t,u)$ may be singular at $u=0$ and $q(t)$ may be singular at $t=0,1$.

Most of the above results told us that the boundary value problems had at least single and multiple positive solutions, there is no result on the uniqueness of solution in them.

In this paper, we consider the following singular three-point boundary value problem

$$\begin{cases} u''(t)+f(t,u)=0, & t\in(0,1),\\ u(0)=0,\ u(1)=\alpha u(\eta), \end{cases} \tag{1.1}$$

where $\alpha>0,\ 0<\eta<1,\ f\in C((0,1)\times(0,+\infty),(0,+\infty))$ and $f(t,u)$ may be singular at $u=0$. Our idea comes from the fixed point theorems for mixed monotone operators ([12-15]). By mixed monotone method, the existence and uniqueness are established.

For the sake of simplicity, let us denote some properties which will be used in next theorems and propositions:

(H_1) $0<\eta<1,\ 0<\alpha<\frac{1}{\eta}$;

(H_2) $f\in C((0,1)\times(0,+\infty),(0,+\infty)),\ f(t,x)=q(t)[g(x)+h(x)]$ on $(0,1)\times(0,+\infty)$, where $g:[0,+\infty)\to[0,+\infty)$ is continuous and nondecreasing, $h:(0,+\infty)\to(0,+\infty)$ is continuous and nonincreasing, $q\in C(0,1)\cap L^1[0,1]$, $q(t)>0$ on $(0,1)$.

2. Preliminaries

Consider the Banach space $E=C[0,1]$ with the norm $||u||=\max\limits_{0\le t\le 1}|u(t)|$, P be a normal cone of the Banach space E, and $e\in P$ with $||e||\le 1,\ e\ne\theta$. Define

$$Q_e=\{x\in P|\, x\ne\theta, \text{ there exists constants } m,\ M>0 \text{ such that } me\le x\le Me\}.$$

Definition 1. ([15]) *Assume* $A: Q_e \times Q_e \to Q_e$. *A is said to be mixed monotone if* $x_1 \le x_2(x_1, x_2 \in Q_e)$ *implies* $A(x_1, y) \le A(x_2, y)$ *for any* $y \in Q_e$, *and* $y_1 \le y_2(y_1, y_2 \in Q_e)$ *implies* $A(x, y_1) \ge A(x, y_2)$ *for any* $x \in Q_e$. $x^* \in Q_e$ *is said to be a fixed point of A if* $A(x^*, x^*) = x^*$.

Theorem 1. ([12]) *Suppose that* $A: Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and there exists constant* β, $0 \le \beta < 1$, *such that*

$$A(tx, \frac{1}{t}y) \ge t^\beta A(x, y), \ \forall x, y \in Q_e, \ 0 < t < 1. \tag{2.1}$$

Then A has a unique fixed point $x^* \in Q_e$. *Moreover, for any* $(x_0, y_0) \in Q_e \times Q_e$, $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, ...$, *satisfy* $x_n \to x^*$, $y_n \to x^*$, *where*

$$||x_n - x^*|| = o(1 - r^{\beta^n}), \ ||y_n - x^*|| = o(1 - r^{\beta^n}), \ 0 < r < 1,$$

r is a constant from (x_0, y_0).

Theorem 2. ([15])*Suppose that* $A: Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and there exists constant* β, $0 \le \beta < 1$, *such that* (2.1) *holds. If* x_λ^* *is a unique solution of equation* $A(x, x) = \lambda x$, $\lambda > 0$, *in* Q_e, *then* $||x_\lambda^* - x_{\lambda_0}^*|| \to 0$, $\lambda \to \lambda_0$. *If* $0 < \beta < \frac{1}{2}$, *then* $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1}^* \ge x_{\lambda_2}^*$, $x_{\lambda_1}^* \ne x_{\lambda_2}^*$, *and* $\lim\limits_{\lambda \to +\infty} ||x_\lambda^*|| = 0$, $\lim\limits_{\lambda \to 0^+} ||x_\lambda^*|| = +\infty$.

Lemma 1. ([6])*Suppose* (H_1) *hold and* $e \in L^1[0, 1]$, *then linear boundary value problem*

$$\begin{cases} u''(t) + e(t) = 0, & t \in (0, 1), \\ u(0) = 0, \ u(1) = \alpha u(\eta), \end{cases} \tag{2.2}$$

have a unique solution

$$u(t) = \int_0^1 G(t, s) e(s) ds, \tag{2.3}$$

where $G(t, s): [0, 1] \times [0, 1] \to R^+$ *is defined by*

$$G(t, s) = \frac{1}{1 - \alpha\eta} \begin{cases} s(1 - t) + \alpha s(t - \eta), & 0 \le s \le \min\{t, \eta\} < 1, \\ t(1 - s - \alpha\eta + \alpha s), & 0 \le t \le s \le \eta < 1, \\ s(1 - t) + \alpha\eta(t - s), & 0 < \eta \le s \le t \le 1, \\ t(1 - s), & 0 < \max\{t, \eta\} \le s \le 1. \end{cases} \tag{2.4}$$

Lemma 2. *Suppose* (H_1) *hold, then*

$$0 \le cG(s, s)t(1 + \alpha\eta - t) \le G(t, s) \le \frac{1}{1 - \alpha\eta} t(1 + \alpha\eta - t), \ t, s \in [0, 1], \tag{2.5}$$

where

$$G(s, s) = \frac{1}{1 - \alpha\eta} \begin{cases} [1 - \alpha\eta - (1 - \alpha)s]s, \ 0 \le s \le \eta < 1, \\ s(1 - s), \ 0 < \eta \le s \le 1, \end{cases}$$
$$c = \min\{\frac{1}{1 + \alpha\eta}, \ \frac{1 - \eta}{\eta(1 - \alpha\eta)}, \ \alpha\eta\}.$$

Proof. First we show $G(t,s)\ge cG(s,s)t(1+\alpha\eta-t),\ t,s\in[0,1]$. There are four case as follows.

Case 1. $0\le s\le \min\{t,\eta\}<1$, when $1\le\alpha<\frac{1}{\eta}$,

$$\frac{G(t,s)}{G(s,s)}=\frac{1-\alpha\eta+(\alpha-1)t}{1-\alpha\eta+(\alpha-1)s}\ge\frac{1-\alpha\eta+(\alpha-1)s}{1-\alpha\eta+(\alpha-1)s}=1\ge t(1+\alpha\eta-t),$$

when $0<\alpha<1$,

$$\begin{aligned}\frac{G(t,s)}{G(s,s)(1+\alpha\eta-t)} &= \frac{1-\alpha\eta+(\alpha-1)t}{[1-\alpha\eta+(\alpha-1)s](1+\alpha\eta-t)}\\ &\ge \frac{1}{1-\alpha\eta}\frac{1-\alpha\eta+(\alpha-1)t}{1+\alpha\eta-t},\end{aligned}$$

Let $k(t)=\frac{1-\alpha\eta+(\alpha-1)t}{1+\alpha\eta-t}$. So $k'(t)=\frac{\alpha(1+\alpha\eta-2\eta)}{(1+\alpha\eta-t)^2}$, when $2-\frac{1}{\eta}\le\alpha<\frac{1}{\eta},\ k'(t)\ge 0,\ k(0)=\frac{1-\alpha\eta}{1+\alpha\eta}$, when $0<\alpha<2-\frac{1}{\eta},\ k'(t)\le 0,\ k(1)=\frac{1-\eta}{\eta}$. Thus, for any $\frac{1}{\eta}>\alpha>0$,

$$G(t,s)\ge\min\{\frac{1}{1+\alpha\eta},\ \frac{1-\eta}{(1-\alpha\eta)\eta}\}t(1+\alpha\eta-t)G(s,s).$$

Case 2. $0\le t\le s\le\eta<1$,

$$\frac{G(t,s)}{G(s,s)}=\frac{t}{s}\ge t(1+\alpha\eta-t)\Rightarrow G(t,s)\ge t(1+\alpha\eta-t)G(s,s).$$

Case 3. $0<\eta\le s\le t\le 1$,

$$\begin{aligned}\frac{G(t,s)}{G(s,s)t} &= \frac{s(1-t)+\alpha\eta(t-s)}{s(1-s)t}=\frac{(\alpha\eta-s)t+(1-\alpha\eta)s}{s(1-s)t}\\ &= \frac{\alpha\eta-s}{s(1-s)}+\frac{(1-\alpha\eta)s}{s(1-s)t}\ge\frac{\alpha\eta}{s}\ge\alpha\eta,\end{aligned}$$

so, $G(t,s)\ge\alpha\eta G(s,s)t(1+\alpha\eta-t)$.

Case 4. $0<\max\{t,\eta\}\le s\le 1$,

$$\frac{G(t,s)}{G(s,s)}=\frac{t}{s}\ge t(1+\alpha\eta-t)\Rightarrow G(t,s)\ge t(1+\alpha\eta-t)G(s,s).$$

Let $c=\min\{\frac{1}{1+\alpha\eta},\ \frac{1-\eta}{\eta(1-\alpha\eta)},\ \alpha\eta\}$, thus for any $s,t\in[0,1]$, $G(t,s)\ge cG(s,s)t(1+\alpha\eta-t)$.

Next we show $G(t,s)\le\frac{1}{1-\alpha\eta}t(1+\alpha\eta-t),\ t,s\in[0,1]$. There are four case as follows.

Case 1. $0\le s\le\min\{t,\eta\}<1$,

$$\begin{aligned}G(t,s)&=\tfrac{1}{1-\alpha\eta}[t(1-s)-(\eta-s)\alpha t-(t-s)(1-\alpha\eta)]\\ &\le\tfrac{1}{1-\alpha\eta}[t(1-s)-(t-s)(1-\alpha\eta)]\\ &=\tfrac{1}{1-\alpha\eta}[t(1+\alpha\eta-t)-(1-t)(t-s)-\alpha\eta s]\\ &\le\tfrac{1}{1-\alpha\eta}t(1+\alpha\eta-t).\end{aligned}$$

Case 2. $0 \le t \le s \le \eta < 1$,

$$G(t,s) = \frac{1}{1-\alpha\eta}[t(1-s) - \alpha t(\eta - s)] \le \frac{1}{1-\alpha\eta} t(1+\alpha\eta - t).$$

Case 3. $0 < \eta \le s \le t \le 1$,

$$\begin{aligned} G(t,s) &= \tfrac{1}{1-\alpha\eta}[t(1-s) - (1-\alpha\eta)(t-s)] \\ &= \tfrac{1}{1-\alpha\eta}[t(1+\alpha\eta - t) - (1-t)(t-s) - \alpha\eta s] \\ &\le \tfrac{1}{1-\alpha\eta} t(1+\alpha\eta - t). \end{aligned}$$

Case 4. $0 < \max\{t, \eta\} \le s \le 1$,

$$G(t,s) = \frac{1}{1-\alpha\eta} t(1-s) \le \frac{1}{1-\alpha\eta} t(1+\alpha\eta - t).$$

In a word, for any $t, s \in [0,1]$,

$$0 \le cG(s,s)t(1+\alpha\eta - t) \le G(t,s) \le \frac{1}{1-\alpha\eta} t(1+\alpha\eta - t).$$

The proof is completed. □

3. Existence and uniqueness of positive solution

Let $P = \{x \in E |\ x(t) \ge 0,\ t \in [0,1]\}$. Obviously, P is a normal cone of Banach space E.

Theorem 3. *Suppose that there exists $\gamma \in (0,1)$ such that*

$$g(tx) \ge t^{\gamma} g(x), \tag{3.1}$$

$$h(\frac{1}{t}x) \ge t^{\gamma} h(x), \tag{3.2}$$

for any $t \in (0,1)$, $x > 0$, and $q \in C((0,1),(0,+\infty))$ satisfies

$$\int_0^1 s^{-\gamma}(1+\alpha\eta - s)^{-\gamma} q(s) ds < \infty. \tag{3.3}$$

Then (1.1) has a unique positive solution $x^*_\lambda(t)$. And moreover, $0 < \lambda_1 < \lambda_2 < 1$ implies $x^*_{\lambda_1} \le x^*_{\lambda_2}$, $x^*_{\lambda_1} \ne x^*_{\lambda_2}$. If $\gamma \in (0, \frac{1}{2})$, then $\lim\limits_{\lambda \to +\infty} ||x^*_\lambda|| = +\infty$, $\lim\limits_{\lambda \to 0^+} ||x^*_\lambda|| = 0$.

Proof. Since (3.2) holds, letting $\frac{1}{t}x = y$, one has

$$h(y) \ge t^{\gamma} h(ty) \Rightarrow h(ty) \le \frac{1}{t^{\gamma}} h(y),\ t \in (0,1),\ y > 0. \tag{3.4}$$

Let $y = 1$, the above inequality is

$$h(t) \le \frac{1}{t^{\gamma}} h(1),\ t \in (0,1). \tag{3.5}$$

From (3.2), (3.4) and (3.5), one has

$$h(\frac{1}{t}x) \ge t^{\gamma} h(x) \Rightarrow h(\frac{1}{t}) \ge t^{\gamma} h(1),$$

$$h(tx) \leq \frac{1}{t^\gamma}h(x) \Rightarrow h(t) \leq \frac{1}{t^\gamma}h(1), t \in (0,1),\ x > 0. \quad (3.6)$$

Similarly, from (3.1), one has

$$g(tx) \geq t^\gamma g(x) \Rightarrow g(t) \geq t^\gamma g(1),\ t \in (0,1),\ x > 0. \quad (3.7)$$

Letting $t = \frac{1}{x}$, $x > 1$, one has

$$g(x) \leq x^\gamma g(1),\ x > 1. \quad (3.8)$$

Let $e(t) = t(1 + \alpha\eta - t)$, and we define

$$Q_e = \{x \in E |\ \frac{1}{M}t(1 + \alpha\eta - t) \leq x(t) \leq Mt(1 + \alpha\eta - t),\ t \in [0,1]\},$$

where $M > 1$ is chosen such that

$$M > \max\{(\tfrac{1}{1-\alpha\eta}\int_0^1 \lambda q(s)[g(1) + s^{-\gamma}(1 + \alpha\eta - s)^{-\gamma}h(1)]ds)^{\frac{1}{1-\gamma}},$$
$$(\int_0^1 c\lambda G(s,s)q(s)[s^\gamma(1 + \alpha\eta - s)^\gamma g(1) + h(1)]ds)^{-\frac{1}{1-\gamma}}\}.$$

For any $x, y \in Q_e$, we define

$$A_\lambda(x,y)(t) = \lambda \int_0^1 G(t,s)q(s)[g(x(s)) + h(y(s))]ds,\ t \in [0,1]. \quad (3.9)$$

First we show that $A_\lambda :\ Q_e \times Q_e \to Q_e$. Letting $x, y \in Q_e$, from (3.7), (3.8) and (H_2), one has

$$g(x(t)) \leq g(Mt(1 + \alpha\eta - t)) \leq g(M) \leq M^\gamma g(1),\ t \in (0,1),$$

and from (3.6) and (H_2), we have

$$\begin{aligned} h(y(t)) &\leq h(\frac{1}{M}t(1 + \alpha\eta - t)) \leq t^{-\gamma}(1 + \alpha\eta - t)^{-\gamma}h(\frac{1}{M}) \\ &\leq M^\gamma t^{-\gamma}(1 + \alpha\eta - t)^{-\gamma}h(1),\ t \in (0,1). \end{aligned}$$

Then from Lemma 2 and (3.3), we have

$$\begin{aligned} &A_\lambda(x,y)(t) = \lambda \int_0^1 G(t,s)q(s)[g(x(s)) + h(y(s))]ds \\ &\leq t(1 + \alpha\eta - t)M^\gamma \tfrac{1}{1-\alpha\eta}\int_0^1 \lambda q(s)[g(1) + s^{-\gamma}(1 + \alpha\eta - s)^{-\gamma}h(1)]ds \\ &\leq Mt(1 + \alpha\eta - t),\ t \in [0,1]. \end{aligned}$$

On the other hand, for any $x, y \in Q_e$, from (3.6) and (3.7), we have

$$g(x(t)) \geq g(\frac{1}{M}t(1 + \alpha\eta - t)) \geq t^\gamma(1 + \alpha\eta - t)^\gamma g(\frac{1}{M}) \geq t^\gamma(1 + \alpha\eta - t)^\gamma \frac{1}{M^\gamma}g(1)$$

and $h(y(t)) \geq h(Mt(1 + \alpha\eta - t)) \geq h(M) \geq \frac{1}{M^\gamma}h(1),\ t \in (0,1)$.

Thus, from Lemma 2, we have

$$\begin{aligned} &A_\lambda(x,y)(t) = \lambda \int_0^1 G(t,s)q(s)[g(x(s)) + h(y(s))]ds \\ &\geq t(1 + \alpha\eta - t)\tfrac{1}{M^\gamma}\int_0^1 c\lambda G(s,s)q(s)[s^\gamma(1 + \alpha\eta - s)^\gamma g(1) + h(1)]ds \\ &\geq \tfrac{1}{M}t(1 + \alpha\eta - t),\ t \in [0,1]. \end{aligned}$$

So, A_λ is well defined and $A_\lambda(Q_e \times Q_e) \subset Q_e$.

Next, for any $\mu \in (0,1)$, one has

$$\begin{aligned} &A_\lambda(\mu x, \tfrac{1}{\mu}y)(t) = \lambda \int_0^1 G(t,s)q(s)[g(\mu x(s)) + h(\tfrac{1}{\mu}y(s))]ds \\ &\geq \lambda \int_0^1 G(t,s)q(s)[\mu^\gamma g(x(s)) + \mu^\gamma h(y(s))]ds \\ &= \mu^\gamma A_\lambda(x,y)(t), \ t \in [0,1]. \end{aligned}$$

So the conditions of Theorem 1 and (2.1) holds. Therefore there exists a unique $x_\lambda^* \in Q_e$ such that $A_\lambda(x^*, x^*) = x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (1.1) for given $\lambda > 0$. Moreover, Theorem 2 means that if $0 < \lambda_1 < \lambda_2$, then $x_{\lambda_1}^*(t) \leq x_{\lambda_2}^*(t)$, $x_{\lambda_1}^*(t) \neq x_{\lambda_2}^*(t)$, and if $\gamma \in (0, \frac{1}{2})$, then $\lim\limits_{\lambda\to+\infty} ||x_\lambda^*|| = +\infty$, $\lim\limits_{\lambda\to 0^+} ||x_\lambda^*|| = 0$. This completes the proof. □

4. Example

In this section, we give some explicit example to illustrate our main results.

Example. Consider three-point boundary value problem

$$\begin{cases} u'' + \lambda(u^{-\frac{1}{3}} + u^{\frac{1}{2}}) = 0, & 0 < t < 1, \ \lambda > 0, \\ u(0) = 0, \ \ u(1) = 3u(\frac{1}{4}) = 0, \end{cases} \tag{4.1}$$

where $\alpha = 3$, $\eta = \frac{1}{4}$, $f(t,u) = u^{-\frac{1}{3}} + u^{\frac{1}{2}}$.

Conclusion. *The boundary value problem* (4.1) *has a unique positive solution. In addition,* $0 < \lambda_1 < \lambda_2 < 1$ *implies* $x_{\lambda_1}^* \leq x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$, *and*

$$\lim_{\lambda\to+\infty} ||x_\lambda^*|| = +\infty, \ \lim_{\lambda\to 0^+} ||x_\lambda^*|| = 0.$$

.

Proof. We apply Theorem 3, then clearly (H_1) and (H_2) holds. Let $\gamma = \frac{1}{4}$, then

$$g(tx) = t^{\frac{1}{2}}x^{\frac{1}{2}} \geq t^{\frac{1}{4}}g(x), \ h(\frac{1}{t}x) = t^{\frac{1}{3}}x^{-\frac{1}{3}} \geq t^{\frac{1}{4}}h(x), \ t \in (0,1), \ x > 0,$$

and

$$\int_0^1 s^{-\frac{1}{4}}(\frac{7}{4} - s)^{-\frac{1}{4}}ds < \infty.$$

Therefore, by Theorem 3, we can obtain the boundary value problem (4.1) has a unique positive solution. In addition, $0 < \lambda_1 < \lambda_2 < 1$ implies $x_{\lambda_1}^* \leq x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$, and $\lim\limits_{\lambda\to+\infty} ||x_\lambda^*|| = +\infty$, $\lim\limits_{\lambda\to 0^+} ||x_\lambda^*|| = 0$.. □

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Chunmei Miao received her BS from Northeast Normal University(NENU). She is now studying for her Ph.D at the Department of Applied Mathematics of Beijing Institute of Technology(BIT) under the direction of Weigao Ge. Since 2004 she has been at the Changchun University. Her research interests focus on the biomathcmatics and boundary value problem. Also she does statistical consulting. This work is sponsored by the National Natural Science Foundation of China (no. 10671012) and the Doctoral Program Foundation of Education Ministryof China (20050007011).

Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, PR China
College of Science, Changchun University, Changchun 130022, PR China.
e-mail: miaochunmei@yahoo.com.cn

Weigao Ge
Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, PR China
e-mail : gew@bit.edu.cn