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THE ALMOST SURE CONVERGENCE OF WEIGHTED AVERAGES UNDER NEGATIVE QUADRANT DEPENDENCE

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ABSTRACT. In this paper we study the strong law of large numbers for weighted average of pairwise negatively quadrant dependent random variables. This result extends that of Jamison et al.(Convergence of weight averages of independent random variables Z. Wahrsch. Verw Gebiete(1965) **4** 40-44) to the negative quadrant dependence.

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\Big\{a_n, n \geq 1\Big\}$ be a sequence positive weights; put $A_n = \sum_{k=1}^{n} a_k$ and $T_n^{-1} = A_n^{-1} \sum_{k=1}^{n} a_k(X_k - EX_k)$. Using an extension of the well known Rademacher-Mensov inequality, Chandra and Ghosal(1995) derived strong laws of large numbers for weighted averages under various dependence assumptions. Using the similar inequality we will establish the almost sure convergence of $\{T_n\}$ to zero, where $\{X_n, n \geq 1\}$ is a sequence of pairwise negatively quadrant dependent random variables. Our result extends considerably that of Jamison et al.(1965), Prutt(1966), Rohatgi(1971) and Etemadi(1983).

Jamison et al.(1965) established the strong law of large number for independent and identically distributed random variables as follows.

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $E|X_1| < \infty$ and $EX_1 = 0$ and let $\{a_n, n \geq 1\}$*

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be a sequence of positive numbers. If

$$A_n = \sum_{k=1}^{n} a_k \uparrow \infty \text{ as } n \to \infty \text{ and } \sharp\{i : A_i/a_i \le n\} = O(n), \ n \ge 1,$$

then

$$T_n = \sum_{k=1}^{n} a_k X_k / A_n \to 0 \ a.s.$$

Definition 1.2. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be pairwise negatively quadrant dependent(NQD) if

$$P\Big(X_i > x_i, X_j > x_j\Big) \le P(X_i > x_i)P(X_j > x_j) \text{ for } x_i, x_j \in \mathbb{R} \text{ and } i \ne j.$$

This concept of NQD was introduced by Lehmann(1966).

Definition 1.3. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P\Big(|X_n| > x) \le CP(|X| > x\Big) \text{ for all } x \ge 0 \text{ and } n \ge 1$$

In this paper we derive the strong law of large numbers for weighted average of pairwise negatively quadrant dependent random variables.

2. Preliminaries

From an extension of the well-known Rademacher-Mensov inequality(see Chandra and Ghosal(1996)) we have:

Lemma 2.1. *Let $\{X_n, n \ge 1\}$ be a pairwise NQD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Then there exists a positive constant C such that*

$$E\left(\max_{1\le k\le n}\left(\sum_{i=1}^{k} X_i\right)^2\right) \le C(\log n)^2 \sum_{i=1}^{n} EX_i^2. \tag{2.1}$$

Lemma 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise NQD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Assume*

$$\sum_{n=1}^{\infty} (\log n)^2 EX_n^2 < \infty. \tag{2.2}$$

Then

$$\sum_{k=1}^{\infty} (X_n - EX_n) \text{ converges almost surely.}$$

Proof. We use the method of subsequences(see Stout(1974)). Note that (2.2) implies that $\sum_{n=1}^{\infty} EX_n^2 < \infty$. Put $S_n = \sum_{i=1}^{n} X_i$. Let $m < n$ be positive integers. Then by pairwise NQD assumption

$$E(S_n - S_m)^2 \leq \sum_{k=m+1}^{n} EX_k^2 \leq \sum_{k=m+1}^{\infty} EX_k^2 \to 0 \text{ } as \text{ } m \to \infty. \tag{2.3}$$

So, $\{S_n, n \geq 1\}$ is a Cauchy sequence with respect to the L^2 norm . Hence, by the completeness of L^2, there exists a random variable $S = \sum_{i=1}^{\infty} X_i$ such that $ES^2 < \infty$ and

$$E(S_n - S)^2 \to 0. \tag{2.4}$$

By Chebyshev's inequality and NQD condition we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} P(|S_{2^k} - S| \geq \epsilon) \\
\leq\ & \epsilon^{-2} \sum_{k=1}^{\infty} E(S_{2^k} - S)^2 \\
\leq\ & C \sum_{k=1}^{\infty} \lim_{n\to\infty} \sup_{n \geq 2^k} E(S_n - S_{2^k})^2 \\
\leq\ & C \sum_{k=1}^{\infty} \sum_{i=2^k+1}^{\infty} EX_i^2 (\log i)^2 / (\log i)^2 \\
\leq\ & C \sum_{k=1}^{\infty} \frac{1}{(\log 2^k)^2} \sum_{i=2^k+1}^{\infty} EX_i^2 (\log i)^2 \\
\leq\ & C \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\end{aligned} \tag{2.5}$$

Hence, we obtain by Borel-Cantelli lemma and (2.5)

$$S_{2^k} \to S \text{ } a.s. \tag{2.6}$$

It remains to show that

$$\max_{2^{k-1} < i \leq 2^k} |S_i - S_{2^{k-1}}| \to 0 \text{ } a.s. \tag{2.7}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\left(\max_{2^{k-1}<i\leq 2^k} |S_i - S_{2^{k-1}}| > \epsilon\right) \\
\leq\ & C\sum_{k=1}^{\infty} E \max_{2^{k-1}<i\leq 2^k} (S_i - S_{2^{k-1}})^2 \\
\leq\ & C\sum_{k=1}^{\infty} (\log 2^{k-1})^2 \sum_{i=2^{k-1}+1}^{2^k} EX_i^2 \qquad (2.8)\\
\leq\ & C\sum_{k=1}^{\infty} \sum_{i=2^{k-1}}^{2^k} (\log i)^2 EX_i^2 \\
\leq\ & C\sum_{k=1}^{\infty} (\log k)^2 EX_k^2 < \infty.
\end{aligned}$$

Hence by Borel-Cantelli lemma and (2.8), (2.7) follows. By (2.6) and (2.8) we have

$$S_n \to S \ a.s. \tag{2.9}$$

3. Main results

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD and stochastically dominated by a random variable X and $EX_n = 0$. Let $\{a_n, n \geq 1\}$ be a sequence of positive numbers with $A_n = \sum_{i=1}^{n} a_k \uparrow \infty$ as $n \to \infty$, $b_1 = \frac{A_n}{a_1}$ and $b_n = \frac{A_n}{a_n \log n}$, $n \geq 2$. Let set $N(x) = Card\{n, x \geq b_n\}$, $x \in \mathbb{R}$. If*

$$EN(X) < \infty, \tag{3.1}$$

and

$$\int_0^{\infty} tP(|X| > t) \int_t^{\infty} N(y)y^{-3} dy dt < \infty,$$

then

$$\sum_{k=1}^{n} \frac{a_k X_k}{A_n} \to 0 \ a.s. \tag{3.2}$$

Proof. Let

$$Y_k = -b_k I(X_k < -b_k) + X_k I(|X_k| \leq b_k) + b_k I(X_k > b_k). \tag{3.3}$$

Then

$$\sum_{k=1}^{n} a_k X_k / A_n = \sum_{k=1}^{n} a_k (X_k - Y_k)/A_n + \sum_{k=1}^{n} a_k (Y_k - EY_k)/A_n + \sum_{k=1}^{n} a_k EY_k / A_n = I_1 + I_2 + I_3 (say). \tag{3.4}$$

First, we estimate I_1.

Since $\{X_k - Y_k\}$ is still NQD by Chebyshev inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} a_k (X_k - Y_k)/A_n\right| > \epsilon\right) \\
\le\ & \sum_{n=1}^{\infty} \epsilon^{-2} \left(\sum_{k=1}^{n} a_k^2 E(X_k - Y_k)^2 / A_n^2\right) \\
\le\ & \sum_{n=1}^{\infty} \epsilon^{-2} \sum_{k=1}^{n} \frac{a_k^2}{A_n^2} E\Big[(X_k - Y_k) I(X_k \ne Y_k)\Big]^2 \\
\le\ & C \sum_{n=1}^{\infty} P(X_n \ne Y_n) \\
=\ & C \sum_{n=1}^{\infty} P(|X_n| > b_n) \\
\le\ & C \sum_{n=1}^{\infty} \int_0^{\infty} I(b_n \le x) dP(|X| \le x) \qquad (3.5) \\
\le\ & CEN(X) < \infty.
\end{aligned}$$

By Borel-Cantelli Lemma $P(X_n \ne Y_n,\ i.o.) = 0,\ i.e.,$

$$I_1 = \sum_{k=1}^{n} a_k (X_k - Y_k)/A_n \to 0 \ a.s. \tag{3.6}$$

Secondly we will show that

$$I_2 = \sum_{k=1}^{n} a_k (Y_k - EY_k)/A_n \to 0 \ a.s.$$

Note that $a_k(Y_k - EY_k)/A_n$'s are NQD. Thus it is sufficient to show that

$$\sum_{n=1}^{\infty} (\log n)^2 a_n^2 E(Y_n - EY_n)^2 / A_n^2 < \infty.$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(\log n)^2\frac{a_n^2}{A_n^2}E(Y_n-EY_n)^2\\
\le\ &\sum_{n=1}^{\infty}b_n^{-2}EY_n^2\\
=\ &\sum_{n=1}^{\infty}b_n^{-2}(b_n^2P(|X_n|>b_n)+E|X_n|^2I(|X_n|\le b_n))\\
\le\ &C\sum_{n=1}^{\infty}P(|X|>b_n)+C\sum_{n=1}^{\infty}b_n^{-2}E|X|^2I(|X_n|\le b_n))\\
\le\ &CEN(X)+2C\sum_{n=1}^{\infty}b_n^{-2}\int_0^{b_n}tP(|X|>t)dt\\
=\ &CEN(X)+2C\int_0^{\infty}tP(|X|>t)\sum_{n:b_n>t}b_n^{-2}dt\\
\le\ &CEN(X)+4C\int_0^{\infty}tP(|X|>t)\int_t^{\infty}N(y)y^{-3}dydt\\
<\ &\infty,
\end{aligned}
$$

by (3.1) and (3.2), i.e., we obtain $\sum_{n=1}^{\infty}(\log n)^2E[a_n(Y_n-EY_n)/A_n]^2<\infty$ and thus $\sum_{n=1}^{\infty}a_n(Y_n-EY_n)/A_n$ converges almost surely by Lemma 2.2. Hence, it follows from Kronecker's lemma

$$I_2\to 0\ a.s. \tag{3.7}$$

Finally, since $EX_n=0$, we get

$$
\begin{aligned}
|EY_n| &= \Big|E\Big(-b_nI(X_n<-b_n)+X_nI(|X_n|\le b_n)+b_nI(X_n>b_n)\Big)\Big| \qquad (3.8)\\
&\le |EX_nI(|X_n|\le b_n)|+E|X_n|I(|X_n|>b_n)\\
&\le E|X_n|I(|X_n|>b_n)|+E|X_n|I(|X_n|>b_n)\\
&\le 2E|X|I(|X|>b_n)\to 0\ as\ n\to\infty.
\end{aligned}
$$

Since $a_k/A_n>0$ and $\sum_{k=1}^{n}a_k/A_n=1$, by Toeplitz Lemma, we get

$$I_3\to 0\ a.s. \tag{3.9}$$

Therefore, by (3.6), (3.7) and (3.9), (3.2) follows.

Theorem 3.2. *Let $\{X_n,n\ge 1\}$ be a sequence of pairwise NQD and stochastically dominated with random variable X and $EX_n=0$. Let $\{a_n,n\ge 1\}$ be a* *sequence of positive numbers such that* $A_n = \sum_{i=1}^{n} a_i \uparrow \infty$ *as* $n \to \infty$, $b_1 = \dfrac{A_n}{a_1}$ *and* $b_n = \dfrac{A_n}{a_n \log n}$, $n \geq 2$. *Set* $N(x) = Card\{n, x \geq b_n\}$, $x \in \mathbb{R}$. *If*

$$EN(X) < \infty, \tag{3.10}$$

and

$$\max_{1 \leq j \leq n} b_j^2 \sum_{j=n}^{\infty} b_j^{-2} \leq cn, \tag{3.11}$$

then

$$\sum_{k=1}^{n} a_k X_k / A_n \to 0 \ a.s. \tag{3.12}$$

Proof. Let

$$Y_k = -b_k I\Big(X_k < -b_k\Big) + X_k I\Big(|X_k| \leq b_k\Big) + b_k I(X_k > b_k).$$

By Theorem 3.1 we need only to show that

$$\sum_{i=1}^{n} a_i (Y_i - EY_i)/A_n \to 0 \ a.s. \tag{3.13}$$

Let $c_n = \max_{1 \leq j \leq n} b_j$ and $c_0 = 0$. Then

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\log n)^2 \left(\frac{a_n}{A_n}\right)^2 E(Y_n - EY_n)^2 \\
\leq \ & \sum_{n=1}^{\infty} b_n^{-2} EY_n^2 \\
\leq \ & \sum_{n=1}^{\infty} b_n^{-2} (b_n^2 P(|X_n| > b_n) + E|X_n|^2 I(|X_n| \leq b_n)) \qquad (3.14) \\
\leq \ & C \sum_{n=1}^{\infty} P(|X| > b_n) + C \sum_{n=1}^{\infty} b_n^{-2} E|X|^2 I(|X_n| \leq b_n)) \\
= \ & I_4 + I_5 \ (say).
\end{aligned}$$

For I_4, we have

$$\begin{aligned}
I_4 &= C \sum_{n=1}^{\infty} P(|X| > b_n) \qquad (3.15) \\
&\leq CEN(X) < \infty.
\end{aligned}$$

For I_5, we also have

$$\begin{aligned} I_5 &= C\sum_{n=1}^{\infty} b_n^{-2} E|X|^2 I(|X| \le b_n)) \\ &= C\sum_{n=1}^{\infty} b_n^{-2}\sum_{j=1}^{n} E|X|^2 I(c_{j-1} < |X| \le c_j)) \\ &\le C\sum_{j=1}^{n} P(c_{j-1} < |X| \le c_j)\sum_{n=j}^{\infty} b_n^{-2} \\ &\le C\sum_{j=1}^{\infty} jP(c_{j-1} < |X| \le c_j) \\ &= C\sum_{j=1}^{\infty}\sum_{n=1}^{j} P(c_{n-1} < |X| \le c_n) \qquad (3.16) \\ &= C\sum_{n=1}^{\infty} P(|X| > c_{n-1}) \\ &\le C\left(1+\sum_{n=1}^{\infty} P(|X| > b_n)\right) \\ &\le C\Big(1+EN(X)\Big) < \infty. \end{aligned}$$

From (3.15) and (3.16) and Lemma 2.2 we have $\sum_{n=1}^{\infty} a_n(Y_n - EY_n)/A_n$ converges almost surely. Hence by Kronecker's Lemma we obtain (3.13), that is,

$$\sum_{i=1}^{n} a_i(Y_i - EY_i)/A_n \to 0 \ a.s.$$

Thus the proof is complete.

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