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SOME CONSIDERATIONS ABOUT UPPER BOUNDS ON THE SUM OF SQUARES OF DEGREES

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ABSTRACT. Goodman([3]) presented the question of finding a best possible upper bound of the form $t(G) + t(G^c)$, where $t(G)$ denote the number of triangles in given graph G. In this, the form of squares of degrees is appeared and many researches have been pursued as an application related to this. Here, we would like to deal with corollary related to the results of Nikiforov([6]).

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1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices and e edges. Let d_i be the degree of the ith vertex $v_i \in V$ and m_i the average of the degrees of the vertices adjacent to vertex $v_i \in V$. A form $\sum_{i=1}^{n} {d_i}^2$ calls the sum of squares of the degrees of a graph. Hilbert thought on a sum of squares of an integer ordered pairs in his 17th problem[8], and Goodman([3]) used this form in the problem to seek the number of triangles in given graph G. Afterward, many researches have been pursued with respecting to find the least upper bounded of $\sum_{i=1}^{n} d_i^2$.

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In 1959 Goodman used this form in the paper titled by 'on sets of acquaintances and strangers at any party' and this is as following;
When $t(G)$ denote the number of triangles in G, $t(G)+t(G^c)$ expressed by

$$t(G)+t(G^c)=\frac{1}{2}\sum_{v_i\in V}\left(d_i-\frac{n-1}{2}\right)^2+\frac{n(n-1)(n-5)}{24},$$

where G^c denotes the complement of G.
Besides, Goodman raised the question of finding a best possible upper bound of the form $t(G)+t(G^c)\le B(n,e)$([3]), and it was proved by Olpp([7]) on 1996.

In this paper, we would like to show that the condition for

$$\sum_{i=1}^{n} d_i^2 = e\left(\frac{2e}{n-1}+n-2\right)$$

holding, and deal with corollary related to the results of Nikiforov([6]).

2. upper bounds on the sum of squares of degrees

All graphs considered here are simple one. Let $G=(V,E)$ be a simple graph with n vertices and e edges, and we assume that $n\ge 2$ in order to avoid triviality.

We would like to recall some known upper bounds for $\sum_{i=1}^{n} d_i^2$.

1) Székely:

$$\sum_{i=1}^{n} d_i^2 \le \left(\sum_{i=1}^{n}\sqrt{d_i}\right)^2$$

2) Caen([1]):

$$\sum_{i=1}^{n} d_i^2 \le e\left(\frac{2e}{n-1}+n-2\right)$$

3) Das([2]):

$$\sum_{i=1}^{n} d_i^2 \le e\left(\frac{2e}{n-1}+\frac{n-2}{n-1}d_1+(d_1-d_n)(1-\frac{d_1}{n-1})\right)$$

and

$$\sum_{i=1}^{n} d_i^2 \le 2e(d_1+d_n)-nd_1d_n$$

where, d_1, d_n are respectively the highest degree and the lowest degree of a graph G.

4) Nikiforov([6]):

$$\sum_{i=1}^{n} d_i^2 \leq e\sqrt{8e+1} - e$$

if $e \geq n(n-1)/4$, and

$$\sum_{i=1}^{n} d_i^2 \leq F(n,e)$$

where, $F(n,e) = (2e)^{3/2}$ if $e \geq n^2/4$, and $(n^2-2e)^{3/2} + 4en - n^3$ if $e < n^2/4$.

Despite these impressive works, it is remains some research points until now. It is needed that the conciseness of nice upper bound on $\sum_{i=1}^{n} d_i^2$. Caen's result is simple and Das' is some complicated but better than Caen's. Because, the right side of 3) can be written by

$$e\left(\frac{2e}{n-1} + n - 2 - (n-2-(d_1-d_n))(1-\frac{d_1}{n-1})\right),$$

and this is always less than that of 2). On the other hand, Nikiforov's result seems to be nice but it holds only for half of the range of edge e, and this has some vagueness. Of course, he showed that if $e < (n-1)(n-2)/2$,

$$e\sqrt{8e+1} - e \leq e(\frac{2e}{n-1} + n - 2).$$

In this paper, In here, we would like to deal with corollary related to the results of Nikiforov([6]).

A graph $G = (V, E)$ is said to be bipartite if there is a partition $\{V_1, V_2\}$ of V such that if $\{u, v\} \in E$, then either $u \in V_1$ and $v \in V_2$, or $u \in V_2$ and $v \in V_1$([9]). A complete bipartite graph of the form $K_{1,s}$ calls a star graph. The complete graph with n vertices will be denoted by K_n.

Theorem 2.1. *If a given graph G is a star or complete* ($\supset (K_n \cup K_1)$)*, then the equality*

$$\sum_{i=1}^{n} d_i^2 = e\left(\frac{2e}{n-1} + n - 2\right)$$

holds.

Proof. *Case 1)* Assume that the graph G is complete bipartite graph has the form $K_{r,s}$ or $K_{s,r}$. Then $\sum d_i^2$ can be denoted by $\sum_{i=1}^{n} d_i^2 = rs^2 + sr^2 = rs(r+s)$, and from the $n = r + s$ and $e = rs$,

$$\begin{aligned}\sum d_i^2 - e\left(\frac{2e}{n-1} + n - 2\right) &= rs(r+s) - rs\left(\frac{2rs}{r+s-1} + r + s - 2\right) \\ &= rs\left(2 - \frac{2rs}{r+s-1}\right) \\ &= rs \cdot \frac{2(r+s-1) - 2rs}{r+s-1} \\ &= rs \cdot \frac{-2(r-1)(s-1)}{r+s-1}.\end{aligned}$$

Since G is a star graph, the equality clearly holds.

Case 2) If we assume that a graph G is complete, $2e = n(n-1)$ holds and

$$e\left(\frac{2e}{n-1} + n - 2\right) = \frac{n(n-1)}{2}(2n-2) = n(n-1)^2.$$

Since $\sum_{i=1}^{n} d_i^2 \le e\left(\frac{2e}{n-1} + n - 2\right)$ is shown by Caen already, we just show that $\sum_{i=1}^{n} d_i^2 \ge e\left(\frac{2e}{n-1} + n - 2\right)$.

Assume that a graph G has n vertices and $\deg(G) = s$. Then $\sum_{i=1}^{n} d_i = 2e$ and

$$\begin{aligned}\sum_{i=1}^{n} d_i^2 \ge \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2 &= \frac{1}{n} \cdot 4e^2 = \frac{1}{n} \cdot n^2(n-1)^2 \\ &= n(n-1)^2 = e\left(\frac{2e}{n-1} + n - 2\right).\end{aligned}$$

So, the proof is complete. □

In general, a complete graph and a complete bipartite graph are another concept. A complete bipartite graph is bipartite graph and not a complete graph too([4]). Here, A thought occurred to naturally is Theorem 2.1 holds in a bipartite graph? The answer is no; for it does not hold $e = rs$.

A graph is called regular of degree r if all the vertices of it have the same degree r([4]).

Example 2.2. Theorem 2.1 does not hold for a graph G is a complete bipartite or regular.

Proof. For the reason of following, it does not hold; $K_{2,3}$ consists of $e = 6$, $n = 5$ and $d = 3 + 3 + 2 + 2 + 2 = 12$ but, $\sum {d_i}^2 = 30$ and $e\left(\frac{2e}{n-1} + n - 2\right) = 36$.

On the other hand, suppose that a graph G is regular of degree r. Then $2e = nr$, $d_i = r$ for all i and $\sum_{i=1}^{n} d_i^2 = nr^2$ holds. Thus,

$$e\left(\frac{2e}{n-1} + n - 2\right) = \frac{nr}{2}\left(\frac{nr}{n-1} + n - 2\right)$$

and since

$$\left(\frac{nr}{n-1} + n - 2\right) - 2r = \frac{(n-2)(n-r-1)}{n-1} \geq 0,$$

$$e\left(\frac{2e}{n-1} + n - 2\right) \geq \frac{nr}{2} \cdot 2r = nr^2 = \sum_{i=1}^{n} d_i^2$$

holds. □

For $n = 2$ or 3, we can easily know that regular graph G satisfies the equality of theorem 2.1.

We would like to consider an upper bound from the upper bound of Das([2]). Let G be a simple graph with n vertices and e edges, and let d_1, d_n are respectively the highest degree and the lowest degree of a graph G. For m_i is the average of the degree of the vertices adjacent to vertex v_i, Das([2]) proved

$$d_i + m_i \leq \frac{2e}{n-1} + \frac{n-2}{n-1}d_1 + (d_1 - d_n)\left(1 - \frac{d_1}{n-1}\right)$$

holds for each non-isolated vertex v_i.

Multiplying both sides by d_i, taking summation over i and putting $\sum_{i=1}^{n} d_i^2 = x$, we can get

$$2x - (n - 2 - (d_1 - d_n))\frac{x}{n-1} \leq \frac{2e}{n-1}(2e + (n-1)(d_1 - d_n)).$$

Since $n-1$ and $n-(d_1-d_n)$ are both positive, we can obtain

$$x=\sum_{i=1}^{n} d_i^2 \leq \frac{2e(2e+(n-1)(d_1-d_n))}{n-(d_1-d_n)}.$$

In the above, we observed the denominator is $n-d_1+d_n$, and since the right-hand side of this inequality is derived from upper bound of Das, it is clearly less than Caen's.

We would like to think about the great and small sizes between $e\sqrt{8e+1}-e$ and a member of $F(m,n)$. Recall that $F(n,e)=(2e)^{3/2}$ if $e \geq n^2/4$, and $(n^2-2e)^{3/2}+4en-n^3$ if $e<n^2/4$.

Theorem 2.3. $e\sqrt{8e+1}-e<(2e)^{3/2}$.

Proof.

$$(e\sqrt{8e+1}-e)-(2e)^{3/2}=e(\sqrt{8e+1}-1-2^{3/2}\sqrt{e})$$

and let us show that $\sqrt{8e+1}-2^{3/2}\sqrt{e}$ has a value in $(0,1)$.

$$(\sqrt{8e+1})^2-\left(2^{3/2}\sqrt{e}\right)^2=1>0$$

and so,

$$\left(\sqrt{8e+1}-2^{3/2}\sqrt{e}\right)\left(\sqrt{8e+1}+2^{3/2}\sqrt{e}\right)=1.$$

Thus,

$$\sqrt{8e+1}-2^{3/2}\sqrt{e}=\frac{1}{\sqrt{8e+1}+2^{3/2}\sqrt{e}}$$

and because the denominator bigger than 1, we can conclude that $0<\sqrt{8e+1}-2^{3/2}\sqrt{e}<1$. Implies,

$$\left(e\sqrt{8e+1}-e\right)-(2e)^{3/2}=e\left(\sqrt{8e+1}-1-2^{3/2}\sqrt{e}\right)<0.$$

□

Theorem 2.4. *If the graph G is complete, then $e\sqrt{8e+1}-e<(n^2-2e)^{3/2}+4en-n^3$ holds.*

Proof. Since the graph G is complete, $2e=n(n-1)$ holds.

$$\begin{aligned}
&(e\sqrt{8e+1}-e)-\Big((n^2-2e)^{3/2}+4en-n^3\Big)\\
&=e\Big(\sqrt{8e+1}-1-4n\Big)+n^3-(n^2-2e)^{3/2},\\
&=\frac{n(n-1)}{2}\Big(\sqrt{4n(n-1)+1}-1-4n\Big)+n^3-n^{3/2}.
\end{aligned}$$

Since $n \geq 2$, $\sqrt{4n^2-4n+1} = |2n-1| = 2n-1$ and so, we get

$$\begin{aligned}
\frac{n(n-1)}{2}(-2n-2)+n^3-n^{3/2} &= -(n-1)n(n+1)+n^3-n^{3/2}\\
&= n\Big(-(n^2-1)+n^2-\sqrt{n}\Big)\\
&= n(1-\sqrt{n}).
\end{aligned}$$

Since $n \geq 2$, $n(1-\sqrt{n}) < 0$ and so, we can conclude $e\sqrt{8e+1}-e < (n^2-2e)^{3/2}+4en-n^3$. □

Theorem 2.5. (Nikiforov([6])) *If $e < (n-1)(n-2)/2$, then $e\sqrt{8e+1}-e < e(\frac{2e}{n-1}+n-2)$ holds.*

By contraposition, if $e\sqrt{8e+1}-e \geq e\Big(\dfrac{2e}{n-1}+n-2\Big)$, $e \geq (n-1)(n-2)/2$ holding.

Here, we would like to consider the case to holding the equality.

Corollary 2.6. *If $e = (n-1)(n-2)/2$, then $e\sqrt{8e+1}-e = e\Big(\dfrac{2e}{n-1}+n-2\Big)$.*

Proof. We assume that $2e = (n-1)(n-2)$ and show that

$$\sqrt{8e+1} = \frac{2e}{n-1}+n-1.$$

Since $2n-3$ is positive,

$$\sqrt{8e+1} = \sqrt{4(n-1)(n-2)+1} = \sqrt{4n^2-12n+9} = 2n-3$$

and

$$\frac{2e}{n-1}+n-1 = 2n-3.$$

So, the proof is complete. □

By the corollary 2.6, we can restate the statement of Nikiforov as following; If $e \leq (n-1)(n-2)/2$, then $e\sqrt{8e+1}-e \leq e\Big(\dfrac{2e}{n-1}+n-2\Big)$.

Corollary 2.7. *If* $e\sqrt{8e+1}-e=e\Big(\frac{2e}{n-1}+n-2\Big)$, *then* $e=(n-1)(n-2)/2$ *or* $e=n(n-1)/2$.

Proof. Let $e\sqrt{8e+1}-e=e\Big(\frac{2e}{n-1}+n-2\Big)$. Then, by the simple calculation, we can get

$$4e^2-4e(n-1)^2+(n-1)^4-(n-1)^2=0.$$

Implies,

$$\begin{aligned} e &= \frac{2(n-1)^2 \pm \sqrt{4(n-1)^4-4(n-1)^4+4(n-1)^2}}{4} \\ &= \frac{(n-1)^2 \pm (n-1)}{2} \\ &= n(n-1)/2,\ (n-2)(n-1)/2. \end{aligned}$$

□

In the above, both value of e satisfies positivity, and thanks to Nikiforov's result, we can obtain the following result.

Corollary 2.8. *In case of* $\frac{1}{2}(n-2)(n-1)\le e\le \frac{1}{2}(n-1)n$, $e\sqrt{8e+1}-e<e\Big(\frac{2e}{n-1}+n-2\Big)$ *does not hold.*

Proof. Assume that $e\sqrt{8e+1}-e<e\Big(\frac{2e}{n-1}+n-2\Big)$ holds. Dividing both sides with e, we get

$$\frac{2e}{n-1}+n-1>\sqrt{8e+1}$$

and multiplying both sides by $n-1$,

$$2e+(n-1)^2>(n-1)\sqrt{8e+1}$$

follows. Squaring both sides, we get

$$4e^2+(n-1)^4+4e(n-1)^2>(n-1)^2(8e+1)$$

$$(n-1)^2\Big\{(n-1)^2+(-4e-1)\Big\}+4e^2>0$$

So, in order to this inequality becomes always positive, $-4e-1>0$ requiring. Thus,

$$e<-1/4$$

and this is a contradiction. □

Let us consider a simple example as the following.

Consider the complete graph K_3. Then this satisfies the hypothesis of corollary 2.8 and

$$e\sqrt{8e+1}-e=12=e\left(\frac{2e}{n-1}+n-2\right)=12$$

holds.

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