J. Appl. Math. & Informatics Vol. **27**(2009), No. 3 - 4, pp. 857 - 864
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REMARKS ON CONFORMAL TRANSFORMATION ON RIEMANNIAN MANIFOLDS

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ABSTRACT. The special conformally flatness is a generalization of a subprojective space. B. Y. Chen and K. Yano ([4]) showed that every canal hypersurface of a Euclidean space is a special conformally flat space. In this paper, we study the conditions for the base space B is special conformally flat in the conharmonically flat warped product space $B^n \times_f R^1$.

AMS Mathematics Subject Classification : 53B20, 53C21.
Key words and phrases : Conformal transformation, special conformally flat, conharmonically flat.

1. Introduction

The conformal transformation on the Riemannian manifold does not change the angle between two vectors at a point and characterized by a change of a Riemannian metric. Conformal flatness is equivalent to $C = 0$ for $m > 3$ and $D = 0$ for $m = 3$ in the m-dimensional Riemannian manifold(see §2 for definitions of C and D).

On the other hand, conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. The conharmonic curvature tensor is invariant under the conharmonic transformation and conharmonically flat is equivalent to conformally flat and scalar curvature vanishes.

In [4], B. Y. Chen and K. Yano introduced the notion of special conformally flat spaces which generalizes that of subprojective space. Also they showed that every conformally flat hypersurface of a Euclidean space (hence of a conformally flat space) is special, and every canal hypersurface of a Euclidean space is a special conformally flat space.

Received February 2, 2009. Accepted March 23, 2009. *Corresponding author.
†The first author was supported by the Research Grant 2004 of Kyung Hee University .

In this point of a view, we shall study the conharmonically flat warped product space $M = B^n \times_f R^1$ of the n-dimensional Riemannian manifold(B, g) and R^1. We shall investigate the condition for B is special conformally flat, and study the geometric characterization of M and B.

Also we can construct new special conformally flat spaces by use of the Theorems 3 and 9.

2. A conformal transformation and a conformal curvature tensor

A conformal transformation between two Riemannian manifolds (M, g) and (M', g') is a diffeomorphism preserving angle measured by the metrics g and g'. It is characterized by

$$g' = e^{2\rho} g \tag{2.1}$$

where ρ is a scalar function. In this case g and g' are said to be conformally equivalent. If the function ρ is constant, then the conformal transformation is said to be homothetic. The Weyl conformal curvature tensor C in an m-dimensional Riemannian manifold M is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{m-2}\Big\{S(Y,Z)X - g(X,Z)QY + g(Y,Z)QX \\ -S(X,Z)Y\Big\} + \frac{K}{(m-1)(m-2)}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}, \end{aligned} \tag{2.2}$$

where R, S and K are curvature tensor, Ricci curvature tensor and scalar curvature of M respectively and $g(QX, Y) = S(X, Y)$. The Weyl conformal curvature 3-tensor D is defined by

$$\begin{aligned} D(X,Y)Z = \nabla_X S(Y,Z) - \nabla_Y S(X,Z) \\ -\frac{1}{2(m-1)}\Big\{g(Y,Z)(XK) - g(X,Z)(YK)\Big\} \end{aligned} \tag{2.3}$$

or equivalently,

$$D(X,Y)Z = \nabla_X L(Y,Z) - \nabla_Y L(X,Z), \tag{2.4}$$

where we have put

$$L(X,Y) = S(X,Y) - \frac{K}{2(m-1)} g(X,Y).$$

It is well known that ([2,3]) M is conformally flat if and only if $C = 0$ for $m > 3$, $D = 0$ for $m = 3$. In general, the harmonicity of functions is not preserved by the conformal transformation. Related this fact, Y. Ishi ([5]) introduced the conharmonic transformation, which is defined by a conformal transformation preserving the harmonicity of a certain function. It is easily seen that conformally flat manifold is conharmonically flat if and only if the scalar curvature vanishes.

3. Special conformally flat space

Let (B, g) be a n-dimensional Riemannian manifold with Riemannian metric g and let $M = B^n \times_f R^1$ be a warped product Riemannian manifold where $f : B \to R^+$ a warping function and this metric tensor

$$(\tilde{g}_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & f^2 \end{pmatrix}, \tag{3.1}$$

where the range of indices $a, b, c, d, \cdots$ is $\{2, 3, \cdots n+1\}$.

Then the Christoffel symbols $\widetilde{\left\{ \begin{smallmatrix} h \\ ij \end{smallmatrix} \right\}}$ of M are given by ([6,7])

$$\begin{aligned}
\widetilde{\left\{ \begin{matrix} a \\ bc \end{matrix} \right\}} &= \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} \\
\widetilde{\left\{ \begin{matrix} a \\ 11 \end{matrix} \right\}} &= -ff^a \\
\widetilde{\left\{ \begin{matrix} 1 \\ 1a \end{matrix} \right\}} &= \frac{f_a}{f} \\
\widetilde{\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\}} &= \overline{\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\}}
\end{aligned} \tag{3.2}$$

and the others are zero, where $f_b = \nabla_b f$, $f^a = f_b g^{ba}$ and the range of indices $h, i, j, k, \cdots$ is $\{1, 2, \cdots, n, n+1\}$. Let $\tilde{R}$, R and $\bar{R}$ be the curvature tensor of M, B and R^1 respectively. Then we have

$$\begin{aligned}
\tilde{R}_{dcb}{}^{a} &= R_{dcb}{}^{a} \\
\tilde{R}_{d1b}{}^{1} &= \frac{1}{f}\nabla_d f_b
\end{aligned} \tag{3.3}$$

and the others are zero.

Hence the Ricci curvature tensors $\tilde{S}$, S and $\bar{S}$ for M, B and R^1 respectively are given by

$$\begin{aligned}
\tilde{S}_{cb} &= S_{cb} - \frac{1}{f}(\nabla_c f_b), \\
\tilde{S}_{c1} &= 0, \\
\tilde{S}_{11} &= -f(\triangle f)
\end{aligned} \tag{3.4}$$

where $\triangle f$ is the Laplacian of f for g. The scalar curvatures $\tilde{K}$, K and $\bar{K}$ for

M, B and R^1 respectively are related by

$$\tilde{K} = K - \frac{2\triangle f}{f}. \tag{3.5}$$

If $\tilde{K} = 0$ then $K = \frac{2}{f}\triangle f$. If B is compact, then $\int_B fK d\sigma = 2\int_B \mathrm{div}(\nabla_i f) d\sigma = 2\int_B \triangle f d\sigma = 0$ by Green's theorem. Since f is positive on B, we have

Lemma 1. *Let $M = B^n \times_f R^1$ be a warped product Riemannian manifold with $\tilde{K} = 0$. If B is compact and K is constant, then $K = 0$.*

Since $\frac{1}{2}\triangle f^2 = f\triangle f + \|f_a\|^2$, we can see that

$$\int_B (f\triangle f + \|f_a\|^2) d\sigma = 0 \tag{3.6}$$

on the compact manifold B by the Green's Theorem. If $\tilde{K} = 0$ and K is constant on a compact manifold B, then, by Lemma 1, $K = 0$. So, by (3.5), $\triangle f = 0$. Hence the equation (3.6) gives $f_a = 0$, that is, f is a constant function. Thus we have

Theorem 2. *Let $M = B^n \times_f R^1$ be warped product Riemannian manifold and $\tilde{K} = 0$ and B is compact. If K is constant, then M is Riemannian product manifold.*

Next, let $M = B^n \times_f R^1$ be a conharmonically flat warped product space with $K > 0$. Then the Riemannian curvature tensor $\tilde{R}$ on M are given by ([1,5])

$$\tilde{R}_{kji}{}^h = \frac{1}{n-1}\left(\tilde{S}_{ji}\delta_k^h - \tilde{S}_{ki}\delta_j^h + \tilde{S}_k{}^h\tilde{g}_{ji} - \tilde{S}_j{}^h\tilde{g}_{ki}\right). \tag{3.7}$$

Using (3.3), (3.4) and (3.7), we get

$$\begin{aligned} R_{dcb}{}^a &= \frac{1}{n-1}\Big(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^a g_{cb} - S_c{}^a g_{db}\Big) \\ &\quad - \frac{1}{(n-1)f}\Big(\delta_d^a\nabla_c f_b - \delta_c^a\nabla_d f_b + g_{cb}\nabla_d f^a - g_{db}\nabla_c f^a\Big), \end{aligned} \tag{3.8}$$

and that

$$S_{cb} = Kg_{cb} - \frac{n-2}{f}\nabla_c f_b - \frac{\triangle f}{f} g_{cb}, \tag{3.9}$$

$$K = \frac{2\triangle f}{f}. \tag{3.10}$$

Hence it is easily obtained that

$$\nabla_c f_b = \frac{f}{n-2}\left(\frac{K}{2}g_{cb} - S_{cb}\right). \tag{3.11}$$

From (3.8) and (3.11), we get

$$\begin{aligned} R_{dcb}{}^{a} &= \frac{1}{n-2}\left(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^a g_{cb} - S_c{}^a g_{db}\right) \\ &- \frac{K}{(n-1)(n-2)}\left(g_{cb}\delta_d^a - g_{db}\delta_c^a\right), \end{aligned} \tag{3.12}$$

that is, B is conformally flat if $n > 3$.

On the other hand, L on B is defined by

$$L_{cb} = -\frac{S_{cb}}{n-2} + \frac{K}{2(n-1)(n-2)}g_{cb} \tag{3.13}$$

and, using (3.9) and (3.11), L is reduced to

$$L_{cb} = -\frac{g_{cb}}{2(n-1)}K + \frac{1}{f}\nabla_c f_b\ . \tag{3.14}$$

If there exist, on a conformally flat space B, two functions α and β such that α is positive and

$$L_{cb} = -\frac{\alpha^2}{2}g_{cb} + \beta\alpha_c\alpha_b \tag{3.15}$$

then B is called a special conformally flat space[4], where we have put $\alpha_c = \partial_c\alpha$.

If we put

$$\alpha = \sqrt{\frac{K}{n-1}}, \qquad \beta = \frac{4(n-1)K}{fK_cK_b}\nabla_c f_b, \tag{3.16}$$

and considering (3.14), then (3.15) is satisfied.

Thus we obtain the following theorem.

Theorem 3. *Let $M = B^n \times_f R^1$ be a conharmonically flat warped product space with $n > 3$. If $K > 0$, then B is special conformally flat space.*

Since a Riemannian manifold has the harmonic curvature if and only if the scalar curvature is constant and $D = 0$ ([8]), we can state

Propositon 4. *Let $M = B^3 \times_f R^1$ be a conharmonically flat warped product space. If B has the harmonic curvature and $K > 0$, then B is special conformally flat space.*

If f is concircular, then we get

$$\nabla_c f_b = \frac{fK}{2n}g_{cb} \tag{3.17}$$

from (3.10)and (3.11). So (3.9) and (3.12) induce

$$S_{cb} = \frac{K}{n} g_{cb}. \tag{3.18}$$

If we substitute (3.18) into (3.12), we can see that

$$R_{dcb}{}^{a} = \frac{K}{n(n-1)} (g_{cb}\delta_d^a - g_{db}\delta_c^a). \tag{3.19}$$

Conversely, if B is a space of constant curvature, then the equations (3.12) and (3.19) imply

$$\nabla_c f_b = \lambda g_{cb},$$

that is, f is concircular. Hence we have

Lemma 5. *Let $M = B^n \times_f R^1$ be a conharmonically flat warped product space with $n > 2$. Then B is a space of constant curvature if and only if f is concircular.*

If $n = 2$, then we have

$$S_{cb} = \left(K - \frac{\Delta f}{f}\right) g_{cb}$$

and that

$$\frac{K}{2} = \frac{\Delta f}{f}.$$

Hence $S_{cb} = \frac{K}{2} g_{cb}$, that is, B is Einstein if K is constant. Since 2-dimensional Einstein space is a space of constant curvature, we have

Propositon 6. *Let $M = B^2 \times_f R^1$ be a conharmonically flat warped product space. If K is constant, then B is a space of constant curvature.*

If B is compact and K is constant, then the conharmonically flat warped product space $M = B^n \times_f R^1$ is Riemannian products by Theorem 2, that is f is a constant function. Hence $R = 0$ by (3.8) and that M is locally Euclidean. Thus we have

Theorem 7. *Let $M = B^n \times_f R^1$ be a conharmonically flat warped product space. If K is constant and B is compact, then M is locally Euclidean.*

The fact that $\tilde{K} = 0$ on a conharmonically flat space and Lemma 1 give the following proposition.

Proposition 8. *Let $M = B^n \times_f R^1$ be a conharmonically flat warped product space with $n > 1$. If B is a compact and K is constant, then $K = 0$.*

By Lemma 5, if B is a space of constant curvature in the conharmonically flat warped product space $M = B^n \times_f R^1$ $(n > 2)$, then f is concircular. From this fact, (3.10) and (3.14) give

$$L_{cb} = -\frac{1}{2}\left(\frac{K}{n-1} - \frac{2\Delta f}{nf}\right) g_{cb} \tag{3.20}$$

and

$$\frac{K}{n-1} - \frac{2\Delta f}{nf} = \frac{K}{n(n-1)}. \tag{3.21}$$

Since $K > 0$, if we put

$$\alpha = \sqrt{\frac{K}{n-1}} \tag{3.22}$$

then $L_{cb} = -\frac{\alpha^2}{2} g_{cb}$. Since B is conformally flat from (3.12), B is a special conformally flat space. Conversely, if B is a special conformally flat space with $\beta = 0$, then B is conformally flat and f is concircular by use of (3.14) and (3.15). So,by Lemma 5, B is a space of constant curvature. Thus we have

Theorem 9. *Let $M = B^n \times_f R^1$ be a conharmonically flat warped product space with $n > 3$ and $K > 0$. Then B is a special conformally flat space with $\beta = 0$ if and only if B is a space of constant curvature.*

By Proposition 4 and equations (3.20)-(3.22), we see that

Propositon 10. *Let $M = B^3 \times_f R^1$ be a conharmonically flat warped product space and let B has the harmonic curvature and $K > 0$. Then B is a special conformally flat space with $\beta = 0$ if and only if B is a space of constant curvature.*

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