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NEW GENERALIZED MINTY'S LEMMA

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ABSTRACT. In this paper, we introduce new pseudomonotonicity and proper quasimonotonicity with respect to a given function, and show some existence results for strong implicit vector variational inequalities by considering new generalized Minty's lemma. Our results generalize and extend some results in [1].

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1. Introduction

In the last 40 years, variational inequalities for numerical functions, which were originated from Hartman and Stampacchia [2], have made much development in theory and applications. Minty [3] showed the linearization for the scalar case, which have played useful roles in variational inequalities. In fact, the classical Minty's inequality and Minty's Lemma have been shown to be important tools in the regularity results of the solution for a generalized nonhomogeneous boundary value problem [4] and, when the operator is a gradient, also a minimum principle for convex optimization problems [5]. And Behera and Panda [6] obtained a nonlinear generaliztion of Minty's Lemma. Furthermore, they applied the result to obtain a solution of a certain variational-like inequality. Kassay and Kolumban [7] considered the Minty-type problem for set-valued mapping with

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two variables for the scalar case. For the vector-valued case, some extensions of Minty's Lemma were obtained by many authors [8-23]. Recently, Lee *et al.* [21], Khan *et al.*[23] and Zhao *et al.*[24] considered generalized Minty's Lemmas by extending it to the vector case under certain new pseudomonotone-type or certain new hemicontonuity conditions. On the other hand, it is well-known that monotonicity concepts with continuity concepts have an important role in variational inequality problems[1, 4-23].

In this paper, we introduce new pseudomonotonicity and proper quasimonotonicity with respect to a given function, and consider new generalized Minty's Lemma for strong implicit vector variational inequalities, which extend results in [1].

2. Preliminaries

Definition 2.1. Let X be a topological vector space. A nonempty subset C of X is said to be convex cone if

$$C + C \subseteq C \text{ and } \lambda C \subseteq C, \text{ for all } \lambda \geq 0.$$

A cone C is said to be pointed if

$$C \cap (-C) = \{0\},$$

where 0 denotes the zero vector.

Note that in this paper, $a \geq_C b$ and $a \not\leq_C b$ mean $a - b \in C$ and $b - a \notin C$, respectively.

Throughout this paper, unless other specified, X and Y are real Banach spaces, $K \subset X$ is a nonempty, closed and convex set, $C \subset Y$ a pointed, closed and convex cone in Y with $intC \neq \emptyset$, where $intC$ denotes the interior of C. Denote by $L(X,Y)$ the space of all the continuous linear mappings from X into Y. For any given $l \in L(X,Y)$, $x \in X$, $\langle l, x\rangle$ denotes the value of l at x. Let $T : K \to L(X,Y)$ and $h : K \times K \to X$ be mappings.

Consider the following strong implicit vector variational inequalities of Stampacchia type (**SIVVI**) and Minty type (**MIVVI**):

(**SIVVI**) Find $x \in K$ such that

$$\Big\langle Tx, h(x,y)\Big\rangle \geq_C 0, \ \ \forall y \in K$$

and

(**MIVVI**) Find $x \in K$ such that

$$\Big\langle Ty, h(y,x)\Big\rangle \leq_C 0, \ \ \forall y \in K.$$

Definition 2.2 [1]. A mapping $h : K \to Y$ is said to be hemicontinuous if, for any fixed x, $y \in K$, a mapping $L : [0,1] \to Y$ defined by $L(t) = h((1-t)x + ty)$ is continuous at 0^+, i.e., $\lim_{t \to 0^+} L(t) = L(0)$.

Lemma 2.1 [1]. *Let C be a pointed, closed and convex cone of a real Banach space E. Then for any, $a \in -C$ and $b \notin C$, we have $t_1 a + t_2 b \notin C$ for all t_1, $t_2 > 0$.*

Theorem 2.1 [24]. **(Fan-KKM Theorem)** *Let M be a nonempty subset of a Hausdorff topological vector space E and $G : M \to 2^E$ be a KKM mapping. If $G(x)$ is closed for every x in M and compact for some $x \in M$, then $\bigcap_{x \in M} G(x) \neq \emptyset$.*

3. Main results

In this section, we introduce new pseudomonotonicity and new properly quasi-monotonicity with respect to a given function, and we prove some existence results for strong implicit vector variational inequalities.

Definition 3.1. Let $T : K \to L(X, Y)$ and $h : K \times K \to X$ be mappings. T is said to be pseudomonotone with respect to h if for any x, $y \in K$,

$$\Big\langle Tx, h(x,y) \Big\rangle \geq_C 0 \;\Rightarrow\; \Big\langle Ty, h(y,x) \Big\rangle \leq_C 0.$$

Example 3.1. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $Tx = \begin{pmatrix} x^2 \\ x^4 \end{pmatrix}$, $h(x,y) = y - (x+1)^2$ for all $x, y \in K$.

Let x, $y \in K$ such that

$$\begin{aligned} \langle Tx, h(x,y) \rangle &= \begin{pmatrix} x^2 \\ x^4 \end{pmatrix} (y - (x+1)^2) \\ &= \begin{pmatrix} x^2(y - (x+1)^2) \\ x^4(y - (x+1)^2) \end{pmatrix} \geq_C 0. \end{aligned}$$

The inequality above implies

$$\begin{aligned} y - (x+1)^2 \geq 0 \;&\Rightarrow y \geq (x+1)^2 \\ &\Rightarrow (y+1)^2 \geq y + 1 \geq y \geq (x+1)^2 \geq x \end{aligned}$$

It follows that

$$\begin{aligned}\langle Ty, h(y,x)\rangle &= \begin{pmatrix} y^2 \\ y^4 \end{pmatrix} (x-(y+1)^2) \\ &= \begin{pmatrix} y^2(x-(y+1)^2) \\ y^4(x-(y+1)^2) \end{pmatrix} \leq_C 0.\end{aligned}$$

Hence T is pseudomonotone with respect to h.

Now, we introduce more general quasimonotonicity.

Definition 3.2. Let $T: K \to L(X,Y)$ and $h: K \times K \to X$ be mappings.

(1) T is said to be properly quasimonotone of Stampacchia type with respect to h if for all $m \in \mathbb{N}$, for all vectors $v_1, \cdots, v_m \in K$, and scalars $\lambda_1, \cdots, \lambda_m > 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $u := \sum_{i=1}^{m} \lambda_i v_i$, $\langle Tu, h(u,v_i)\rangle \geq_C 0$ holds for all i.

(2) T is said to be properly quasimonotone of Minty type with respect to h if for all $m \in \mathbb{N}$, for all vectors $v_1, \cdots, v_m \in K$ and scalars $\lambda_1, \cdots, \lambda_m > 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $u := \sum_{i=1}^{m} \lambda_i v_i$, $\langle Tv_i, h(v_i,u)\rangle \leq_C 0$ holds for all i.

Example 3.2. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $Tx = \begin{pmatrix} x^2 \\ x^3 \end{pmatrix}$, $h(x,y) = -x + x^2 - y^2$, for all $x, y \in K$. Suppose that there exist $x_1, \cdots, x_m \geq 0$ and $\lambda_1, \cdots, \lambda_m > 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\Big\langle Tx, h(x,x_i)\Big\rangle \not\geq_C 0,$$

where $x = \sum_{i=1}^{m} \lambda_i x_i$. It follows that

$$\begin{aligned}\langle Tx, h(x,x_i)\rangle &= \begin{pmatrix} x^2 \\ x^3 \end{pmatrix} (-x_i^2 - x + x^2) \\ &= \begin{pmatrix} x^2(-x_i^2 - x + x^2) \\ x^3(-x_i^2 - x + x^2) \end{pmatrix} \not\geq_C 0, \quad i = 1, \cdots, m,\end{aligned}$$

which is a contradiction since $x^2(-x_i^2 - x + x^2) \geq 0$ and $x^3(-x_i^2 - x + x^2) \geq 0$ for at least one i. Hence T is properly quasimonotone of Stampacchia type with respect to h.

And suppose that there exist $x_1, \cdots, x_m \geq 0$ and $\lambda_1, \cdots, \lambda_m > 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\langle Tx_i, h(x_i, x)\rangle \not\leq_C 0.$$

where $x = \sum_{i=1}^{m} \lambda_i x_i$. It follows that

$$\begin{aligned}\langle Tx_i, h(x_i, x)\rangle &= \begin{pmatrix} x_i^2 \\ x_i^3 \end{pmatrix} (-x^2 - x_i + x_i^2) \\ &= \begin{pmatrix} x_i^2(-x^2 - x_i + x_i^2) \\ x_i^3(-x^2 - x_i + x_i^2) \end{pmatrix} \not\leq_C 0, \quad i = 1, \cdots, m,\end{aligned}$$

which is also a contradiction since $x_i^2(-x^2-x_i+x_i^2) \leq 0$ and $x_i^3(-x^2-x_i+x_i^2) \leq 0$ for some i. Hence T is properly quasimonotone of Minty type with respect to h.

Lemma 3.1. *Let $T : K \to L(X, Y)$ and $h : K \times K \to X$ be mappings. Suppose that T is pseudomonotone and properly quasimonotone of Stampacchia type with respect to h. Then T is properly quasimonotone of Minty type with respect to h.*

Proof. By the pseudomonotonicity of T with respect to h, it is easily proved. □

We consider a new generalized Minty's Lemma, which extends some results in [1].

Theorem 3.1. *Let $T : K \to L(X, Y)$ and $h : K \times K \to X$ be mappings satisfying the following conditions;*

(1) *T is pseudomonotone with respect to h;*
(2) *for any fixed $v \in K$, the mapping $u \mapsto \langle Tu, h(u, v)\rangle$ is hemicontinuous;*
(3) *$\langle Tu, h(u, u)\rangle \in C$ for all $u \in K$;*
(4) *h is bilinear.*

Then for a given point $x_0 \in K$, the following conclusions are equivalent

(*i*) *$\langle Tx_0, h(x_0, x)\rangle \geq_C 0$, $\forall x \in K$;*
(*ii*) *$\langle Tx, h(x, x_0)\rangle \leq_C 0$, $\forall x \in K$.*

Proof. (ii) is easily shown from (i) by the condition (1).

Conversely, for any given $z \in K$ and $t \in (0, 1)$, let $z_t = x_0 + t(z - x_0)$. It follows from (ii) that

$$\Big\langle T(z_t), h(z_t, x_0)\Big\rangle \leq_C 0.$$

Now we show that $\langle T(z_t), h(z_t, z)\rangle \geq_C 0$ holds for all $t \in (0,1)$. Suppose that there exists some $s \in (0,1)$ such that

$$\Big\langle T(z_s), h(z_s, z)\Big\rangle \not\geq_C 0.$$

By Lemma 2.1 and the bilinearity of h, we have

$$\Big\langle T(z_s), h(z_s, z_s)\Big\rangle = \Big\langle T(z_s), h(z_s, x_0 + s(z - x_0))\Big\rangle$$

$$\begin{aligned} &= \Big\langle T(z_s), h((1+s-s)z_s, (1-s)x_0 + sz)\Big\rangle \\ &= s\Big\langle T(z_s), h(z_s, z)\Big\rangle + (1-s)\Big\langle T(z_s), h(z_s, x_0)\Big\rangle \\ &\notin C, \end{aligned}$$

which contradicts the condition (3).

Hence $\Big\langle T(z_t), h(z_t, z)\Big\rangle \geq_C 0$, $\forall t \in (0,1)$. From the condition (2), a function $L : [0,1] \to L(X,Y)$ defined by

$$L(t) = \Big\langle T(u + t(u - v)), h(u + t(u - v), v)\Big\rangle$$

is continuous at 0^+. Thus

$$\begin{aligned} \Big\langle Tx_0, h(x_0, x)\Big\rangle &= \lim_{t \to 0^+} \Big\langle T(x_0 + t(x - x_0)), h(x_0 + t(x - x_0), x)\Big\rangle \\ &= \lim_{t \to 0^+} \Big\langle T(z_t), h(z_t, z)\Big\rangle \geq_C 0, \quad \forall x \in K. \quad \square \end{aligned}$$

Now we consider the existence of solutions to (**SIVVI**), which extends some results in [1].

Theorem 3.2. *Let $K \subset X$ be a nonempty compact and convex set. Let $T : K \to L(X,Y)$ and $h : K \times K \to X$ be mappings satisfying the following conditions;*

(1) *For any fixed $v \in K$, the mapping $u \mapsto \langle Tu, h(u,v)\rangle$ is continuous;*

(2) *T is properly quasimonotone of Stampacchia type with respect to h;*

(3) *$\Big\langle Tu, h(u,u)\Big\rangle \geq_C 0$ for all $u \in K$.*

Then there exists $x \in K$ such that

$$\Big\langle Tx, h(x,y)\Big\rangle \geq_C 0, \quad \forall y \in K.$$

Proof. Define a multivalued mapping $S_T : K \to 2^K$ as follows:

$$S_T(z) = \Big\{x \in K : \langle Tx, h(x,z)\rangle \geq_C 0\Big\}, \quad \forall z \in K.$$

Obviously, $S_T(z) \neq \emptyset$ by the condition (3). We claim that S_T is a KKM mapping. Suppose that there exist $\{x_1, \cdots, x_n\} \subset K$, $x = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ such that $x \notin \bigcup_{i=1}^{n} S_T(x_i)$. Hence we have

$$\Big\langle Tx, h(x, x_i)\Big\rangle \not\geq_C 0, \quad i = 1, \cdots, n,$$

which contradicts the condition (2). Therefore, S_T is a KKM mapping. On the other hand, from the condition (1), $S_T(z)$ is closed. Since $S_T(z)$ is closed subset of compact set K, $S_T(z)$ is also compact for all $z \in K$. By Fan-KKM Theorem, $\bigcap_{z \in K} S_T(z) \neq \emptyset$. Hence there exists $x \in K$ such that

$$\Big\langle Tx, h(x, y)\Big\rangle \geq_C 0, \quad \forall y \in K. \quad \square$$

Next, we consider the existence of solutions to (**MIVVI**), which extends some results in [1].

Theorem 3.3. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y a real Banach space. Let $T : K \to L(X, Y)$ and $h : K \times K \to X$ be mappings satisfying the following conditions;*
(1) *For any fixed $v \in K$, the mapping $u \mapsto \langle Tu, h(u, v)\rangle$ is continuous;*
(2) *T is properly quasimonotone of Minty type with respect to h;*
(3) *$\Big\langle Tu, h(u, u)\Big\rangle \leq_C 0$ for all $u \in K$.*
Then there exists $x \in K$ such that

$$\Big\langle Ty, h(y, x)\Big\rangle \leq_C 0, \quad \forall y \in K.$$

Proof. Define a multivalued mapping $M_T : K \to 2^K$ as follows:

$$M_T(z) = \Big\{x \in K : \langle Tz, h(z, x)\rangle \leq_C 0\Big\}, \quad \forall z \in K.$$

Obviously, $M_T(z) \neq \emptyset$ by the condition (3). By the same method in the proof of Theorem 3.2, it is easily shown that M_T is KKM mapping by the condition (2). From the condition (1), $M_T(z)$ is closed. Since X is reflexive, $M_T(z)$ is weakly compact for all $z \in K$([25]). By Fan-KKM Theorem, $\bigcap_{z\in K} M_T(z) \neq \emptyset$. Hence there exists $x \in K$ such that

$$\Big\langle Ty, h(y,x)\Big\rangle \leq_C 0, \ \ \forall y \in K. \quad \square$$

By Lemma 3.1, Theorem 3.1 and Theorem 3.3, we have the following result, which extends some results in [1].

Theorem 3.4. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y a real Banach space. Let $T : K \to L(X,Y)$ and $h : K \times K \to X$ be mappings satisfying the following conditions:*

(1) *T is pseudomonotone and properly quasimonotone of Stampacchia type with respect to h;*
(2) *for any fixed $v \in K$, the mapping $u \mapsto \langle Tu, h(u,v)\rangle$ is continuous;*
(3) *$\Big\langle Tu, h(u,u)\Big\rangle \in C$ for all $u \in K$;*
(4) *h is bilinear and for any fixed $x \in K$, $h(\cdot,x)$ is continuous on K.*

*Then problems (***SIVVI***) and (***MIVVI***) have the same nonempty solution set.*

References

1. Y. P. Fang and N. J. Huang, *Existence results for system of stromg implicit vector variational inequalities*, Acta Math. Hungar., **103(4)** (2004), 265–277
2. P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential functional equations*, Acta Math. **115** (1966), 271–310
3. G. Minty, *Monotone (nonlinear) operators in Hilbert spaces*, Duke Math. J. **29** (1962), 341–346
4. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities, Applications to Free Boundary Problems*, John Wiley and Sons, New york(1984)
5. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities*, Academic Press, New York(1980)
6. A. Behera and G.K. Panda, *A generalization of Minty's lemma*, Indian J. Pure Appl. Math.**28**(1997), 897–903
7. G. Kassay, J. Kolumbán and Z. Páles, *Factorization of Minty and Stampacchia variational inequality system*, European J. Oper. Res., **143(2)** (2002), 377–389
8. Q.H. Ansari, A.H. Siddiqi and J.C. Yao, *Generalized vector variational-like inequalities and their scalarizations, In vector variational inequalities and vector equilibria*,(Edited by F. Giannessi), Kluwer Academic, Boston (2000), 17–37
9. G.Y. Chen, *Existence of solutions for a vector variational inequality: an extension of Hartman-Stampacchia theorem*, J. Optim. Theory and Appl., **74** (1992), 445–456

10. G.Y. Chen and X.Q. Yang, *The vector complementarity problem and its equivalences with the weak minimal element in ordered spaces*, J. Math. Anal. Appl., **153** (1990), 136–158
11. F. Giannessi, *On Minty variational principle*, in new trends in mathematical programming, (Edited by F. Giannessi, S. Komlosi and T. Rapcsak), Kluwer Academic, Dordrecht, 1998
12. I.V. Konnov and J.C. Yao, *On the generalized vector variational inequality problems*, J. Math. Anal. Appl., **206** (1997), 42–58
13. B.S. Lee and G.M. Lee, *A vector version of Minty's lemma and application*, Appl. Math. Lett **12** (1999), 43–50
14. B.S. Lee and S.J. Lee, *A vector extension to Bhera and Panda's generalization of Minty's lemma*, Indian J. Pure Math. **31** (2000), 1483–1489
15. G.M Lee And S. Kum, *Vector variational inequalities in a Hausdorff topological vactor space In vector variational inequalities and vector equilibria*,(Edited by F. Giannessi), Kluwer Academic, Boston (2000), 307–320
16. G. Mastroeni, *On Minty vector variational inequality, In vector variational inequalities and vector equilibria*,(Edited by F. Giannessi), Kluwer Academic, Boston (2000), 351–361
17. S.J. Yu and J.C. Yao, *On vector variational inequalities*, J. Optim. Theory and Appl., **89** (1996), 749–769
18. B.S. Lee, G.M. Lee and D.S. Kim, *Generalized vector variational-like inequalities on locally convex Hausdorff topological vector spaces*, Indian J. Pure Appl, Math. **28(1)** (1997), 33–41
19. A.H. Siddiqi, Q.H. Ansari and A. Khaliq, *On vector variational inequalities*, J. Optim. Theory and Appl., **84** (1995), 171–180
20. F. Giannessi, *Theorems of alterative, quadratic programs and complementarity problems*, in: Variational Inequalities and Complementarity Problems (Edited by R.W. Cottle, F. Giannessi, and J.L. Lions), John Wiley and Sons, New York, 1980
21. B.S. Lee, S.S. Chang, J.S. Jung and S.J. Lee, *Generalized vector version of Minty's lemma and applications*, Comp. Math. with Appl. **45** (2003), 647–653
22. M.F. Khan and Salahuddin, *On generalized vector variational-like inequalities*, Nonlinear Analysis **59** (2004), 879–889
23. Y. Zhao and Z. Xia, *On the existence of solutions to generalized vector variational-like inequalities*, Nonlinear Analysis **64** (2006), 2075–2083
24. K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann., **266** (1984), 519–537
25. P.R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer-Velag, New York Inc.(1982)

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