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DISJOINT SMALL CYCLES IN GRAPHS

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ABSTRACT. We call a cycle C be a small cycle if the length of C equals to 3 or 4. In this paper, we obtain two sufficient conditions to ensure the existence of vertex-disjoint small cycles in graph and propose several problems.

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1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges and we use Bondy and Murty [1] for terminology and notation not defined here. Let $G = (V(G), E(G))$ be a graph, denote the order of G by $n = |V(G)|$. A set of subgraphs of G is said to be vertex-disjoint if no two of them have any common vertex in G, and we use disjoint to stand for vertex-disjoint throughout this paper. We call a cycle of length 3 a triangle and of length 4 a quadrilateral. We use $\delta(G)$ to denote the minimum degree of G. Let G_1 and G_2 be two subgraphs of G or subsets of $V(G)$. If G_1 and G_2 have no any common vertex in G, we define $E(G_1, G_2)$ to be the set of edges of G between G_1 and G_2, and let $e(G_1, G_2) = |E(G_1, G_2)|$.

Let H be a subgraph of G and $u \in V(G)$ a vertex, $N(u, H)$ is the set of neighbors of u contained in H. We let $d(u, H) = |N(u, H)|$. Clearly, $d(u, G) = d(u)$ denotes the degree of u in G. Let $N(U, H) = \cup_{u \in U} N(u, H)$ for a subset U of $V(G)$. If there is no fear of confusion, we often identify a subgraph H of G with its vertex set $V(H)$. For example, we often write $N(U, H)$ instead of $N(V(U), H)$. For a noncomplete graph G, let $\sigma_2(G) := \min\{d(x) + d(y) | x \in V(G), y \in V(G), xy \notin E(G)\}$; if G is a complete graph, let $\sigma_2(G) := \infty$.

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For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. Let C and P be a cycle and a path, respectively, we use $l(C)$ and $l(P)$ to denote the length of C and P, respectively. That is, $l(C) = |C|$ and $l(P) = |P| - 1$. Let Z denote the subgraph obtained from K_4 by removing two edges which have a common vertex. A graph G is said to be Z-free if G does not contain an induced subgraph isomorphic to Z. A Hamiltonian cycle of G is a cycle which contains all vertices of G, and a Hamiltonian path of G is a path of G which contains every vertex in G.

Corrádi and Hajnal [3] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order $n \geq 3k$ with $\delta(G) \geq 2k$, then G contains k disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k disjoint triangles. Erdős and Faudree [5] conjectured that if G is a graph with $n = 4k$ and $\delta(G) \geq 2k$, then G contains k disjoint quadrilaterals. With respect to this conjecture, Johansson [8] proved that G contains $k-1$ disjoint quadrilaterals and a path such that all the quadrilaterals and the path are disjoint. Randerath etc. [9] showed that G contains $k-1$ quadrilaterals and a subgraph of order 4 with at least four edges such that all of them are disjoint. Recently, Wang [10] gave a further result states that if G is a graph of order n with $4k+1 \leq n \leq 4k+4$ and $\delta(G) \geq 2k+1$, then G contains k disjoint quadrilaterals. However, this conjecture is still open. Brandt etc. [2] considered the disjoint triangles and quadrilaterals in a graph and obtained the following result.

Theorem 1 [2]. *Suppose $s \leq k$, $n \geq 3s + 4(k-s)$ and $\sigma_2(G) \geq n + s$. Then G contains k disjoint cycles $C_1, \cdots, C_k$ satisfying $|C_i| = 3$ for $1 \leq i \leq s$ and $|C_i| \leq 4$ for $s < i \leq k$.*

Note that in the above result, the length of C_i is not specified for $s < i \leq k$. J. Yan [11] proved that the length of C_i is four for each $s < i \leq k$ if $n \geq 3s + 4(k-s) + 3$.

Theorem 2 [11]. *Suppose $s \leq k$, $n \geq 3s + 4(k-s) + 3$ and $\sigma_2(G) \geq n + s$. Then G contains k disjoint cycles $C_1, \cdots, C_k$ such that $|C_i| = 3$ for $1 \leq i \leq s$ and $|C_i| = 4$ for $s < i \leq k$.*

Very recently, we improve the result of Theorem 2 by the following Theorem.

Theorem 3 [6]. *Suppose $s \leq k$, $n \geq 3s + 4(k-s) + 1$ and $\sigma_2(G) \geq n + k$. Then G contains k disjoint cycles $C_1, \cdots, C_k$ such that $|C_i| = 3$ for $1 \leq i \leq s$ and $|C_i| = 4$ for $s < i \leq k$.*

Clearly, to ensure the existence of s triangles and $k-s$ quadrilaterals such that all of them are disjoint, $n \geq 3s + 4(k-s)$ is sufficient. Decreasing the lower bound of n by one in Theorem 3 will link it to the famous conjecture proposed by Erdős and Faudree [5]. So some revolutionary idea should be needed to solve the following conjecture.

Conjecture. *Suppose $1 \leq s \leq k$, $n \geq 3s + 4(k-s)$ and $\sigma_2(G) \geq n + s$. Then G contains k disjoint cycles $C_1, \cdots, C_k$ such that $|C_i| = 3$ for $1 \leq i \leq s$ and $|C_i| = 4$ for $s < i \leq k$.*

In this paper, we obtain the following result.

Theorem 4. *Let s and k be two positive integers with $s \leq k$. Let G be a Z-free graph with order $n \geq 3s + 4(k-s)$. If $\sigma_2(G) \geq n + s$, then G contains k disjoint cycles $C_1, \cdots, C_k$ such that $|C_i| = 3$ for $1 \leq i \leq s$ and $|C_i| = 4$ for $s < i \leq k$.*

2. Lemmas

Lemma 2.1 [9]. *Let $C = a_1a_2a_3a_4a_1$ be a quadrilateral of G and u and v be two nonadjacent vertices of G not on C. If $d(u, C) + d(v, C) \geq 5$, then $G[V(C) \cup \{u, v\}]$ contains a quadrilateral $C^{'}$ and an edge e such that $C^{'}$ and e are disjoint and e is incident with exactly one of u and v.*

Lemma 2.2. *Let $C = a_1a_2a_3a_1$ be a triangle of G and v a vertex not on C. If $d(v, C) \geq 2$, then $G[V(C) \cup \{v\}]$ contains a quadrilateral.*

Lemma 2.3 [9]. *Let $C = a_1a_2a_3a_4a_1$ be a quadrilateral an P_1 and P_2 be two paths in G with $l(P_1) = l(P_2) = 1$. Suppose C, P_1 and P_2 are disjoint and $e(C, P_1 \cup P_2) \geq 9$. Then $G[V(C \cup P_1 \cup P_2)]$ contains a quadrilateral $C^{'}$ and a path P with $l(P) = 3$ such that $C^{'}$ and P are disjoint.*

Lemma 2.4 [9]. *Let P_1 and P_2 be two disjoint paths in G with $l(P_1) = 1$ and $1 \leq l(P_2) \leq 2$. If $e(P_1, P_2) \geq 3$, then $G[V(P_1 \cup P_2)]$ contains a quadrilateral.*

Lemma 2.5 [9]. *Let C be a quadrilateral and P a path with $l(P) = 3$. Suppose C and P are disjoint and $e(C, P) \geq 13$. Then $G[V(C \cup P)]$ contains two independent quadrilaterals.*

Lemma 2.6 [9]. *Let C be a quadrilateral and P a path of length 3 in G such that C and P are disjoint. Suppose that $G[V(C \cup P)]$ does not contain a quadrilateral $C^{'}$ and a path of length 3 such that $C^{'}$ and $P^{'}$ are disjoint and $e(G[V(C^{'})]) > e(G[V(C)])$. If $e(C, P) \geq 9$, then $G[V(C \cup P)]$ contains a quadrilateral $C^{''}$ and a subgraph H of order 4 such that $C^{''}$ and H are disjoint and H has at least four edges.*

Lemma 2.7 [8]. *Let C_1 be a quadrilateral and C_2 be a cycle of length 6 such that they are disjoint. If $e(C_1, C_2) \geq 13$, then $G[V(C_1 \cup C_2)]$ contains two disjoint quadrilaterals.*

Lemma 2.8 [4]. *Let $P = u_1u_2 \ldots u_s$ $(s \geq 2)$ be a path in G, $u \in V(G) - V(P)$, when $uu_1 \notin E(G)$, if $d(u, P) + d(u_s, P) \geq s$, then G has a path $P^{'}$ with vertex set $V(P^{'}) = V(P) \cup \{u\}$ whose end vertices are u and u_1. When $uu_1 \in E(G)$, if $d(u, P) + d(u_s, P) \geq s + 1$, then G has a path $P^{'}$ with vertex set $V(P^{'}) = V(P) \cup \{u\}$ whose end vertices are u and u_1.*

Lemma 2.9 [4]. *Let $P = u_1u_2\ldots u_s$ be a path with $s \geq 3$ in G. If $d(u_s, P) + d(u_1, P) \geq s$, then G has a cycle C with $V(C) = V(P)$.*

3. Proof of Theorem 4

Proof. Suppose that s, k and n be three positive integers such that $s \leq k$, $n \geq 3s + 4(k - s)$. Let G be a Z-free graph with order n and $\sigma_2(G) \geq n + s$. According to Theorem 1, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|C_i| = 3$ for each $1 \leq i \leq s$ and $|C_i| \leq 4$ for each $s < i \leq k$. We choose k disjoint cycles $C_1, \ldots, C_k$ such that

$$\text{The number of quadrilaterals in } \{C_{s+1}, \ldots, C_k\} \text{ is as large as possible.} \quad (1)$$

Subject to (1), we choose k disjoint cycles $C_1, \ldots, C_k$ such that

$$\text{The length of the longest path of } G - V(\bigcup_{i=1}^{k} C_i) \text{ is maximum.} \quad (2)$$

Let $P = x_1 \cdots x_p$ be a longest path of $G - V(\bigcup_{i=1}^{k} C_i)$. Subject to (1) and (2), we choose k disjoint cycles $C_1, \ldots, C_k$ and P of G such that

$$G - V(\cup_{i=1}^{k} C_i) \cup V(P) \text{ has the maximum matching.} \quad (3)$$

Suppose that there are l triangles and $k-l$ quadrilaterals in $C_1, \ldots, C_k$. If $l = s$, then we have nothing to prove. Hence, we may assume that $l \geq s + 1$. Let $H = \bigcup_{i=1}^{k} C_i$, $D = G - V(H)$ and $|D| = d$. Furthermore, let $M = \{y_1z_1, \ldots, y_rz_r\}$ be a maximum matching of $D - V(P)$. Our proof includes several cliams.

For convenience, in the following, let $T_1, \ldots, T_l$ denote l disjoint triangles and $Q_{l+1}, \ldots, Q_k$ denote $k-l$ disjoint quadrilaterals in $C_1, \ldots, C_k$, let $H_T = \bigcup_{i=1}^{l} T_i$ and $H_Q = \bigcup_{i=l+1}^{k} Q_i$. Since $l \geq s + 1$ and $n = 3l + 4(k - l) \geq 3s + 4(k - s)$, we see that $d = l - s \geq 1$. Under the choice (1), (2) and (3), Yan [11] obtained the following results.

Claim 3.1 (Yan. [11] Claim 3.4). *If $d \geq 7$, then there is a cycle C of length 6 and a quadrilateral $Q_j \in H_Q$ such that $e(C, Q_j) \geq 13$, where $C \in G[V(D)]$.*

By Claim 3.1 Lemma 2.7, $G[V(C \cup Q_j)]$ contains two disjoint quadrilaterals Q_j' and Q_j''. If we replace T_1 and Q_j with Q_j' and Q_j'', respectively, we arrive at a contradiction to (1). So $l = s$. We have completed the proof of Theorem 4. Therefore, in the following proof, we just consider the case when $d \leq 6$.

Claim 3.2. *There exist k disjoint cycles $T_1, \cdots, T_l$ and $Q_{l+1}, \cdots, Q_k$ such that $G[V(D \cup T_m)]$ contains a path of order 4, where $T_m \in H_T$.*

Proof. On the contrary, suppose the claim is false. Label $T_1 = a_1a_2a_3a_1$. Since $D \neq \emptyset$, let $x \in V(D)$. For each $T_i \in H_T$, if $e(x, T_i) \geq 1$, then we observe that $G[V(D \cup T_i)]$ contains a path of length 4, a contradiction. So, $e(x, T_i) = 0$ for each $T_i \in H_T$. Therefore, we have $d(x, D \cup T_1) + d(a_1, D \cup T_1) \leq d + 1$. By Lemma 2.2 and (1), $d(x, D - T_1) + d(a_1, D - T_1) \leq 3(l-1)$. Then $e(\{x, a_1\}, H_Q) \geq n + s - 3(l - 1) - (d + 1) > 4(k - l) + 1$, which implies that there exists some

$Q_i \in H_Q$ such that $e(\{x, a_1\}, Q_i) \geq 5$. By Lemma 2.1, $G[V(Q_i) \cup \{x, a_1\}]$ contains a quadrilateral Q' and an edge e such that Q' and e are disjoint and e is incident with exactly one of x and a_1. If a_1 is an end of e, we may assume the other end of e is x_0, then replace Q_i with Q', we obtain a path $P' = x_0a_1a_2a_3$ of order 4 which disjoint with Q', a contradiction. Hence, x is an end of e, say $e = xx_0$ with $x_0 \in V(Q')$. Clearly, $e(xx_0, a_2a_3) = 0$.

The above argument allow us to see that we can choose $T_1, \cdots, T_l, Q_{l+1}, \cdots, Q_k$ such that $G[V(D \cup T_m)]$ contains two disjoint edges for some $T_m \in H_T$. Now, let y_1y_2 and z_1z_2 be the two disjoint edges in $D \cup T_m$. Set $S = \{y_1, y_2, z_1, z_2\}$ and $M' = (D \cup T_m) - S$. As $G[V(D \cup T_m)]$ contains none of P_4 and C_4, we see that $N(y_i, M') \cap N(z_j, M') = \emptyset$ for each $\{i, j\} \subseteq \{1, 2\}$, $|N(y_1, M') \cap N(y_2, M')| \leq 1$, $|N(z_1, M') \cap N(z_2, M')| \leq 1$ and $e(y_1y_2, z_1z_2) = 0$. It follows that $\sum_{x \in S} e(x, D \cup T_m) \leq d + 4$. By Lemma 2.2 and (1), $e(S, H_T - T_1) \leq 4(l - 1)$. Therefore, $e(S, H_Q) \geq 2(n + s) - 4(l - 1) - (d + 4) > 8(k - l) + 1$, which implies that there exists some $Q_j \in H_Q$ such that $e(S, Q_j) \geq 9$. By Lemma 2.3, $G[V(Q_j \cup S)]$ contains a quadrilateral Q' and a path with $l(P) = 3$ such that Q' and P are disjoint. If we replace Q_j with Q', we see that $G[V(D \cup T_m)]$ contains a path of order 4, a contradiction. □

Claim 3.3. *There exist k disjoint cycles $T_1, \cdots, T_l$ and $Q_{l+1}, \cdots, Q_k$ such that $G[V(D \cup T_m)]$ contains a subgraph of order 4 with at least 4 edges, where $T_m \in H_T$.*

Proof. On the contrary, suppose this claim is false. By claim 3.2, there exist k disjoint cycles $T_1, \cdots, T_l$ and $Q_{l+1}, \cdots, Q_k$ such that $G[V(D \cup T_m)]$ contains a path of order 4, where $T_m \in H_T$. We choose k disjoint cycles $T_1, \cdots, T_l$ and $Q_{l+1}, \cdots, Q_k$ with $\sum_{i=l+1}^{k} e(G[V(Q_i)])$ as large as possible such that $G[V(D \cup T_m)]$ contains a path P of order 4, where $T_m \in H_T$. If $e(P) \geq 4$, then we are done. So assume $e(P) = 3$. Let $P = x_1x_2x_3x_4$ and $D' = D - V(P \cap D)$. Clearly, $x_1x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Note that $|V(P \cap T_m)| \leq 2$. Otherwise, $e(P) \geq 4$, a contradiction. Furthermore, we may assume that $|V(P \cap T_m)| \leq 1$. Otherwise, say $|V(P \cap T_m)| = 2$ and let $u \in V(T_m - P)$. If $\{x_1, x_3\} = N(u, P)$, then $ux_1x_2x_3u$ is a quadrilateral. Replace T_m by $ux_1x_2x_3u$, we arrive at a contradiction to (1). Hence, by symmetry, we may assume that $\{x_1, x_2\} = N(u, P)$. Then $\{u, x_1, x_2, x_3\}$ isomorphic to Z, a contradiction.

Consequently, it follows from Lemma 2.2 and (1) that $e(P, H_T - T_m) \leq 6(l - 1)$. As $G[V(D \cup T_m)]$ does not contain a quadrilateral, we see that $d(x, D') \leq 2$ for each $x \in V(D')$. Therefore, $e(P, D \cup T_m) \leq 6 + 2(d - 1) = 2d + 4$, and so $e(P, H_Q) \geq 2(n + s) - 6(l - 1) - (2d + 4) = 8(l - l) + 2s + 2$, which implies there exists $Q_i \in H_Q$ such that $e(P, Q_i) \geq 9$. By Lemma 2.6, $G[V(Q_i \cup P)]$ contains a quadrilateral C'' and a subgraph H of order 4 such that C'' and H are disjoint and H has at least four edges. Replace Q_i with C'', we see that $G[V(D \cup T_m)]$ contains a subgraph of order 4 with at least 4 edges. □

By Claim 3.3, there exists $T_m \in H_T$ such that $G[V(D \cup T_m)]$ contains a subgraph H of order 4 with at least four edges. Since G is a Z-free graph, the only possibility is that H is isomorphic to a quadrilateral, denoted by C_4. However, if we replace T_m with C_4, we obtain a contradiction to (1). This proves Theorem 4. □

4. Disjoint small cycles passing through specified vertices

Many people have studied the problems of cyclability, i.e., for a given subset S of vertices, there exist a cycle or several independent cycles, covering S. We are interested in such problems, in particular, for any k independent vertices $v_1, \ldots, v_k$, what ensures that there exists k disjoint triangles $C_1, \ldots, C_k$ with respect to $\{v_1, \ldots, v_k\}$. For disjoint triangles covering, Li et al [7] obtained the following result.

Theorem 5. (Li et al [7]). *Let k, n be two positive integers and let G be a graph of order $n \geq 3k$, X a set of any k distinct vertices of G. If the minimum degree $\delta(G) \geq (n + 2k)/2$, then G contains k disjoint triangles such that each triangle contains exactly one vertex of X.*

Let $v_1, \ldots, v_k$ be k distinct vertices in G, and let $C_1, \ldots, C_k$ be k disjoint cycles passing through $v_1, \ldots, v_k$, respectively, in G. Then we say that G has k disjoint cycles $C_1, \ldots, C_k$ with respect to $\{v_1, \ldots, v_k\}$. Let v be a vertex and H is a subgraph of G. We say H is an v-subgraph if $v \in V(H)$. Let P be an v-path, we define $\lambda(v, P) = \min\{|P_1|, |P_2|\}$, where P_1 and P_2 be two subpaths of $P - v$.

Theorem 6. *Let k be a positive integer and let G be a graph of order $n \geq 3k+1$, X a set of any k distinct vertices of G. If $\sigma_2(G) \geq n + 2k - 2$, then G contains k disjoint cycles $T_1, \ldots, T_k$ such that each cycle contains exactly one vertex in X, and $|T_i| = 3$ for each $1 \leq i \leq k$ or $|T_k| = 4$ and the rest are all triangles.*

Proof. Suppose that G does not contains k disjoint $T_1, \ldots, T_k$ such that each cycle contains exactly one vertex in X and $|T_k| = 4$ and the rest are triangles. We prove that G contains k disjoint triangles $T_1, \ldots, T_k$ such that each cycle contains exactly one vertex in X. Suppose this is false, let G be an edge-maximal counterexample. Since a complete graph of order $n \geq 3k + 1$ contains k disjoint triangles such that each triangle contains exactly one vertex of X, thus, G is not a complete graph. Let u and v be nonadjacent vertices of G and define $G' = G + uv$, the graph obtained from G by adding the edge uv. Then G' is not a counterexample by the maximality of G, that is, for any $X = \{v_1, \ldots, v_k\} \subseteq V(G)$, G' contains k disjoint triangles $T_1, \ldots, T_k$ with respect to $\{v_1, \ldots, v_k\}$.

Claim 4.1. $k \geq 2$.

Proof. Otherwise, suppose $k = 1$. By the classical result of Ore, G contains a Hamiltonian cycle $C = y_1y_2 \ldots y_ny_1$. We may assume that $v_1 = y_1$, otherwise, we can relabel the index of C.

We consider the path $P = y_1y_2y_3y_4$. Since $G[V(P)]$ contains no quadrilateral, therefore, $y_1y_3 \notin E(G)$, $y_2y_4 \notin E(G)$, $N(y_1, C - V(P)) \cap N(y_3, C - V(P)) =$

$\emptyset$ and $N(y_2, C - V(P)) \cap N(y_4, C - V(P)) = \emptyset$. Then it follows that $2n \leq \sum_{x \in V(P)} d(x, G) \leq 6 + 2(n-4) = 2n - 2$, a contradiction. □

By the choice of G, there exists $v \in \{v_1, v_2 \ldots, v_k\}$ such that G contains $k-1$ triangles $T_1, \ldots, T_{k-1}$ with respect to $\{v_1, v_2, \ldots, v_k\} - \{v\}$, $v \notin V(\bigcup_{i=1}^{k-1} T_i)$. Subject to this, we choose $v \in \{v_1, v_2, \ldots, v_k\}$ and $k-1$ triangles $T_1, \ldots, T_{k-1}$ with respect to $\{v_1, \ldots, v_k\} - \{v\}$ such that

$$\text{The length of the longest } v\text{-path in } G - V(\bigcup_{i=1}^{k-1} T_i) \text{ is maximum.} \tag{4}$$

Let $P = u_1 \ldots u_s$ be a longest v-path in $G - V(\bigcup_{i=1}^{k-1} T_i)$. Subject to (4), we choose $v \in \{v_1, v_2, \ldots, v_k\}$, $k-1$ vertex disjoint triangles $T_1, \ldots, T_{k-1}$ with respect to $\{v_1, \ldots, v_k\} - \{v\}$ and P such that

$$\lambda(v, P) \text{ is maximum.} \tag{5}$$

Without loss of generality, suppose that $v = v_k$ and $v_i \in V(T_i)$ for each $i \in \{1, 2, \ldots, k-1\}$. Let $H = \bigcup_{i=1}^{k-1} T_i$, $D = G - H$ and $|D| = d$. Clearly, $d \geq 4$ as $n \geq 3k+1$. Furthermore, by the choice of G, $s \geq 3$.

Claim 4.2. *P is a Hamiltonian path of D.*

Proof. Suppose $s < d$. We choose an arbitrary vertex $x_0 \in D - V(P)$. Clearly, $x_0 u_1 \notin E(G)$ and $x_0 u_s \notin E(G)$. Note $s \geq 3$, by (4) and Lemma 2.8, $d(x_0, P) + d(u_1, P) \leq s - 1$. Since $d(x_0, D - V(P)) \leq d - s - 1$ and $d(u_1, D - V(P)) = 0$, it follows that $d(x_0, D) + d(u_1, D) \leq d - 2$. By the assumption on the degree condition of G, we have

$$d(x_0, H) + d(u_1, H) \geq (n + 2k - 2) - (d - 2) = 5(k-1) + 2.$$

This implies that there exists $T_i \in H$, say T_1, such that $d(x_0, T_1) + d(u_1, T_1) = 6$. Let $T_1 = v_1 w_1 w_2 v_1$. If we replace T_1 with $x_0 w_2 v_1 x_0$, we obtain a path $P' = w_1 u_1 \ldots u_s$ with $|P'| = |P| + 1$, contradicting (4). □

Claim 4.3. *If $\lambda(v_k, P) = 0$ or 1, then D is Hamiltonian.*

Proof. By Claim 4.2, D contains a hamiltonian path $P = u_1 \ldots u_d$ passing through v_k. If $u_1 u_d \in E(G)$, then we have nothing to prove. So, $u_1 u_d \notin E(G)$. By symmetry, if $\lambda(v_k, P) = 0$, we may assume that $v_k = u_1$. If $\lambda(v_k, P) = 1$, we assume that $v_k = u_2$.

If there exists $T_i \in H_T$ such that $d(u_1, T_i) + d(u_d, T_i) = 6$, then there exits $w \in V(T_i)$ with $u_1 w \in E(G)$ such that $T_i - w + u_d$ contains a triangle T_i' passing through v_i. If we replace T_i with T_i', we see that D contains a v_k-path $P' = P - u_d + w$. However, $\lambda(v_k, P') = \lambda(v_k, P) + 1$, contradicting (5) while (4) still holds. Hence, $d(u_1, T_i) + d(u_d, T_i) \leq 5$ for each $T_i \in H$ and so $d(u_1, H) + d(u_d, H) \leq 5(k-1)$. It follows that

$$d(u_1, D) + d(u_d, D) \geq n + 2k - 2 - 5(k-1) = d.$$

By Lemma 2.9, D contains a hamiltonian cycle. This proves the claim. □

Case 1. $d = 4$. By Claim 4.3, D contains a hamiltonian cycles C. Then G contains $k-1$ disjoint triangles $T_1, T_2, \ldots, T_{k-1}$ and a quadrilateral C with respect to $\{v_1, v_2, \ldots, v_k\}$ such that all of them are disjoint, a contradiction.

Case 2. $d \geq 5$. By Claims 4.2 and 4.3, for each v_k-path P' of length 4 in D, we may assume that $\lambda(v_k, P') = 2$. Let $P' = y_1y_2y_3y_4y_5$ be an arbitrary v_k-path of length 4, then $v_k = y_3$. Since D does not contain a triangle passing through $y_3 = v_k$, hence, $y_1y_3 \notin E(G)$ and $y_2y_4 \notin E(G)$. Let $P'' = P' - y_5$. Since D contains no quadrilateral passing through $v_k = y_3$, then $N(y_1, D - V(P'')) \cap N(y_3, D - V(P'')) = \emptyset$ and $N(y_2, D - V(P'')) \cap N(y_4, D - V(P'')) = \emptyset$. So, $\sum_{x \in V(P'')} d(x, D) \leq 2(d-4) + 6 = 2d - 2$. This gives that

$$\sum_{x \in V(P'')} d(x, H) \geq 2(n + 2k - 2) - (2d - 2) = 10(k-1) + 2.$$

This implies that there exists $T_i \in H$, without loss of generality, say T_1, such that $e(P'', T_1) \geq 11$. That is, there is at most one edge absent between P'' and T_1. Let $T_1 = v_1w_1w_2v_1$.

We claim that $d(y_3, T_1) = 2$. Otherwise, say $d(y_3, T_1) = 3$. If $G[\{y_1, y_2, v_1\}]$ contains a triangle, denoted by T_1', then G contains k disjoint triangles $T_1', T_2, \ldots, T_{k-1}, y_3w_1w_2y_3$ with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction. Hence, it follows that $e(v_1, y_1y_2) \leq 1$ and so $e(P'', T_1) \leq 11$, which yields to $d(y_4, T_1) = 3$ and $y_2w_2 \in E(G)$. Consequently, G contains k disjoint triangles $y_4w_1v_1y_4, T_2, \ldots, T_{k-1}$ $y_2y_3w_2y_2$ with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction.

Since $d(y_3, T_1) = 2$, without loss of generality, say $y_3w_2 \in E(G)$. Furthermore, we have $d(y_i, T_1) = 3$ for each $i \in \{1, 2, 4\}$. Then G contains k disjoint triangles $y_1y_2v_1y_1, T_2, \ldots, T_{k-1}$ and $y_3y_4w_2y_3$ with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction. We have completed the proof of Theorem 6. □

5. Concluding remarks

The condition $n \geq 3k+1$ in Theorem 6 is necessary. To see this, we construct the following graph G of order $3k$. For two subgraphs G_1 and G_2, we use $G_1 \vee G_2$ to denote the join of G_1 and G_2. Construct $G = (K_{k-1} \cup \{u\}) \vee K_{2k}$. Let $X = K_{k-1} \cup \{u\}$. Clearly, there is at most $k-1$ disjoint triangles such that one of them passing through u and $\sigma_2(G) = 5k - 2$.

To conclude this paper, we propose the following problem.

Problem 1. *Let k, n be two positive integers and let G be a graph of order $n \geq 3k+1$, X a set of any k distinct vertices of G. If $\sigma_2(G) \geq n + 2k - 1$, does G contain k disjoint triangles $T_1, \ldots, T_k$ such that each triangle contains exactly one vertex in X?*

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