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FUZZY IDEALS OF PSEUDO BCI-ALGEBRAS

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ABSTRACT. The concepts of fuzzy pseudo ideals (resp. fuzzy pseudo p-ideals, associative fuzzy pseudo ideals, fuzzy pseudo q-ideals and fuzzy pseudo a-ideals) in a pseudo BCI-algebra are introduced, and related properties are investigated. Conditions for a fuzzy pseudo ideal to be a fuzzy pseudo p-ideal (resp. fuzzy pseudo q-ideal) are provided. A characterization and properties of an associative fuzzy pseudo ideal are given.

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1. Introduction

In [3], G. Georgescu and A. Iorgulescu introduced the notion of a pseudo BCK-algebra as an extended notion of BCK-algebras. Y. B. Jun [4] gave a characterization of pseudo BCK-algebra, and provided conditions for a pseudo BCK-algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered). In [6], Y. B. Jun et al. introduced the notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra, and then they investigated some of their properties. W. A. Dudek and Y. B. Jun [2] introduced the notion of pseudo BCI-algebras as an extension of BCI-algebras, and investigated some properties.

In [5], Y. B. Jun et al. introduced the concepts of pseudo-atoms, pseudo ideals and pseudo BCI-homomorphisms in pseudo BCI-algebras. They displayed characterizations of a pseudo ideal, and provided conditions for a subset to be a pseudo ideal. Y. L. Liu et al. [8] extended the ideal and congruence theory to pseudo BCK-algebras, and investigated the connections between pseudo BCK-algebras and PD (GPD)-posets. In [7], K. J. Lee and C. H. Park introduced the concepts of (associative) pseudo ideals, pseudo p-ideals, pseudo q-ideals and pseudo a-ideals in a pseudo BCI-algebra, and investigated relative properties.

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In this paper, we introduce the concepts of fuzzy pseudo ideals (resp. fuzzy pseudo p-ideals, associative fuzzy pseudo ideals, fuzzy pseudo q-ideals and fuzzy pseudo a-ideals) in a pseudo BCI-algebra, and investigate related properties. We provide conditions for a fuzzy pseudo ideal to be a fuzzy pseudo p-ideal (resp. fuzzy pseudo q-ideal). We give characterizations and properties of an associative fuzzy pseudo ideal.

2. Preliminaries

The study of BCK/BCI-algebra was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then many researches worked extensively in this area .

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ \Big(((x * y) * (x * z)) * (z * y) = 0\Big)$,

(II) $(\forall x, y \in X)\ \Big((x * (x * y)) * y = 0\Big)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ \Big(x * y = 0\ \&\ y * x = 0 \Rightarrow x = y\Big)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,

(a3) $(\forall x, y, z \in X)\ \Big((x * y) * z = (x * z) * y\Big)$,

(a4) $(\forall x, y, z \in X)\ \Big((x * z) * (y * z) \leq x * y\Big)$

where $x \leq y$ if and only if $x * y = 0$.

A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I \tag{1}$$

and

$$(\forall x, y \in X)\ \Big(x * y \in I\ \&\ y \in I \Longrightarrow x \in I\Big). \tag{2}$$

A non-empty subset I of a BCI-algebra X is called a *p-ideal* of X (see [10]) if it satisfies (1) and

$$(\forall x, y, z \in X)\ \Big((x * z) * (y * z) \in I\ \&\ y \in I \Longrightarrow x \in I\Big). \tag{3}$$

A non-empty subset I of a BCI-algebra X is called a *q-ideal* of X (see [9]) if it satisfies (1) and

$$(\forall x, y, z \in X)\ \Big(x * (y * z) \in I\ \&\ y \in I \Longrightarrow x * z \in I\Big). \tag{4}$$

A non-empty subset I of a BCI-algebra X is called an *a-ideal* of X (see [9]) if it satisfies (1) and

$$(\forall x, y, z \in X)\Big((x * z) * (0 * y) \in I \ \&\ z \in I \implies y * x \in I\Big). \tag{5}$$

Definition 1. [3] A *pseudo BCK-algebra* is a structure $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$, where "$\preceq$" is a binary relation on a set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

$$(x * y) \diamond (x * z) \preceq z * y, \ \ (x \diamond y) * (x \diamond z) \preceq z \diamond y, \tag{6}$$

$$x * (x \diamond y) \preceq y, \ \ x \diamond (x * y) \preceq y, \tag{7}$$

$$x \preceq x, \tag{8}$$

$$0 \preceq x, \tag{9}$$

$$x \preceq y \ \&\ y \preceq x \implies x = y, \tag{10}$$

$$x \preceq y \iff x * y = 0 \iff x \diamond y = 0. \tag{11}$$

Definition 2. [2] A *pseudo BCI-algebra* is a structure $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$, where "$\preceq$" is a binary relation on a set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms (6), (7), (8), (10) and (11).

Example 1. [5] Let $X = [0, \infty]$ and let $\leq$ be the usual order on X. Define binary operations "$*$" and "$\diamond$" on X by

$$x*y := \begin{cases} 0 & \text{if } x \leq y, \\ \dfrac{2x}{\pi} \arctan\left(\ln(\dfrac{x}{y})\right) & \text{if } y < x, \end{cases} \quad \text{and} \quad x \diamond y := \begin{cases} 0 & \text{if } x \leq y, \\ x e^{-\tan(\frac{\pi y}{2x})} & \text{if } y < x, \end{cases}$$

for all $x, y \in X$. Then $\mathfrak{X} := (X, \leq, *, \diamond, 0)$ is a pseudo BCK-algebra, and hence a pseudo BCI-algebra.

Proposition 1. [2, 5] *In a pseudo BCI-algebra $\mathfrak{X}$ the following holds:*

(b1) $x \preceq 0 \Rightarrow x = 0.$
(b2) $x \preceq y \Rightarrow z * y \preceq z * x, \ z \diamond y \preceq z \diamond x.$
(b3) $x \preceq y, \, y \preceq z \Rightarrow x \preceq z.$
(b4) $(x * y) \diamond z = (x \diamond z) * y.$
(b5) $x * y \preceq z \Leftrightarrow x \diamond z \preceq y.$
(b6) $(x * y) * (z * y) \preceq x * z, \ (x \diamond y) \diamond (z \diamond y) \preceq x \diamond z.$
(b7) $x \preceq y \Rightarrow x * z \preceq y * z, \ x \diamond z \preceq y \diamond z.$
(b8) $x * 0 = x = x \diamond 0.$
(b9) $x * \big(x \diamond (x * y)\big) = x * y$ *and* $x \diamond \big(x * (x \diamond y)\big) = x \diamond y.$
(b10) $0 * (x \diamond y) \preceq y \diamond x.$
(b11) $0 \diamond (x * y) \preceq y * x.$
(b12) $0 * (x * y) = (0 \diamond x) \diamond (0 * y).$

(b13) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$.

Proposition 2. [7] *In a pseudo BCI-algebra $\mathfrak{X}$, we have $0 * x = 0 \diamond x$ for all $x \in X$.*

In what follows, let $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$ be a pseudo BCI-algebra unless otherwise specified.

For any non-empty subset J of X and any element y of X, we denote

$$*(y, J) := \{x \in X \mid x * y \in J\} \text{ and } \diamond (y, J) := \{x \in X \mid x \diamond y \in J\}.$$

A non-empty subset J of $\mathfrak{X}$ is called a *pseudo ideal* of $\mathfrak{X}$ (see [5]) if it satisfies:

$$0 \in J, \tag{12}$$

$$(\forall y \in J)\ (*(y, J) \subseteq J \ \&\ \diamond (y, J) \subseteq J). \tag{13}$$

A non-empty subset J of $\mathfrak{X}$ is called a *pseudo p-ideal* of $\mathfrak{X}$ (see [7]) if it satisfies (12) and

$$\begin{array}{l} (x * z) \diamond (y * z) \in J \ \&\ y \in J \implies x \in J, \\ (x \diamond z) * (y \diamond z) \in J \ \&\ y \in J \implies x \in J \end{array} \tag{14}$$

for all $x, y, z \in X$.

A non-empty subset J of $\mathfrak{X}$ is called an *associative pseudo ideal* of $\mathfrak{X}$ (see [7]) if it satisfies (12) and

$$\begin{array}{l} (x * y) \diamond z \in J \ \&\ y \diamond z \in J \implies x \in J, \\ (x \diamond y) * z \in J \ \&\ y * z \in J \implies x \in J \end{array} \tag{15}$$

for all $x, y, z \in X$.

A non-empty subset J of $\mathfrak{X}$ is called a *pseudo q-ideal* of $\mathfrak{X}$ (see [7]) if it satisfies (12) and

$$\begin{array}{l} x * (y \diamond z) \in J \ \&\ y \in J \implies x * z \in J, \\ x \diamond (y * z) \in J \ \&\ y \in J \implies x \diamond z \in J \end{array} \tag{16}$$

for all $x, y, z \in X$.

A non-empty subset J of $\mathfrak{X}$ is called a *pseudo a-ideal* of $\mathfrak{X}$ (see [7]) if it satisfies (12) and

$$\begin{array}{l} (x * y) \diamond (0 * z) \in J \ \&\ y \in J \implies z \diamond x \in J, \\ (x \diamond y) * (0 \diamond z) \in J \ \&\ y \in J \implies z * x \in J \end{array} \tag{17}$$

for all $x, y, z \in X$.

3. Fuzzy pseudo ideals

Definition 3. A fuzzy set μ in $\mathfrak{X}$ is called a *fuzzy pseudo ideal* of $\mathfrak{X}$ if it satisfies:

(F1) $(\forall x \in X)\ (\mu(0) \geq \mu(x))$,

(F2) $(\forall x, y \in X)\ \Big(\mu(x) \geq \min\Big\{\mu(x*y),\ \mu(y)\Big\} \ \&\ \mu(x) \geq \min\{\mu(x \diamond y),\ \mu(y)\}\Big)$.

Proposition 3. *Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$. Then*

$$\mu(0 * (0 \diamond x)) = \mu(0 \diamond (0 * x)) \geq \mu(x)$$

for all $x \in X$.

Proof. Let $x \in X$. By Proposition 2, $\mu(0 * (0 \diamond x)) = \mu(0 \diamond (0 * x))$. It follows from (8), (11), (b4), (F1) and (F2) that

$$\begin{aligned}\mu(0 * (0 \diamond x)) &\geq \min\Big\{\mu((0 * (0 \diamond x)) \diamond x),\, \mu(x)\Big\} \\ &= \min\Big\{\mu((0 \diamond x) * (0 \diamond x)),\, \mu(x)\Big\} = \min\{\mu(0),\, \mu(x)\} = \mu(x).\end{aligned}$$

This completes the proof. □

Proposition 4. *Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$. If $y \preceq x$, then $\mu(y) \geq \mu(x)$.*

Proof. If $y \preceq x$, then $y * x = 0$, and hence $\mu(y) \geq \min\{\mu(y * x),\, \mu(x)\} = \min\{\mu(0),\, \mu(x)\} = \mu(x)$. □

Theorem 1. *Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$. If $\mu^\sharp$ is a fuzzy set in $\mathfrak{X}$ defined by $\mu^\sharp(x) = \mu(0 * (0 \diamond x))$ for all $x \in X$, then $\mu^\sharp$ is a fuzzy pseudo ideal of $\mathfrak{X}$ and $\mu \subseteq \mu^\sharp$.*

Proof. For any $x \in X$, by using (F1), $\mu^\sharp(0) = \mu(0) \geq \mu(0 * (0 \diamond x)) = \mu^\sharp(x)$. Let $x, y \in X$. It follows from (b12), (b13) and Proposition 2 that

$$\begin{aligned}\mu^\sharp(x * y) &= \mu(0 * (0 \diamond (x * y))) = \mu(0 \diamond (0 * (x * y))) \\ &= \mu\Big(0 \diamond ((0 \diamond x) \diamond (0 * y))\Big) = \mu((0 * (0 \diamond x)) * (0 \diamond (0 * y)))\end{aligned}$$

and $\mu^\sharp(y) = \mu(0 * (0 \diamond y)) = \mu(0 \diamond (0 * y))$. Then

$$\begin{aligned}\min\{\mu^\sharp(x * y),\, \mu^\sharp(y)\} &= \min\Big\{\mu((0 * (0 \diamond x)) * (0 \diamond (0 * y))),\, \mu(0 \diamond (0 * y))\Big\} \\ &\leq \mu(0 * (0 \diamond x)) = \mu^\sharp(x).\end{aligned}$$

Now, by using (b12) and (b13) we have

$$\begin{aligned}\mu^\sharp(x \diamond y) &= \mu(0 * (0 \diamond (x \diamond y))) = \mu\Big(0 * ((0 * x) * (0 \diamond y))\Big) \\ &= \mu((0 \diamond (0 * x)) \diamond (0 * (0 \diamond y)))\end{aligned}$$

and $\mu^\sharp(y) = \mu(0 * (0 \diamond y))$, and so

$$\begin{aligned}\min\{\mu^\sharp(x \diamond y),\, \mu^\sharp(y)\} &= \min\Big\{\mu((0 \diamond (0 * x)) \diamond (0 * (0 \diamond y))),\, \mu(0 * (0 \diamond y))\Big\} \\ &\leq \mu(0 \diamond (0 * x)) = \mu(0 * (0 \diamond x)) = \mu^\sharp(x).\end{aligned}$$

Hence $\mu^\sharp$ is a fuzzy pseudo ideal of $\mathfrak{X}$. By Proposition 3,

$$\mu(x) \leq \mu(0 * (0 \diamond x)) = \mu^\sharp(x)$$

for all $x \in X$, i.e., $\mu \subseteq \mu^\sharp$. This completes the proof. □

Definition 4. A fuzzy set μ in $\mathfrak{X}$ is called a *fuzzy pseudo p-ideal* of $\mathfrak{X}$ if it satisfies (F1) and

$$\begin{aligned} \mu(x) &\geq \min\left\{\mu((x * z) \diamond (y * z)),\, \mu(y)\right\}, \\ \mu(x) &\geq \min\left\{\mu((x \diamond z) * (y \diamond z)),\, \mu(y)\right\} \end{aligned} \tag{18}$$

for all $x, y, z \in X$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notions of a fuzzy pseudo p-ideal and a fuzzy p-ideal coincide.

Theorem 2. *Every fuzzy pseudo p-ideal of $\mathfrak{X}$ is a fuzzy pseudo ideal of $\mathfrak{X}$.*

Proof. Let μ be a fuzzy pseudo p-ideal of $\mathfrak{X}$. For any $x, y \in X$, it follows from (18) and (b8) that

$$\mu(x) \geq \min\left\{\mu((x * 0) \diamond (y * 0)),\, \mu(y)\right\} = \min\{\mu(x \diamond y),\, \mu(y)\},$$

$$\mu(x) \geq \min\left\{\mu((x \diamond 0) * (y \diamond 0)),\, \mu(y)\right\} = \min\{\mu(x * y),\, \mu(y)\},$$

and hence μ is a fuzzy pseudo ideal of $\mathfrak{X}$. □

Proposition 5. *Let μ be a fuzzy pseudo p-ideal of $\mathfrak{X}$. Then*

$$\mu(x) \geq \mu(0 * (0 \diamond x)) = \mu(0 \diamond (0 * x))$$

for all $x \in X$.

Proof. Let $x \in X$. By Proposition 2, $\mu(0 * (0 \diamond x)) = \mu(0 \diamond (0 * x))$. Taking $y := 0$ and $z := x$ in (18) implies that

$$\mu(x) \geq \min\left\{\mu((x \diamond x) * (0 \diamond x)),\, \mu(0)\right\} = \mu(0 * (0 \diamond x)).$$

This completes the proof. □

Combining Propositions 3 and 5, we have the following corollary.

Corollary 1. *Let μ be a fuzzy pseudo p-ideal of $\mathfrak{X}$, and let $\mu^\sharp$ be as before in Theorem 1. Then $\mu = \mu^\sharp$.*

We give a condition for a fuzzy pseudo ideal to be a fuzzy pseudo p-ideal.

Theorem 3. *Every fuzzy pseudo ideal μ of $\mathfrak{X}$ satisfying the following conditions:*

$$\begin{aligned} \mu(x * y) &\geq \mu\Big((x \diamond z) * (y \diamond z)\Big), \\ \mu(x \diamond y) &\geq \mu\Big((x * z) \diamond (y * z)\Big) \end{aligned} \tag{19}$$

for all $x, y, z \in X$, is a fuzzy pseudo p-ideal of $\mathfrak{X}$.

Proof. Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$ that satisfies (19). For any $x, y, z \in X$, it follows from (F2) and (19) that

$$\mu(x) \geq \min\Big\{\mu(x * y),\ \mu(y)\Big\} \geq \min\{\mu((x \diamond z) * (y \diamond z)),\ \mu(y)\},$$

$$\mu(x) \geq \min\Big\{\mu(x \diamond y),\ \mu(y)\Big\} \geq \min\{\mu((x * z) \diamond (y * z)),\ \mu(y)\}.$$

Then μ is a fuzzy pseudo p-ideal of $\mathfrak{X}$. □

Definition 5. A fuzzy set μ in $\mathfrak{X}$ is called an *associative fuzzy pseudo ideal* of $\mathfrak{X}$ if it satisfies (F1) and

$$\begin{aligned} \mu(x) &\geq \min\Big\{\mu((x * y) \diamond z),\ \mu(y \diamond z)\Big\}, \\ \mu(x) &\geq \min\Big\{\mu((x \diamond y) * z),\ \mu(y * z)\Big\} \end{aligned} \tag{20}$$

for all $x, y, z \in X$.

We provide a characterization of associative fuzzy pseudo ideals.

Theorem 4. *A fuzzy set μ in $\mathfrak{X}$ is an associative fuzzy pseudo ideal of $\mathfrak{X}$ if and only if the level set $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$ is an associative pseudo ideal of $\mathfrak{X}$ for all $t \in [0, 1]$ when it is nonempty.*

Proof. Let μ be an associative fuzzy pseudo ideal of $\mathfrak{X}$ and let $t \in [0, 1]$ be such that $U(\mu; t) \neq \emptyset$. Then there exists $x \in U(\mu; t)$, and so $\mu(0) \geq \mu(x) \geq t$. Thus $0 \in U(\mu; t)$. Let $x, y, z \in X$ be such that $(x * y) \diamond z \in U(\mu; t)$ and $y \diamond z \in U(\mu; t)$. Then $\mu(x) \geq \min\Big\{\mu((x * y) \diamond z),\ \mu(y \diamond z)\Big\}$, and so $x \in U(\mu; t)$. Similarly, if $(x \diamond y) * z \in U(\mu; t)$ and $y * z \in U(\mu; t)$, then $x \in U(\mu; t)$. Therefore $U(\mu; t)$ is an associative pseudo ideal of $\mathfrak{X}$.

Conversely, assume that $U(\mu; t)$ is an associative pseudo ideal of $\mathfrak{X}$ when $U(\mu; t) \neq \emptyset$. If $\mu(0) < \mu(x)$ for some $x \in X$, then there exists $t \in (0, 1)$ such that $\mu(0) < t \leq \mu(x)$. It follows that $0 \notin U(\mu; t)$, a contradiction. Hence $\mu(0) \geq \mu(x)$ for all $x \in X$. Now, suppose there exist $a, b, c \in X$ such that

$$\mu(a) < \min\Big\{\mu((a * b) \diamond c), \mu(b \diamond c)\Big\}.$$

We can select $t \in (0, 1)$ such that $\mu(a) < t \leq \min\Big\{\mu((a * b) \diamond c), \mu(b \diamond c)\Big\}$. Then $(a * b) \diamond c \in U(\mu; t)$ and $b \diamond c \in U(\mu; t)$, but $a \notin U(\mu; t)$. This is a contradiction. Hence $\mu(x) \geq \min\Big\{\mu((x * y) \diamond z),\ \mu(y \diamond z)\Big\}$ for all $x, y, z \in X$. Similarly, we can induce the inequality $\mu(x) \geq \min\{\mu((x \diamond y) * z),\ \mu(y * z)\}$ for all $x, y, z \in X$. Therefore μ is an associative fuzzy pseudo ideal of $\mathfrak{X}$. □

Proposition 6. *If μ is an associative fuzzy pseudo ideal of $\mathfrak{X}$, then*

$$\mu(x) \geq \mu\Big((x * y) \diamond y\Big) = \mu((x \diamond y) * y) \tag{21}$$

for all $x, y \in X$.

Proof. Assume that μ is an associative fuzzy pseudo ideal of $\mathfrak{X}$. Let $x, y \in X$. By taking $z := y$ in (20), we have

$$\mu(x) \geq \min\Big\{\mu((x * y) \diamond y),\ \mu(y \diamond y)\Big\} = \mu((x * y) \diamond y),$$

and the last equality in (21) holds by (b4). □

Proposition 7. *Let μ be an associative fuzzy pseudo ideal of $\mathfrak{X}$. Then*

$$\mu(x) \geq \mu(0 * x) = \mu(0 \diamond x)$$

for all $x \in X$.

Proof. For any $x \in X$, we have $\mu(x) \geq \mu((x * x) \diamond x) = \mu(0 \diamond x) = \mu(0 * x)$ from (21) and Proposition 2. □

Theorem 5. *Every associative fuzzy pseudo ideal of $\mathfrak{X}$ is a fuzzy pseudo ideal of $\mathfrak{X}$.*

Proof. Let μ be an associative fuzzy pseudo ideal of $\mathfrak{X}$, and let $x, y \in X$. Then

$$\mu(x) \geq \min\Big\{\mu((x * y) \diamond 0),\ \mu(y \diamond 0)\Big\} = \min\{\mu(x * y),\ \mu(y)\},$$

$$\mu(x) \geq \min\Big\{\mu((x \diamond y) * 0),\ \mu(y * 0)\Big\} = \min\{\mu(x \diamond y),\ \mu(y)\},$$

by using (20) and (a8). This means that μ is a fuzzy pseudo ideal of $\mathfrak{X}$. □

Proposition 8. *Every associative fuzzy pseudo ideal μ of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned} \mu(y) &\geq \min\{\mu(x * y),\ \mu(x)\}, \\ \mu(y) &\geq \min\{\mu(x \diamond y),\ \mu(x)\} \end{aligned} \tag{22}$$

for all $x, y \in X$.

Proof. Let $x, y \in X$. It follows from (20), (b4), Propositions 3 and 7 that

$$\begin{aligned} \mu(y) &\geq \min\Big\{\mu((y * x) \diamond y),\ \mu(x * y)\Big\} = \min\{\mu((y \diamond y) * x),\ \mu(x * y)\} \\ &= \min\Big\{\mu(0 * x),\ \mu(x * y)\Big\} \geq \min\{\mu(0 \diamond (0 * x)),\ \mu(x * y)\} \\ &\geq \min\{\mu(x),\ \mu(x * y)\}, \end{aligned}$$

$$\begin{aligned}\mu(y) &\geq \min\Big\{\mu((y\diamond x)*y),\ \mu(x\diamond y)\Big\} = \min\{\mu((y*y)\diamond x),\ \mu(x\diamond y)\}\\ &= \min\Big\{\mu(0\diamond x),\ \mu(x\diamond y)\Big\} \geq \min\{\mu(0*(0\diamond x)),\ \mu(x\diamond y)\}\\ &\geq \min\{\mu(x),\ \mu(x\diamond y)\}.\end{aligned}$$

This completes the proof. □

Definition 6. A fuzzy set μ in $\mathfrak{X}$ is called a *fuzzy pseudo q-ideal* of $\mathfrak{X}$ if it satisfies (F1) and

$$\begin{aligned}\mu(x*z) &\geq \min\Big\{\mu(x*(y\diamond z)),\ \mu(y)\Big\},\\ \mu(x\diamond z) &\geq \min\Big\{\mu(x\diamond(y*z)),\ \mu(y)\Big\}\end{aligned} \tag{23}$$

for all $x, y, z \in X$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x*y = x\diamond y$ for all $x, y \in X$, then the notions of a fuzzy pseudo q-ideal and a fuzzy q-ideal coincide.

Theorem 6. *Every fuzzy pseudo q-ideal of $\mathfrak{X}$ is a fuzzy pseudo ideal of $\mathfrak{X}$.*

Proof. Let μ be a fuzzy pseudo q-ideal of $\mathfrak{X}$. Taking $z := 0$ in (23) and using (b8), we have

$$\begin{aligned}\mu(x) = \mu(x*0) &\geq \min\{\mu(x*(y\diamond 0)),\ \mu(y)\} = \min\{\mu(x*y),\ \mu(y)\},\\ \mu(x) = \mu(x\diamond 0) &\geq \min\{\mu(x\diamond(y*0)),\ \mu(y)\} = \min\{\mu(x\diamond y),\ \mu(y)\}\end{aligned}$$

for all $x, y \in X$. Then μ is a fuzzy pseudo ideal of $\mathfrak{X}$. □

Proposition 9. *Every fuzzy pseudo q-ideal μ of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned}\mu(x*y) &\geq \mu\Big(x*(0\diamond y)\Big),\\ \mu(x\diamond y) &\geq \mu\Big(x\diamond(0*y)\Big)\end{aligned} \tag{24}$$

for all $x, y \in X$.

Proof. Let $x, y \in X$. Putting $z := y$ and $y := 0$ in (23) and using (F1), we have

$$\mu(x*y) \geq \min\Big\{\mu(x*(0\diamond y)),\ \mu(0)\Big\} = \mu(x*(0\diamond y)),$$

$$\mu(x\diamond y) \geq \min\Big\{\mu(x\diamond(0*y)),\ \mu(0)\Big\} = \mu(x\diamond(0*y)).$$

This completes the proof. □

Proposition 10. *Every fuzzy pseudo q-ideal μ of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned}\mu((x*y)*z) &\geq \mu(x*(y\diamond z)),\\ \mu((x\diamond y)\diamond z) &\geq \mu(x\diamond(y*z))\end{aligned} \tag{25}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$. From (7), we know that

$$x \diamond (x * (y \diamond z)) \preceq y \diamond z \text{ and } x * (x \diamond (y * z)) \preceq y * z.$$

It follows from (8), (b4), (b7) and Proposition 4 that

$$\begin{aligned}
&\mu\Big(((x * y) * (0 \diamond z)) \diamond (x * (y \diamond z))\Big) = \mu\Big(((x * y) \diamond (x * (y \diamond z))) * (0 \diamond z)\Big) \\
&= \mu(((x \diamond (x * (y \diamond z))) * y) * (0 \diamond z)) \geq \mu(((y \diamond z) * y) * (0 \diamond z)) \\
&= \mu\Big(((y * y) \diamond z) * (0 \diamond z)\Big) = \mu((0 \diamond z) * (0 \diamond z)) = \mu(0)
\end{aligned}$$

and

$$\begin{aligned}
&\mu\Big(((x \diamond y) \diamond (0 * z)) * (x \diamond (y * z))\Big) = \mu\Big(((x \diamond y) * (x \diamond (y * z))) \diamond (0 * z)\Big) \\
&= \mu\Big(((x * (x \diamond (y * z))) \diamond y) \diamond (0 * z)\Big) \geq \mu(((y * z) \diamond y) \diamond (0 * z)) \\
&= \mu\Big(((y \diamond y) * z) \diamond (0 * z)\Big) = \mu((0 * z) \diamond (0 * z)) = \mu(0).
\end{aligned}$$

Then

$$\begin{aligned}
\mu((x * y) * z) &\geq \mu((x * y) * (0 \diamond z)) \\
&\geq \min\Big\{\mu(((x * y) * (0 \diamond z)) \diamond (x * (y \diamond z))),\ \mu(x * (y \diamond z))\Big\} \\
&\geq \min\Big\{\mu(0),\ \mu(x * (y \diamond z))\Big\} \\
&= \mu(x * (y \diamond z))
\end{aligned}$$

and

$$\begin{aligned}
\mu((x \diamond y) \diamond z) &\geq \mu((x \diamond y) \diamond (0 * z)) \\
&\geq \min\Big\{\mu(((x \diamond y) \diamond (0 * z)) * (x \diamond (y * z))),\ \mu(x \diamond (y * z))\Big\} \\
&\geq \min\Big\{\mu(0),\ \mu(x \diamond (y * z))\Big\} \\
&= \mu(x \diamond (y * z)),
\end{aligned}$$

by using Theorem 6 and Proposition 9. This completes the proof. □

Now we provide conditions for a fuzzy pseudo ideal to be a fuzzy pseudo q-ideal.

Theorem 7. *If a fuzzy pseudo ideal μ of $\mathfrak{X}$ satisfies the following conditions:*

$$\begin{aligned}
\mu((x * y) \diamond z) &\geq \mu(x \diamond (y * z)), \\
\mu((x \diamond y) * z) &\geq \mu(x * (y \diamond z))
\end{aligned} \tag{26}$$

for all $x, y, z \in X$, then μ is a fuzzy pseudo q-ideal of $\mathfrak{X}$.

Proof. Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$ that satisfies (26). For any $x, y, z \in X$, it follows from (F2), (b4) and (26) that

$$\begin{aligned}
\mu(x * z) &\geq \min\Big\{\mu((x * z) \diamond y),\ \mu(y)\Big\} = \min\{\mu((x \diamond y) * z),\ \mu(y)\} \\
&\geq \min\Big\{\mu(x * (y \diamond z)),\ \mu(y)\Big\},
\end{aligned}$$

$$\begin{aligned}\mu(x \diamond z) &\geq \min\Big\{\mu((x \diamond z) * y),\, \mu(y)\Big\} = \min\{\mu((x * y) \diamond z),\, \mu(y)\} \\ &\geq \min\Big\{\mu(x \diamond (y * z)),\, \mu(y)\Big\}.\end{aligned}$$

Then μ is a fuzzy pseudo q-ideal of $\mathfrak{X}$. □

Theorem 8. *Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$ which satisfies:*

$$\mu(x * y) \geq \mu(x) \text{ and } \mu(x \diamond y) \geq \mu(x) \tag{27}$$

for all $x, y \in X$. Then μ is a fuzzy pseudo q-ideal of $\mathfrak{X}$.

Proof. Let $x, y, z \in X$. Using (F2) and (27), we have

$$\mu(x * z) \geq \mu(x) \geq \min\Big\{\mu(x * (y \diamond z)),\, \mu(y \diamond z)\Big\} \geq \min\Big\{\mu(x * (y \diamond z)),\, \mu(y)\Big\},$$

$$\mu(x \diamond z) \geq \mu(x) \geq \min\Big\{\mu(x \diamond (y * z)),\, \mu(y * z)\Big\} \geq \min\Big\{\mu(x \diamond (y * z)),\, \mu(y)\Big\},$$

which means that μ is a fuzzy pseudo q-ideal of $\mathfrak{X}$. □

Theorem 9. *Let μ be a fuzzy pseudo ideal of $\mathfrak{X}$ that satisfies (25) and*

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y \;\&\; (x \diamond y) \diamond z = (x \diamond z) \diamond y). \tag{28}$$

Then μ is a fuzzy pseudo q-ideal of $\mathfrak{X}$.

Proof. For any $x, y, z \in X$, we obtain that

$$\begin{aligned}\mu(x * z) &\geq \min\Big\{\mu((x * z) * y),\, \mu(y)\Big\} = \min\{\mu((x * y) * z),\, \mu(y)\} \\ &\geq \min\Big\{\mu(x * (y \diamond z)),\, \mu(y)\Big\}\end{aligned}$$

and

$$\begin{aligned}\mu(x \diamond z) &\geq \min\Big\{\mu((x \diamond z) \diamond y),\, \mu(y)\Big\} = \min\{\mu((x \diamond y) \diamond z),\, \mu(y)\} \\ &\geq \min\Big\{\mu(x \diamond (y * z)),\, \mu(y)\Big\},\end{aligned}$$

by using (F2), (25) and (28). Hence μ is a fuzzy pseudo q-ideal of $\mathfrak{X}$. □

Definition 7. A fuzzy set μ in $\mathfrak{X}$ is called a *fuzzy pseudo a-ideal* of $\mathfrak{X}$ if it satisfies (F1) and

$$\begin{aligned}\mu(z * x) &\geq \min\Big\{\mu((x \diamond y) * (0 \diamond z)),\, \mu(y)\Big\}, \\ \mu(z \diamond x) &\geq \min\Big\{\mu((x * y) \diamond (0 * z)),\, \mu(y)\Big\}\end{aligned} \tag{29}$$

for all $x, y, z \in X$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notions of a fuzzy pseudo a-ideal and a fuzzy a-ideal coincide.

Proposition 11. *If μ is a fuzzy pseudo a-ideal of $\mathfrak{X}$, then*

$$\mu(x) = \mu(0 * x) = \mu(0 \diamond x)$$

for all $x \in X$.

Proof. Let $x \in X$. By using (8), (b8), (F1) and (29), we have

$$\mu(0 * x) \geq \min\Big\{\mu((x \diamond 0) * (0 \diamond 0)),\, \mu(0)\Big\} = \mu(x)$$

and

$$\mu(0 \diamond x) \geq \min\Big\{\mu((x * 0) \diamond (0 * 0)),\, \mu(0)\Big\} = \mu(x),$$

which implies that

$$\begin{aligned} \mu(x) &= \mu(x \diamond 0) \geq \min\Big\{\mu((0 * 0) \diamond (0 * x)),\, \mu(0)\Big\} \\ &= \mu(0 \diamond (0 * x)) \geq \mu(0 * x) \end{aligned}$$

and

$$\begin{aligned} \mu(x) &= \mu(x * 0) \geq \min\Big\{\mu((0 \diamond 0) * (0 \diamond x)),\, \mu(0)\Big\} \\ &= \mu(0 * (0 \diamond x)) \geq \mu(0 \diamond x). \end{aligned}$$

Therefore $\mu(x) = \mu(0 * x) = \mu(0 \diamond x)$ for all $x \in X$. □

Theorem 10. *Every fuzzy pseudo a-ideal of $\mathfrak{X}$ is a fuzzy pseudo ideal of $\mathfrak{X}$.*

Proof. Let μ be a fuzzy pseudo a-ideal of $\mathfrak{X}$, and let $x, y \in X$. Then

$$\mu(x) = \mu(0 \diamond x) \geq \min\Big\{\mu((x * y) \diamond (0 * 0)),\, \mu(y)\Big\} = \min\{\mu(x * y),\, \mu(y)\}$$

and

$$\mu(x) = \mu(0 * x) \geq \min\Big\{\mu((x \diamond y) * (0 \diamond 0)),\, \mu(y)\Big\} = \min\{\mu(x \diamond y),\, \mu(y)\},$$

by using (8), (b8), (29) and Proposition 11. This means that μ is a fuzzy pseudo ideal of $\mathfrak{X}$. □

Proposition 12. *Every fuzzy pseudo a-ideal μ of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned} \mu(y * (x \diamond z)) &\geq \mu\Big((x \diamond z) * (0 \diamond y)\Big), \\ \mu(y \diamond (x * z)) &\geq \mu\Big((x * z) \diamond (0 * y)\Big) \end{aligned} \tag{30}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$. It follows from (b8), (F1) and (29) that

$$\begin{aligned} \mu(y * (x \diamond z)) &\geq \min\Big\{\mu(((x \diamond z) \diamond 0) * (0 \diamond y)),\, \mu(0)\Big\} \\ &= \mu((x \diamond z) * (0 \diamond y)), \\ \mu(y \diamond (x * z)) &\geq \min\Big\{\mu(((x * z) * 0) \diamond (0 * y)),\, \mu(0)\Big\} \\ &= \mu((x * z) \diamond (0 * y)). \end{aligned}$$

This completes the proof. □

Taking $z := 0$ in (30) and using (b8), we have the following corollary.

Corollary 2. *Every fuzzy pseudo a-ideal μ of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned} \mu(y * x) &\geq \mu(x * (0 \diamond y)), \\ \mu(y \diamond x) &\geq \mu(x \diamond (0 * y)) \end{aligned} \tag{31}$$

for all $x, y \in X$.

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