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STRONG CONVERGENCE OF MONOTONE CQ ITERATIVE PROCESS FOR ASYMPTOTICALLY STRICT PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. T.H. Kim, H.K. Xu, [Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions, Nonlinear Anal.(2007),doi:10.1016/j.na.2007.02.029.] proved the strong convergence for asymptotically strict pseudo-contractions by the classical CQ iterative method. In this paper, we apply the monotone CQ iterative method to modify the classical CQ iterative method of T.H. Kim, H.K. Xu, and to obtain the strong convergence theorems for asymptotically strict pseudo-contractions. In the proved process of this paper, Cauchy sequences method is used, so we complete the proof without using the demi-closedness principle, Opial's condition or others about weak topological technologies. In addition, we use a ingenious technology to avoid defining that $F(T)$ is bounded. On the other hand, we relax the restriction on the control sequence of iterative scheme.

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1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a self-mapping of C. We use $F(T)$ to denote the set of fixed points of T ; that is, $F(T) = \{x \in C : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \emptyset$.

Strict pseudo-contractions in Hilbert spaces were introduced by Browder and Petryshyn [2]. Given a closed convex subset C of a Hilbert space H, a mapping $T : C \to C$ is said to be a *strict pseudo-contraction* [2] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

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for all $x, y \in C$. If (1.1) holds, we also say that T is a *$k-$strict pseudo-contraction.* These mappings are extensions of nonexpansive mappings which satisfy the inequality (1.1) with $k = 0$. That is, $T : C \to C$ is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

Given a closed convex subset C of a Hilbert space H, a mapping $T : C \to C$ is said to be an *asymptotically $k-$strict pseudo-contraction* if there exists a constant $0 \leq k < 1$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \tag{1.2}$$

for all $x, y \in C$ and all integer $n \geq 1$, where $\gamma_n \geq 0$ for all n and such that $\gamma_n \to 0$ as $n \to \infty$. Note that if $k = 0$ then T is an asymptotically nonexpansive mapping, a concept introduced by Goebel and Kirk [5] in 1972. That is, T is an *asymptotically nonexpansive* if there exists a sequence $\{\gamma_n\}$ of nonnegative numbers with $\gamma_n \to 0$ and such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2$$

for all $x, y \in C$ and all integer $n \geq 1$.

Iterative methods for nonexpansive mappings and asymptotically nonexpansive mappings have been extensively investigated; see [3, 5, 6, 7, 8, 9, 12, 14, 15, 17] and the references therein.

However, iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn[2] initiated their work in 1967. Needless to say, the development of asymptotically strict pseudo-contractions. The reason is probably that the second term appearing in the right-hand side of (1.1) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction T. On the other hand, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems [16]. Therefore it is interesting to develop the theory of iterative methods for strict pseudo-contractions.

As a matter of fact, Marino and Xu [13] recently show that if a k-strict pseudo-contraction T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the following *Mann's algorithm* [1]:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, n \geq 0 \tag{1.3}$$

converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$.

In order to find a fixed point of an asymptotically k-strict pseudo-contraction T, the *modified Mann's iteration method* is studied in [7, 17, 18] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n \quad n \geq 0 \tag{1.4}$$

It is well known that Mann's iteration method (1.3) has only weak convergence, in general, even for nonexpansive mappings(see the example in [4]). Similarly, the modified Mann's iteration method (1.4) is not convergent strongly for asymptotically strict pseudo-contractions in general. So in order to get strong convergence, one has to modify the iteration method (1.4). Some such modifications for Mann's iteration methods (1.3) can be found in [1, 8, 9, 12, 13, 14].

Quite recently, T. H. Kim and H.K.Xu [10] proposed the following modification of modified Mann's iteration method for an asymptotically $k-$strick pseudo-contraction on a closed convex subset C in a Hilbert space H.

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - v\|^2 \\ \qquad\qquad + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.5}$$

where $\theta_n = \triangle_n^2(1-\alpha_n)\gamma_n \to 0$ $(as\ n \to \infty)$, $\triangle_n = \sup\{\|x_n - z\| : z \in F(T)\} < \infty$.

In the iterative method (1.5), the fixed points set $F(T)$ is assumed bounded. T.H.Kim and H.K.Xu have proven the following convergence theorem.

Theorem KX.[10] *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically k-strict pseudo-contraction for some $0 \leq k < 1$. Assume that the fixed point set $F(T)$ of T is nonempty and bounded. Let $\{x_n\}_{n=0}^{\infty}$be the sequence generated by the (CQ) algorithm (1.5). Assume that the control sequence $\{\alpha_n\}$ is chosen so that* $\limsup_{n\to\infty} \alpha_n < 1-k$. *Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.*

In this article, we will also propose a modification for the algorithm (1.4). The modified algorithm is obtained by applying additional projections onto the intersections of two closed convex subsets which satisfying monotone condition and is guaranteed to have strong convergence.

Our modification for the algorithm (1.4) produces a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad\qquad -(1-\alpha_n)(\alpha_n - k)\|x_n - T^n x_n\|^2 + \theta_n\}, \\ C_0 = \{z \in C : \|y_0 - z\|^2 \leq \|x_0 - z\|^2 \\ \qquad\qquad -(1-\alpha_0)(\alpha_0 - k)\|x_0 - T^n x_0\|^2 + \theta_0\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \end{cases} \tag{1.6}$$

$$\begin{cases} Q_0 = C, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where

$$\theta_n = (1-\alpha_n)\gamma_n(\sup_{z \in A} \|x_n - z\|)^2 \to 0 \ (\ n \to \infty),$$
$$A = \{y \in F(T) : \|y - P_{F(T)}x_0\| \le 1\}.$$

It is easy to see that the algorithm (1.6) is different from the algorithm (1.5). In the proved process of this article, the Cauchy sequence method is used, so that without using the demi-closedness principle, Opial's condition or others about weak topological technologies, we complete the proof. In particular, in the proved process, we use a ingenious technology to avoid defining that $F(T)$ is bounded. It is also worth mentioning that in our algorithm (1.6), the choice of the control sequence $\{\alpha_n\}$ is quite free than that in T. H. Kim and H. K. Xu [10]. That is, relax the restriction on the control sequence $\{\alpha_n\}$ from the $\limsup_{n\to\infty} \alpha_n < 1-k$ to the $\limsup_{n\to\infty} \alpha_n < 1$.

2. Preliminaries

We will use the notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω−limit set of $\{x_n\}$.

In order to prove our main results, we shall make use of the following Lemmas. (see [12] for necessary proof of Lemma 2.2)

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities.*
(*i*) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \ \forall x, y \in H$
(*ii*) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \ \forall x, y \in H$, $t \in [0,1]$.

Lemma 2.2 *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$ and given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

Recall that given a closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted by $P_K x$ in K from x to K; that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \le \|x - y\| \quad \text{for all } y \in K.$$

Lemma 2.3. *Let K be a closed convex subset of a real Hilbert space H.Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \le 0 \quad \textit{for all } y \in K.$$

The following proposition lists a useful property for asymptotically strict pseudo-contractions.(See [10] for necessary proof)

Proposition 2.4 *Assume that C is a closed convex subset of a Hilbert space H. If $T : C \to C$ is an asymptotically $k-$strict pseudo-contraction, for each $n \geq 1$, T^n satisfies the Lipschitz condition:*

$$\|T^n x - T^n y\| \leq L_n \|x - y\| \quad \forall x, y \in C. \text{ where } L_n = \frac{k + \sqrt{1 + \gamma_n(1-k)}}{1-k}.$$

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically $k-$strict pseudo-contraction for some $0 \leq k < 1$. Assume that the fixed point set $F(T)$ of T is nonempty and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequences in $(0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define the sequence $\{x_n\}$ in C generated by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad -(1-\alpha_n)(\alpha_n - k)\|x_n - T^n x_n\|^2 + \theta_n\}, \\ C_0 = \{z \in C : \|y_0 - z\|^2 \leq \|x_0 - z\|^2 \\ \qquad -(1-\alpha_0)(\alpha_0 - k)\|x_0 - T^n x_0\|^2 + \theta_0\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3.1}$$

where

$$\theta_n = (1-\alpha_n)\gamma_n (\sup_{z\in A} \|x_n - z\|)^2 \to 0 \ (\ n \to \infty),$$

$$A = \{y \in F(T) : \|y - P_{F(T)} x_0\| \leq 1\}.$$

Then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

Proof. Firstly, we observe that C_n is convex and closed for every $n \geq 0$ by Lemma 2.2. It is easy to see that $A = \{y \in F(T) : \|y - p_0\| \leq 1\}$ is a bounded closed convex subset of H, where $p_0 = P_{F(T)} x_0$. So we can obtain that $A \subset F(T)$ and $p_0 = P_A x_0$.

Now,we show that $A \subset C_n$ for all $n \geq 0$. Indeed, for all $p \in A$ and $n \geq 0$, we have

$$\begin{aligned} \|y_n - p\|^2 =& \|\alpha_n(x_n - p) + (1-\alpha_n)(T^n x_n - p)\|^2 \\ =& \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|T^n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\ \leq& \alpha_n \|x_n - p\|^2 + (1-\alpha_n)[(1+\gamma_n)\|x_n - p\|^2 + k\|x_n - T^n x_n\|^2] \\ &- \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \end{aligned}$$

$$=(1+(1-\alpha_n)\gamma_n)\|x_n-p\|^2-(1-\alpha_n)(\alpha_n-k)\|x_n-T^nx_n\|^2$$
$$\leq\|x_n-p\|^2-(1-\alpha_n)(\alpha_n-k)\|x_n-T^nx_n\|^2+\theta_n.$$

So that $p\in C_n$ for all $n\geq 0$. That is $A\subset C_n$ for all $n\geq 0$.

Next, we prove that $A\subset C_n\cap Q_n$ for all $n\geq 0$. It suffices to show $A\subset Q_n$ for all $n\geq 0$. We prove this by induction. For $n=0$, $A\subset F(T)\subset C=Q_0$ holds. Assume that $A\subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n\cap Q_n$, by Lemma2.3, we have

$$\langle x_{n+1}-z,x_0-x_{n+1}\rangle\geq 0\ \ \forall z\in C_n\cap Q_n.$$

As $A\subset C_n\cap Q_n$ by the assumption of induction, the last inequality holds. In particular, for all $z\in A$. This together with the definition of Q_{n+1} implies that $A\subset Q_{n+1}$. Hence, $A\subset C_n\cap Q_n$ for all $n\geq 0$.

Note that the definition of Q_n actually implies $x_n=P_{Q_n}x_0$. This together with the fact $A\subset Q_n$ further implies

$$\|x_n-x_0\|\leq\|p-x_0\|\quad \textit{for all } p\in A.$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n-x_0\|\leq\|p_0-x_0\|\ \ p_0=P_Ax_0. \tag{3.2}$$

Furthermore $x_n=P_{Q_n}x_0$ which together with the fact that $x_{n+1}\in C_n\cap Q_n$ implies that

$$\|x_n-x_0\|\leq\|x_{n+1}-x_0\|. \tag{3.3}$$

This further implies that the sequence $\{\|x_n-x_0\|\}$ is increasing. Since $\{x_n\}$ is bounded, so that $\lim_{n\to\infty}\|x_n-x_0\|$ exists.

Note again that $x_n=P_{Q_n}x_0$, hence for any positive integer m, we have $x_{n+m}\in Q_{n+m-1}\subset Q_n$ which implies that $\langle x_{n+m}-x_n,x_n-x_0\rangle\geq 0$. This together with Lemma2.1(i) follows that

$$\begin{aligned}\|x_{n+m}-x_n\|^2=&\|(x_{n+m}-x_0)-(x_n-x_0)\|^2\\=&\|x_{n+m}-x_0\|^2-\|x_n-x_0\|^2-2\langle x_{n+m}-x_n,x_n-x_0\rangle\\\leq&\|x_{n+m}-x_0\|^2-\|x_n-x_0\|^2.\end{aligned} \tag{3.4}$$

From the result (3.4), we know that $\{x_n\}$ is a Cauchy sequence in C, so that there exists a point $p\in C$ such that $\lim_{n\to\infty}x_n=p$.

Next, we prove

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \tag{3.5}$$

In fact, on the one hand, by the fact $x_{n+1}\in C_n$, we get

$$\|x_{n+1}-y_n\|^2\leq\|x_{n+1}-x_n\|^2-(1-\alpha_n)(\alpha_n-k)\|x_n-T^nx_n\|^2+\theta_n. \tag{3.6}$$

On the other hand, since $y_n=\alpha_nx_n+(1-\alpha_n)T^nx_n$, using (3.6) we have

$$
\begin{aligned}
&(1-\alpha_n)^2\|x_n - T^n x_n\|^2 = \|y_n - x_n\|^2 \\
\le&\|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\
\le&(k-\alpha_n)(1-\alpha_n)\|x_n - T^n x_n\|^2 + \theta_n \\
&+ 2(\|x_{n+1} - x_n\|^2 + \|y_n - x_{n+1}\|\|x_{n+1} - x_n\|).
\end{aligned}
$$

It follows that

$$
(1-k)(1-\alpha_n)\|x_n - T^n x_n\|^2 \le 2(\|x_{n+1} - x_n\|^2 + \|y_n - x_{n+1}\|\|x_{n+1} - x_n\|) + \theta_n. \tag{3.7}
$$

Since $\alpha_n \le 1-\varepsilon$ for some $\varepsilon > 0$, $\theta_n \to 0$ and

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}
$$

So that $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ holds. By using (3.8) again, we get

$$
\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.9}
$$

Next we show (3.5) implies that

$$
\lim_{n\to\infty} \|x_n - T x_n\| = 0. \tag{3.10}
$$

As a matter of fact, we obtain

$$
\begin{aligned}
\|x_n - Tx_n\| \le&\|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\| \\
\le&(1+L_1)\|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\|.
\end{aligned} \tag{3.11}
$$

By the definition of y_n, we have

$$
\begin{aligned}
\|T^n x_n - T^{n+1} x_n\| \le&\|T^n x_n - y_n\| + \|y_n - x_{n+1}\| \\
&+ \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| \\
\le&\alpha_n\|x_n - T^n x_n\| + \|y_n - x_{n+1}\| \\
&+ \|x_{n+1} - T^{n+1} x_{n+1}\| + L_{n+1}\|x_{n+1} - x_n\|.
\end{aligned} \tag{3.12}
$$

Combining(3.11) with (3.12), yields

$$
\begin{aligned}
\|x_n - Tx_n\| \le& (1+\alpha_n+L_1)\|x_n - T^n x_n\| + \|y_n - x_{n+1}\| \\
&+\|x_{n+1} - T^{n+1} x_{n+1}\| + L_{n+1}\|x_{n+1} - x_n\|.
\end{aligned}
$$

Now, together (3.5)(3.9) and (3.8) implies (3.10). We have proved that $\{x_n\}$ converges in norm to point $p \in C$ which together with (3.10)implies that p is a fixed point of T. we claim that $p = p_0 = P_{F(T)} x_0$. If not, we have $\|x_0 - p\| > \|x_0 - p_0\|$. There must exists a positive integer N, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - p_0\|$, which leads to

$$
\begin{aligned}
\|x_0 - p_0\|^2 =&\|x_0 - x_n + x_n - p_0\|^2 \\
=&\|x_0 - x_n\|^2 + \|x_n - p_0\|^2 + 2\langle x_n - p_0, x_0 - x_n\rangle.
\end{aligned}
$$

It follows that $\langle x_n - p_0, x_0 - x_n\rangle < 0$, implies that $p_0 \notin Q_n$ This is a contradiction, hence $p = p_0$ and the proof is complete. □

Remark. Theorem 3.1 improves and extends Theorem KX of T. H. Kim and H. K. Xu [10] in several respects:

(1). From CQ iterative method modified to monotone CQ iterative method, so that new method of proof is used. In addition, since monotone CQ method satisfies the following relation

$$C_{n+1} \cap Q_{n+1} \subset C_n \cap Q_n, \quad \forall\, n \geq 0,$$

so that the monotone CQ iterative method has more desirable property;

(2). Without assuming that fixed points set is bounded;

(3). Relax the restriction on the control sequence $\{\alpha_n\}$ from the $\limsup_{n\to\infty} \alpha_n < 1-k$ to the $\limsup_{n\to\infty} \alpha_n < 1$.

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