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DYNAMICS OF A HIGHER ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT. In this paper, we investigate the invariant interval, periodic character, semicycle and global attractivity of all positive solutions of the equation $y_{n+1} = \frac{p+qy_{n-k}}{1+y_n+ry_{n-k}}$, $n = 0, 1, \ldots$, where the parameters p, q, r and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are positive real numbers, $k \in \{1, 2, 3, \ldots\}$.It is worth to mention that our results solve the open problem proposed by Kulenvic and Ladas in their monograph [Dynamics of Second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman & Hall/CRC, Boca Raton, 2002]

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1. Introduction and preliminaries

Our aim in this paper is to study the dynamical behavior of the following difference equation

$$y_{n+1} = \frac{p + qy_{n-k}}{1 + y_n + ry_{n-k}}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters p, q, r and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are positive real numbers, $k \in \{1, 2, 3, \ldots\}$.

The study of properties of rational difference equations has been an area of intense interest in recent years; see [1-10] and the references therein.

For the sake of convenience, we recall some notations and results which will be useful in the sequel. Let I be some interval of real numbers and let f be a continuously differentiable function defined on $I \times I$.Then, for initial conditions $x_{-k}, \ldots, x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad , n = 0, 1, \ldots, \tag{1.2}$$

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has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x}$ is called an equilibrium point of (1.2) if $\overline{x} = f(\overline{x}, \overline{x})$. That is , $x_n = \overline{x}$, for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of f.

An interval $J \subseteq I$ is called an invariant interval of the difference Eq. (1.2) if $y_{N-k+1}, \cdots, y_{N-1}, y_N \in J \Rightarrow x_n \in J$ for some $N \geq 0$.
That is, every solution of Eq.(1.2) with initial conditions in J remains in J.

Let $a = \frac{\partial f}{\partial u}(\overline{x}, \overline{x})$ and $b = \frac{\partial f}{\partial v}(\overline{x}, \overline{x})$, $f(u, v)$ is the function in Eq.(1.2) and $\overline{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = ay_n + by_{n-k}, \quad , n = 0, 1, \ldots \tag{1.3}$$

is called the linearized equation associated with Eq.(1.2) about the equilibrium point $\overline{x}$. Its characteristic equation is

$$\lambda^{k+1} - a\lambda^k - b = 0. \tag{1.4}$$

Lemma 1.1. [1,4,6,7] *Assume that α, $\beta \in \boldsymbol{R}$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$|\alpha| + |\beta| < 1, \tag{1.5}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} - \alpha x_n + \beta x_{n-k} = 0, \quad n = 0, 1, 2, \ldots. \tag{1.6}$$

Suppose in addition that one of the following two cases holds:
(a) k is odd and $\beta < 0$; (b) k is even and $\alpha\beta < 0$.
Then (1.5) is also a necessary condition for the asymptotic stability of equation (1.6).

Lemma 1.2. [8] *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots \tag{1.7}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non-increasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is non-decreasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) The difference equation (1.7) has no solutions of prime period two in $[a, b]$.

Then Eq.(1.7) has a unique equilibrium $\overline{y}$ and every solution of Eq.(1.7) converges to $\overline{y}$.

Lemma 1.3. [8] *Assume that $f \in [(0, \infty) \times (0, \infty), (0, \infty)]$ is such that: $f(x, y)$ is decreasing in x for each fixed y, and $f(x, y)$ is increasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of Eq. (1.2). Then except possibly for the first semicycle, every oscillatory solution of Eq. (1.2) has semicycle of length k.*

Theorem 1.1. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots \tag{1.7}$$

where $k \in \{1, 2, \dots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that* $f : [a, b] \times [a, b] \to [a, b]$ *is a continuous function satisfying the following properties:*

(i) $f(x, y)$ *is non-increasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, *and* $f(x, y)$ *is non-decreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$;

(ii) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system* $f(m, M) = M$ *and* $f(M, m) = m$, *then* $m = M$.

Then Eq.(1.7) has a unique equilibrium $\overline{y} \in [a, b]$ *and every solution of Eq.(1.7) converges to* $\overline{y}$.

Proof. Set $m_0 = a$ and $M_0 = b$ and for $i = 1, 2, \dots$, set $m_i = f(M_{i-1}, m_{i-1})$ and $M_i = f(m_{i-1}, M_{i-1})$.

Now, we observe that for $i \geq 0$, $m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0$ and $m_i \leq y_n \leq M_i$ for $n \geq (i-1)k + i$. set $m = \lim_{i\to\infty} m_i$ and $M = \lim_{i\to\infty} M_i$.

Then clearly $m \leq \lim_{i\to\infty} \inf y_i \leq \lim_{i\to\infty} \sup y_i \leq M$ and by continuity of f, $m = f(M, m)$ and $M = f(m, M)$. and so $m = M = \overline{y}$. The proof is complete. □

2. Linearized stability and periodic character

In the section, we consider the linearized stability and periodic character of the positive solutions of Eq.(1.1). Eq.(1.1) has the unique positive equilibrium $\overline{y}$ given by

$$\overline{y} = \frac{q - 1 + \sqrt{(q-1)^2 + 4p(1+r)}}{2(1+r)}$$

The linearized equation associated with Eq.(1.1) about $\overline{y}$ is

$$Z_{n+1} + \frac{p + q\overline{y}}{(1 + (1+r)\overline{y})^2} Z_n - \frac{q - pr + q\overline{y}}{(1 + (1+r)\overline{y})^2} Z_{n-k} = 0, \quad n = 0, 1, \dots,$$

and it's characteristic equation is

$$\lambda^{k+1} + \frac{p + q\overline{y}}{(1 + (1+r)\overline{y})^2} \lambda^k - \frac{q - pr + q\overline{y}}{(1 + (1+r)\overline{y})^2} = 0.$$

From this and Lemma1.1., we have the following result.

Theorem 2.1. *The positive equilibrium* $\overline{y}$ *of Eq.(1.1) is locally asymptotically stable provided that one of the following two conditions is satisfied:*

(i) $q - pr + q\overline{y} > 0$ *and* $p + q - pr + 2q\overline{y} < (1 + (1+r)\overline{y})^2$;

(ii) $q - pr + q\overline{y} \leq 0$ *and* $p - q + pr < (1 + (1+r)\overline{y})^2$.

Theorem 2.2. *The positive equilibrium* $\overline{y}$ *of Eq.(1.1) is unstable provided that one of the following two conditions is satisfied:*

(a) k *is even,* $q - pr + q\overline{y} < 0$ *and* $p - q + pr \geq (1 + (1+r)\overline{y})^2$;

(b) k *is odd,* $q - pr + q\overline{y} > 0$ *and* $p + q - pr + 2q\overline{y} \geq (1 + (1+r)\overline{y})^2$.

In the following, we will consider the periodic character of Eq.(1.1).

Theorem 2.3. *The following results are true:*

(1) *If $q > 1$ and $0 < r < 1$, then Eq.(1.1) has prime period-two solutions $\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$ iff k is odd and $(q-1)^2(1-r) > 4pr^2$, where the values of Φ and Ψ are the (positive and distinct) solutions of the quadratic equation*

$$t^2 - \frac{q-1}{r}t + \frac{p}{1-r} = 0.$$

(2) *If $0 < q \leq 1$ and $r \geq 1$, then Eq.(1.1) has no nonnegative prime period-two solutions.*

Proof. (1) Let $q > 1$, $0 < r < 1$. Assume that there exist distinct nonnegative real number Φ and Ψ, such that

$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$

is a prime period-two solution of Eq.(1.1). There are two cases to be considered.

Case(a): k is odd.

In this case, $y_{n+1} = y_{n-k}$, Φ and Ψ satisfy

$$\Phi = \frac{p + q\Phi}{1 + \Psi + r\Phi} \quad \text{and} \quad \Psi = \frac{p + q\Psi}{1 + \Phi + r\Psi}$$

then we have

$$\Phi + \Psi = \frac{q-1}{r}, \quad \text{and} \quad \Phi\Psi = \frac{p}{1-r}, \quad r \neq 0, 1$$

Now, consider the quadratic equation $t^2 - \frac{q-1}{r}t + \frac{p}{1-r} = 0$. So, the value of Φ and Ψ are the (positive and distinct) solutions of the above quadratic equation, i.e.,

$$t = \frac{\frac{q-1}{r} \pm \sqrt{(\frac{q-1}{r})^2 - \frac{4p}{1-r}}}{2}.$$

Case(b): k is even.

In this case, $y_n = y_{n-k}$, Φ and Ψ satisfy

$$\Phi = \frac{p + q\Psi}{1 + \Psi + r\Psi} \quad \text{and} \quad \Psi = \frac{p + q\Phi}{1 + \Phi + r\Phi}$$

then we have $(\Phi - \Psi)(1 + q) = 0$, So, $\Phi = \Psi$. This contradicts that Φ and Ψ distinct nonnegative real number.

(2) According to the assumption, there are four cases to be considered. (i) $0 < q < 1$, $r > 1$; (ii) $0 < q < 1$, $r = 1$; (iii) $q = 1$, $r > 1$; (iv) $q = 1, r = 1$.

Now , we just give the proof of the theorem for case (i), the other three cases are similar and we omitted them. Let $0 < q < 1$, $r > 1$. Assume that there exist distinct nonnegative real number Φ and Ψ, such that $\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$ is a prime period-two solution of Eq.(1.1). There are two cases to be considered.

Case(a): k is odd.

In this case, $y_{n+1} = y_{n-k}$, Φ and Ψ satisfy

$$\Phi = \frac{p + q\Phi}{1 + \Psi + r\Phi} \quad \text{and} \quad \Psi = \frac{p + q\Psi}{1 + \Phi + r\Psi}$$

then we have

$$\Phi + \Psi = \frac{q-1}{r}, \quad \text{and} \quad \Phi\Psi = \frac{p}{1-r}, \quad r \neq 0, 1$$

Since $\Phi\Psi > 0$, Φ, Ψ is distinct nonnegative real number, this implies that

$$p(1-r) > 0,$$

that is, $r < 1$, which contradicts the hypothesis that $r > 1$.

Case(b): k is even.

In this case, $y_n = y_{n-k}$, Φ and Ψ satisfy

$$\Phi = \frac{p + q\Psi}{1 + \Psi + r\Psi} \quad \text{and} \quad \Psi = \frac{p + q\Phi}{1 + \Phi + r\Phi}$$

then we have $(\Phi - \Psi)(1 + q) = 0$, So, $\Phi = \Psi$. This contradicts that Φ and Ψ distinct nonnegative real number. The proof is complete. □

3. Analysis of semicycles and oscillations

In this section, we give the character of semicycles, oscillations and invariant intervals.

Theorem 3.1. *Assume that $q - pr \geq 0$, then either $\{y_n\}$ oscillates about the equilibrium $\overline{y}$ with semicycles of length k after the first semicycle, or y_n converges monotonically to $\overline{y}$.*

Proof. The proof follows from Lemma 1.3 by observing that the condition $q - pr \geq 0$ implies that the function

$$f(x, y) = \frac{p + qy}{1 + x + ry},$$

is decreasing in x and increasing in y. The proof is complete. □

We now examine the existence of intervals which attract all solution of Eq.(1.1).

Theorem 3.2. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1). Then the following statements are true:*

(1) *Suppose $q - pr \geq 0$, $0 < q \leq 1$, $r \geq 1$, and assume that for some $N \geq 0$,*

$$y_{N-k+1}, \cdots, y_{N-1}, y_N \in [\frac{p}{2+q-p}, p],$$

then $y_n \in [\frac{p}{2+q-p}, p]$ for all $n > N$. That is, the interval $[\frac{p}{2+q-p}, p]$ is an invariant interval of Eq.(1.1).

(2) *Suppose $q - pr \geq 0$, $q > 1$, $0 < r < 1$, and assume that for some $N \geq 0$,*

$$y_{N-k+1}, \cdots, y_{N-1}, y_N \in [\frac{pr}{q+r}, \frac{q}{r}],$$

then $y_n \in [\frac{pr}{q+r}, \frac{q}{r}]$, for all $n > N$. That is, the interval $[\frac{pr}{q+r}, \frac{q}{r}]$ is an invariant interval of Eq.(1.1).

Proof. (1) If for some $N>0$, $\frac{p}{2+q-p}\le y_N\le p$, then

$$y_{n+1}=\frac{p+qy_{n-k}}{1+y_n+ry_{n-k}}\le\frac{p+ry_{n-k}}{1+y_n+ry_{n-k}}\le p,\ \text{since } q<r$$

also, it follows that $0<\frac{p}{2+q-p}\le y_N\le p<1$, then

$$y_{n+1}\ge\frac{p+py_{n-k}}{1+y_n+ry_{n-k}}\ge\frac{p(1+\frac{p}{2+q-p})}{2+rp}\ge\frac{p(1+\frac{p}{2+q-p})}{2+q}\ge\frac{p}{2+q-p}$$

(2) If for some $N>0$, $\frac{pr}{q+r}\le y_N\le\frac{q}{r}$, then

$$y_{n+1}=\frac{p+qy_{n-k}}{1+y_n+ry_{n-k}}\le\frac{\frac{q}{r}+qy_{n-k}}{1+ry_{n-k}}=\frac{q}{r},$$

On the other hand,

$$y_{n+1}=\frac{p+qy_{n-k}}{1+y_n+ry_{n-k}}\ge\frac{p+q\frac{pr}{q+r}}{1+\frac{q}{r}+q}=\frac{pr}{q+r}$$

The proof is complete. □

4. Global stability

In this section, we will give the global stability for the equilibrium of the Eq.(1.1).

Theorem 4.1. *Assume that* $0<q\le1,\ r\ge1,\ \frac{q}{1+r}\le p\le\frac{q}{r}$, *then the positive equilibrium of Eq.(1.1) is globally asymptotically stable.*

Proof. Set

$$f(x,y)=\frac{p+qy}{1+x+ry},$$

when $q-pr\ge0$, the function $f(x,y)$ is decreasing in x for each fixed y, and increasing in y for each fixed x, also, clearly

$$\frac{p}{2+q-p}\le f(x,y)\le p,\qquad \text{for all } x,\ y>0.$$

and when $\frac{q}{1+r}\le p\le\frac{q}{r}$, $\frac{p}{2+q-p}\le\overline{y}\le p$.

Secondly, if $0<q\le1,\ r\ge1$, the only solution of the system

$$m=\frac{p+qm}{1+M+rm},\quad M=\frac{p+qM}{1+m+rM}$$

is $m=M$.

Finally, when $0<q\le1,\ r\ge1$, Eq.(1.1) has no solutions of prime period two . Now the result is a consequence of Theorem 1.1. □

Theorem 4.2. *Assume that* $q>1,\ 0<r<1, q-pr\ge0$ *and* k *odd, then the positive equilibrium of Eq.(1.1) is globally asymptotically stable when* $(q-1)^2(1-r)<4pr^2$.

Proof. Set

$$f(x,y)=\frac{p+qy}{1+x+ry},$$

when $q-pr\geq 0$, the function $f(x,y)$ is decreasing in x for each fixed y, and increasing in y for each fixed x, also, clearly

$$\frac{pr}{q+r}\leq f(x,y)\leq\frac{q}{r}\text{ for all }x,y>0.$$

Finally, since $(q-1)^2(1-r)<4pr^2$, Eq.(1.1) has no prime period-two solution. Now the conclusion of the theorem follows as a consequence of Lemma 1.2. □

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