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THE GENERALIZATION OF STYAN MATRIX INEQUALITY ON HERMITIAN MATRICES

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ABSTRACT. We point out: to make Hermtian matrices A and B satisfy Styan matrix inequality, the condition "positive definite property" demanded in the present literatures is not necessary. Furthermore, on the premise of abandoning positive definite property, we derive Styan matrix inequality of Hadamard product for inverse Hermitian matrices and the sufficient and necessary conditions that the equation holds in our paper.

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1. Introduction

Let $C^{n\times n}$, $H(n)$, $H_0^+(n)$ and $H^+(n)$ be the sets of $n\times n$ complex, Hermitian, positive semi-definite and positive definite matrices, respectively. When the matrices $A, B \in H(n)$ satisfy $A - B \in H_0^+(n)$, we call A, B to have Löwner partial ordering inequality $A \geq B$ or $B \leq A$. Therefore, if $A \in H_0^+(n)$ ($H^+(n)$), write it as $A \geq 0 (> 0)$. $A(\alpha, \beta)$ is the submatrix of A lying in rows indexed by $\alpha \subset < n >= \{1, 2, \cdots, n\}$ and columns indexed by $\beta \subset < n >$, and $A(\alpha, \alpha) = A(\alpha)$. $A^* = (\overline{A})^T$ is the conjugate transpose matrix of A, I is identity matrix. $E_{ii} \in C^{n\times n}$ is diagonal matrix that its ith diagonal element is 1 otherwise 0, and $Z_n = [E_{11}, E_{22}, \cdots, E_{nn}]^* \in C^{n^2\times n}$. Hadamard and Kronecker products of matrices $A = (a_{ij})$, $B = (B_{ij})$ are denoted by $A \circ B = (a_{ij}b_{ij})$ and $A \otimes B = (a_{ij}B)$, respectively. If every diagonal element of $R \in H_0^+(n)$ is 1, we call R as a correlation matrix (see [8] or [2, 5.5.10]). $CH^+(n)$ denotes the set of $n \times n$ positive definite correlation matrices.

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In 1973, using the method of multivariate analysis, Styan (see [8, Theorem 4.1 and Corollary 4.2(4.21)]) proved the matrix inequalities as follows

$$R\circ R-2(R^{-1}\circ R+I)^{-1}\geq 0,\ \ R\in CH^{+}(n); \tag{1}$$

$$R^{-1}\circ R+I-2(R\circ R)^{-1}\geq 0,\ \ R\in CH^{+}(n). \tag{2}$$

Moreover, the general forms of (1) and (2) are obtained as well (see[8, Corollary 4.3] or [12, pp174-175, (6.1.11), (6.1.12)])

$$A\circ A-2(A\circ I)(A^{-1}\circ A+I)^{-1}(A\circ I)\geq 0,\ \ A\in H^{+}(n); \tag{3}$$

$$A^{-1}\circ A+I-2(A\circ I)(A\circ A)^{-1}(A\circ I)\geq 0,\ \ A\in H^{+}(n). \tag{4}$$

Meanwhile, Styan [8] pointed out "a matrix-theoretic proof of Theorem 4.1 would be of interest".

The researches on Styan matrix inequalities (1) and (2) have been noticeable all the while. [1,4,5,9,11] etc gave many generalized results by applying the matrix method. In 1979, for $A, B\in H^{+}(n)$, T. Anto derived the following conclusion(see [1, Theorem 20 and pp239] or [12, pp175])

$$A\circ B\geq (A\circ I+B\circ I)(A\circ B^{-1}+A^{-1}\circ B+2I)^{-1}(A\circ I+B\circ I), \tag{5}$$

In 2000, S. Liu generalized (4) to the positive semi-definite matrix (see [4, Proposition 1]), and in 2002, S. Liu also introduced this conclusion on Khatri-Rao product (see [5, Theorem 5]). In 2000, as an application of the obtained results, Visick gave (4) and (2) (see [9, Theorem 20]). In the same year, when $A, B\in H^{+}(n)$, F. Zhang derived the further results in [11, Application 4 and (15)]) as follows

$$A\circ B^{-1}+A^{-1}\circ B+2I\geq (A\circ I+B\circ I)(A\circ B)^{-1}(A\circ I+B\circ I), \tag{6}$$

$$A\circ B^{-1}+A^{-1}\circ B+2I\geq 4(A\circ B)^{-1},\ \ A, B\in CH^{+}(n). \tag{7}$$

In view of the ideas of [4] and [5], Al Zhour and Kilicman studied the generalization of Styan matrix inequality for Khatri-Rao product on the basis of [9] in 2006 (see [14, Theorems 4.6-4.8] or [15, Theorem 2.11]), these results in [14] are all about positive definite matrices, the researched object in [14] and that in [1,2,4,5,8,9,11] are all positive (semi)definite matrices.

Example 1. *Let* $A=\frac{1}{2}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$, $B=\frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\in H(2)$, *then the inequality* (6) *holds, it is because*

$$(A\circ B^{-1}+A^{-1}\circ B+2I)-(A\circ I+B\circ I)(A\circ B)^{-1}(A\circ I+B\circ I)=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\in H_0^{+}(2).$$

Example 1 indicates "positive definite property" is not necessary condition that Styan matrix inequality (6) holds.

In our paper, we will give Styan matrix inequality of the inverse Hermitian matrices, our results summarize the corresponding ones about positive definite matrices in the present literatures. The discussion on equation conditions of matrix inequalities is very important (see [6], [7, Corollary 1.3], [10] etc), but [1,2,4,5,8,9,11] do not deal with this problem, however, our results include equation conditions. By the following Example 2, we know that not all invertible Hermitian matrices can satisfy the inequality (6), thus it is natural to add some restricted conditions.

Example 2. *Let* $A=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \in H(2)$, *then the inequality* (6) *does not hold, this is because*

$$(A\circ B^{-1}+A^{-1}\circ B+2I)-(A\circ I+B\circ I)(A\circ B)^{-1}(A\circ I+B\circ I)=\frac{1}{6}\begin{bmatrix} -29 & -31 \\ -31 & -34 \end{bmatrix}.$$

2. Prepared knowledge

From [2, Problem 4.2.14] and [9], for complex matrices A,B,C,D with appropriate order, we have

$$(A\otimes C)\otimes(B\otimes D)=AB\otimes CD; \tag{8}$$

$$(A+B)\otimes C=A\otimes C+B\otimes C,\ \ A\otimes(B+C)=A\otimes B+A\otimes C; \tag{9}$$

$$\begin{gathered}(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}\ (\text{where } A \text{ and } B \text{ are inveritible}),\\ (A\otimes B)^*=A^*\otimes B^*;\end{gathered} \tag{10}$$

$$(A\circ B)=Z_n^*(A\otimes B)Z_n,\ \ A,B\in C^{n\times n},\ \ Z_n=[E_{11},E_{22},\cdots,E_{nn}]^*\in C^{n^2\times n}; \tag{11}$$

$$(A\otimes B)\in H(n^2)(\text{or } H_0^+(n^2),H^+(n^2)),\ \ A\circ B\in H(n)(\text{or } H\text{or }{}_0^+(n),H^+(n)), \tag{12}$$

where $A,\ B\in H(n)$ (or $H_0^+(n)$, $H^+(n)$) in (12). For $M\in C^{n\times n}$ and $M(\alpha')$ is invertible (appointing $[M(\alpha')]^{-1}=M(\alpha')^{-1}$), we call

$$M/\alpha'(=M/M(\alpha'))=M(\alpha)-M(\alpha,\alpha')M(\alpha')^{-1}M(\alpha',\alpha),$$

as the general Schur complement of $M(\alpha')$ in M, where $\alpha\subset<n>$, $\alpha'=<n>-\alpha$. In view of [13], there is a permutation matrix U such that

$$U^*MU=\begin{bmatrix} M(\alpha) & M(\alpha,\alpha') \\ M(\alpha',\alpha) & M(\alpha') \end{bmatrix}\in C^{n\times n}, \tag{13}$$

thus

$$U^*MU/M(\alpha')=M/\alpha'=M(\alpha)-M(\alpha,\alpha')M(\alpha')^{-1}M(\alpha',\alpha). \tag{14}$$

Lemma 1. *Let $C \in C^{n\times n}$, $M \in H(n)$ be invertible and appoint $C(\alpha)^* = (C(\alpha))^*$, $\alpha \subset < n >$, $\alpha' = < n > -\alpha$. If $M^{-1}(\alpha') > 0$, then $M(\alpha)$ is invertible and*

$$C(\alpha)^* M(\alpha)^{-1} C(\alpha) \leq (C^* M^{-1} C)(\alpha), \tag{15}$$

moreover, the equality in (15) *holds if and only if*

$$M^{-1}(\alpha', \alpha) C(\alpha) + M^{-1}(\alpha') C(\alpha', \alpha) = 0.$$

Proof. In view of [12], there exists a permutation matrix U such that (13) holds. Noting that

$$U^*(C^* M^{-1} C) U = (U^* C U)^* (U^* M^{-1} U)(U^* C U),$$

thus, from (13), we get

$$U^* M^{-1} U = \begin{bmatrix} M^{-1}(\alpha) & M^{-1}(\alpha, \alpha') \\ M^{-1}(\alpha', \alpha) & M^{-1}(\alpha') \end{bmatrix} \in H(n),$$

$$U^* C U = \begin{bmatrix} C(\alpha) & C(\alpha, \alpha') \\ C(\alpha', \alpha) & C(\alpha') \end{bmatrix},$$

$$U^*(C^* M^{-1} C) U = \begin{bmatrix} (C^* M^{-1} C)(\alpha) & (C^* M^{-1} C)(\alpha, \alpha') \\ (C^* M^{-1} C)(\alpha', \alpha) & (C^* M^{-1} C)(\alpha') \end{bmatrix} \in H(n). \tag{16}$$

According to [3, 7.7.5] or [13, Application 1], we have

$$(F/\alpha)^{-1} = F^{-1}(\alpha') \ ((F/\alpha')^{-1} = F^{-1}(\alpha))$$

when $F \in H(n)$ and $F(\alpha)(F(\alpha'))$ are invertible .

From the known conditions, it shows that $M^{-1} \in H(n)$ and $M^{-1}(\alpha')$ are invertible, then by the above facts, it follows that $M(\alpha)$ is invertible and $M^{-1}/\alpha' = M(\alpha)^{-1}$.

Moreover, let $W = \begin{bmatrix} I & 0 \\ -A^{-1}(\alpha')^{-1} A^{-1}(\alpha', \alpha) & I \end{bmatrix}$, from (13), (14) and (16), we have

$$\begin{aligned} W^* U^* M^{-1} U W &= diag(M^{-1}/\alpha', M^{-1}(\alpha')) \\ &= M^{-1}/\alpha' \oplus M^{-1}(\alpha') \\ &= M(\alpha)^{-1} \oplus M^{-1}(\alpha'), \end{aligned} \tag{17}$$

$$W^{-1} U^* C U = \begin{bmatrix} C(\alpha) & C(\alpha, \alpha') \\ X & Y \end{bmatrix}, \tag{18}$$

where $X = M^{-1}(\alpha')^{-1} M^{-1}(\alpha', \alpha) C(\alpha) + C(\alpha', \alpha)$. From the fact $X^* M^{-1}(\alpha') X \geq 0$ obtained by using $M^{-1}(\alpha') > 0$ and the identity

$$U^* C^* M^{-1} C U = (W^{-1} U^* C U)^* (W^* U^* M^{-1} U W)(W^{-1} U^* C U),$$

thus from (16)∼(18), we have

$$\begin{aligned}(C^*M^{-1}C)(\alpha) &= (C(\alpha)^*, X^*)(M(\alpha)^{-1} \oplus M^{-1}(\alpha'))(C(\alpha)^*, X^*)^* \\ &= C(\alpha)^*M(\alpha)^{-1}C(\alpha) + X^*M^{-1}(\alpha')X \\ &\geq C(\alpha)^*M(\alpha)^{-1}C(\alpha),\end{aligned}$$

namely, (15) holds. And

$$\text{the equation holds } \Leftrightarrow X^*M^{-1}(\alpha')X = 0 \Leftrightarrow X = 0,$$

which is equivalent to

$$M^{-1}(\alpha')X = M^{-1}(\alpha', \alpha)C(\alpha) + M^{-1}(\alpha')C(\alpha', \alpha) = 0,$$

this completes the proof of Lemma 1. □

When $M \in H^+(n)$, then $M^{-1}(\alpha') > 0$ holds automatically, hence [11, (7) in Theorem 1] can be obtained by Lemma 1.

Lemma 2. *Let $F, G \in H^+(n)$, for any $T \in C^{n\times n}$, then*

$$F \geq TG^{-1}T^* \Leftrightarrow G \geq T^*F^{-1}T \text{ and } F = TG^{-1}T^* \Leftrightarrow G = T^*F^{-1}T.$$

Proof. Meantime, we have $M = \begin{bmatrix} F & T \\ T^* & G \end{bmatrix} \in H(2n)$, and there exist $P = \begin{bmatrix} I & 0 \\ -G^{-1}T^* & I \end{bmatrix}$ and $Q = \begin{bmatrix} I & -F^{-1}T \\ 0 & I \end{bmatrix}$, such that

$$P^*MP = (F - TG^{-1}T^*) \oplus G, \qquad Q^*MQ = F \oplus (G - T^*F^{-1}T),$$

which indicates that

$$F \geq TG^{-1}T^* \Leftrightarrow P^*MP \in H_0^+(2n) \Leftrightarrow Q^*MQ \in H_0^+(2n) \Leftrightarrow G \geq T^*F^{-1}T$$

and

$$\begin{aligned}F = TG^{-1}T^* &\Leftrightarrow rankP^*MP = rankG = n \\ &\Leftrightarrow rankQ^*MQ = rankF = n \\ &\Leftrightarrow G = T^*F^{-1}T.\end{aligned}$$

□

3. Main results

By applying (11) and [9], we derive

$$(A \otimes B)(\alpha) = A \circ B, \quad A, B \in C^{n\times n} \tag{19}$$

where $\alpha = \{1, n+2, 2n+3, \cdots, (n-2)n+n-1, (n-1)n+n\} \subset < n^2 >$.

Theorem 1. *Let $A, B \in H(n)$ be invertible, α be determined by (19) and $\alpha' = <n^2> - \alpha$. If $(A \otimes B)^{-1}(\alpha') > 0$, then $A \circ B$ is invertible and*

$$A \circ B^{-1} + A^{-1} \circ B + 2I \geq (A \circ I + B \circ I)(A \circ B)^{-1}(A \circ I + B \circ I). \tag{20}$$

Moreover, the equation in (20) holds if and only if

$$(A \otimes B)^{-1}(\alpha', \alpha)(A \circ I + B \circ I) + (A \otimes B)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha) = 0. \tag{21}$$

Proof. Let $C = A \otimes I + I \otimes B$, $M = A \otimes B$. From (12), we have $C \in H(n^2)$. Through (10), it follows that M is invertible. Applying the above facts and the assumption $M^{-1}(\alpha') = (A \otimes B)^{-1}(\alpha') > 0$, we obtain that C and M satisfy the hypothesis of Lemma 1, furthermore, from (8),(9), (10) and (19), it follows that $A \circ B = (A \otimes B)(\alpha)$ is invertible, and

$$\begin{aligned} C^*M^{-1}C &= (A \otimes I + I \otimes B)(A^{-1} \otimes B^{-1})(A \otimes I + I \otimes B) \\ &= A \otimes B^{-1} + A^{-1} \otimes B + 2(I \otimes I), \end{aligned}$$

however, $C(\alpha) = (A \otimes I + I \otimes B)(\alpha) = (A \otimes I)(\alpha) + (I \otimes B)(\alpha)$, furthermore, from (19), it follows that

$$(C^*M^{-1}C)(\alpha) = A \circ B^{-1} + A^{-1} \circ B + 2I; \tag{22}$$

$$C(\alpha) = A \circ I + I \circ B = A \circ I + B \circ I. \tag{23}$$

Thus, by (15), (22) and (23), we have

$$\begin{aligned} &(A \circ I + B \circ I)(A \circ B)^{-1}(A \circ I + B \circ I) \\ &= C(\alpha)^* M(\alpha)^{-1} C(\alpha) \\ &\leq (C^*M^{-1}C)(\alpha) \\ &= A \circ B^{-1} + A^{-1} \circ B + 2I \end{aligned}$$

i.e., the inequality (20) holds. By applying (8), (9), (10)and (23), it follows that

$$\begin{aligned} &M^{-1}(\alpha', \alpha)C(\alpha) + M^{-1}(\alpha')C(\alpha', \alpha) \\ &= (A \otimes B)^{-1}(\alpha', \alpha)C(\alpha) + (A \otimes B)^{-1}(\alpha')C(\alpha', \alpha) \\ &= (A \otimes B)^{-1}(\alpha', \alpha)(A \circ I + B \circ I) + (A \otimes B)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha), \end{aligned}$$

thereby, we have gotten that

the equation in the inequality (20) $\Leftrightarrow$ the equation (21) holds. □

Corollary 1. *Let $A, B \in H^+(n)$, then the inequalities (20) (i.e., (6)) and (5) hold, and*

the equation in (20) *holds* $\Leftrightarrow$ *the equation in* (5) *holds*
$\Leftrightarrow$ *the equation* (21) *holds,*

particularly, when $A,\ B \in CH^+(n)$, then the inequality (7) *and*

$$A \circ B \geq 4(A \circ B^{-1} + A^{-1} \circ B + 2I)^{-1} \tag{24}$$

hold, and the equation in (7) holds $\Leftrightarrow$ the equation in (24) holds $\Leftrightarrow$

$$2(A \otimes B)^{-1}(\alpha', \alpha) + (A \otimes B)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha) = 0$$

Proof. In terms of (12), it follows that $A \otimes B \in H^+(n^2)$, thus the hypothesis of Theorem 1 is satisfied, it follows that the inequality (20) (that is (6)) holds, and the equality in (20) holds $\Leftrightarrow$ (21) holds.

Let $F = A \circ B^{-1} + A^{-1} \circ B + 2I$, $G = A \circ B$ and $T = A \circ I + B \circ I$. From (12), we have both F and G are positive definite, thus by applying Lemma 2, we derive

$$F \geq TG^{-1}T^* \Leftrightarrow G \geq T^*F^{-1}T,$$

thereby,

the inequality (20) holds $\Leftrightarrow$ the inequality (5) holds

and

the inequality (20) holds $\Leftrightarrow$ the equation in (5) holds.

When $A, B \in CH^+(n)$, then $A \circ I + B \circ I = 2I$, thus the inequality (7) can be obtained by (20). And using (5) or Lemma 2 again, it follows that the inequality (24) holds, according to the same reason, we can easily get the corresponding equation condition. □

Take $A = B$ in Corollary 1, we obtain

Corollary 2. *Let $A \in H^+(n)$, then the inequalities (3) and (4) hold, and the equation in (3) holds $\Leftrightarrow$ the equation in (4) holds $\Leftrightarrow$*

$$2(A \otimes A)^{-1}(\alpha', \alpha)(A \circ I) + (A \otimes A)^{-1}(\alpha')(A \otimes I + I \otimes A)(\alpha', \alpha) = 0; \tag{25}$$

In particular, when $R \in CH^+(n)$, it follows that the inequalities (1) and (2) hold, and the equation in (1) holds $\Leftrightarrow$ the equation in (2) holds $\Leftrightarrow$

$$2(R \otimes R)^{-1}(\alpha', \alpha) + (R \otimes R)^{-1}(\alpha')(R \otimes I + I \otimes R)(\alpha', \alpha) = 0. \tag{26}$$

In light of the proof of Corollaries 1 and 2, we know that (1) and (2), (3) and (4), (5) and (6), (7) and (24) are mutual determined, moreover, they can be obtained from Theorem 1, the basic reason is that: when $A, B \in H^+(n)$, the hypothesis of Theorem 1 is naturally satisfied, meantime, the equation conditions of the inequalities related to [1,2,4,5,8,9,11] can be obtained from Corollaries 1 and 2. From Example 1, we know Theorem 1 generalizes Styan matrix inequality (6) of positive definite matrices in [1,2,4,5,8,9,11] to more extensive range. In fact, $A, B \in H(n)$ in Example 1 are not positive definite, but they can still make the inequality (6) hold, the reason lies on: meantime, from $A^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$, $\alpha = \{1,4\}$, $\alpha' = \{2,3\}$, we derive

$$(A^{-1} \otimes B^{-1})(\alpha') = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} > 0,$$

i.e., which satisfies the hypothesis of Theorem 1.

Example 3. *Let A, B be the ones like Example 1, meantime, although the inequality (6) holds, but (5) does not hold. It is because*

$$(A \circ B) - (A \circ I + I \circ B)(A^{-1} \circ B + B^{-1} \circ A + 2I)^{-1}(A \circ I + B \circ I) = \frac{1}{30}\begin{bmatrix} -10 & 10 \\ 10 & -16 \end{bmatrix}.$$

Example 3 indicates: after abandoning "positive definite property", the inequalities (5) and (6) are not mutual determined anymore.

Theorem 2. *Let $A, B \in H(n)$ be invertible, and α be determined by (19), $\alpha' =< n^2 > -\alpha$. If $A \otimes I + I \otimes B$ is invertible and $(A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha') > 0$, then the inequality (5) holds, moreover, the equation in (5) holds $\Leftrightarrow$*

$$(A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha', \alpha)(A \circ I + B \circ I) +$$
$$(A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha) = 0. \tag{27}$$

Proof. Let $M = A \otimes B^{-1} + A^{-1} \otimes B + 2I$, $C = A \otimes I + I \otimes B \in H(n^2)$. We konw $A \otimes B \in H(n^2)$ is invertible from (10). In this case, by (8), (9), (10) and the fact that $C = C^* = A \otimes I + I \otimes B \in H(n^2)$ is invertible, it follows that

$$\begin{aligned} C^*(A \otimes B)^{-1}C &= (A \otimes I + I \otimes B)(A \otimes B)^{-1}(A \otimes I + I \otimes B) \\ &= (A \otimes I + I \otimes B)(A^{-1} \otimes B^{-1})(A \otimes I + I \otimes B) \\ &= A \otimes B^{-1} + A^{-1} \otimes B + 2I = M \in H(n^2) \end{aligned}$$

is also invertible. And by the known conditions and (19), we have

$$M^{-1}(\alpha') = (A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha') > 0,$$

which shows that the matrices M and C satisfy the hypothesis of Lemma 1. Noting that (8), (9), (10) and (19), we derive

$$A \otimes B = C^*(C^*(A \otimes B)^{-1}C)^{-1}C = C^*M^{-1}C \in H(n^2);$$

$$\begin{aligned} &(A \circ I + B \circ I)(A \circ B^{-1} + A^{-1} \circ B + 2I)^{-1}(A \circ I + B \circ I) \\ &= (A \otimes I + I \otimes B)(\alpha)\,(A \otimes B^{-1} + A^{-1} \otimes B + 2I)(\alpha)^{-1}\,(A \otimes I + I \otimes B)(\alpha) \\ &= C^*(\alpha)M(\alpha)^{-1}C(\alpha). \end{aligned}$$

Thus, in terms of (15) in Lemma 1, we have

$$\begin{aligned} A \circ B &= (A \otimes B)(\alpha) = (C^* M^{-1} C)(\alpha) \\ &\geq C^*(\alpha) M(\alpha)^{-1} C(\alpha) \\ &= (A \circ I + B \circ I)(A \circ B^{-1} + A^{-1} \circ B + 2I)^{-1}(A \circ I + B \circ I), \end{aligned}$$

namely, the inequality (5) holds. Furthermore, by Lemma 1, it follows that the equation in (5) holds $\Leftrightarrow$

$$\begin{aligned} & M^{-1}(\alpha', \alpha) C(\alpha) + M^{-1}(\alpha') C(\alpha', \alpha) \\ &= (A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha', \alpha)(A \circ I + I \circ B) \\ &\quad + (A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha) \\ &= 0, \end{aligned}$$

which indicates that the sufficient and necessary condition that the equation in (5) holds can be determined by the identity (27). □

From (8), (9) , (10) and (12), we know that $A \otimes I + I \otimes B$ and $A \otimes B^{-1} + A^{-1} \otimes B + 2I$ are positive definite when $A, B \in H^+(n)$, so the hypothesis of Theorem 2 is satisfied naturally. Thus, Theorem 2 is viewed as the generalization of Styan matrix inequalities (1), (3) and (5) under abandoning the condition of "positive definite property". As the application of Theorems 1 and 2, we can get the generalization of Styan matrix inequality (1)~(4) about the inverse Hermitian matrix easily .

Corollary 3. *Let $A \in H(n)$ be invertible, α be determined by (19), $\alpha' =< n^2 > -\alpha$, then*

(1) *if $A \otimes I + I \otimes A$ is invertible and $(A \otimes A^{-1} + A^{-1} \otimes A + 2I)^{-1}(\alpha') > 0$, then*

$$A \circ A - 2(A \circ I)(A^{-1} \circ A + I)^{-1}(A \circ I) \geq 0$$

and the equality holds

$$\begin{aligned} &\Leftrightarrow 2(A \otimes A^{-1} + A^{-1} \otimes A + 2I)^{-1}(\alpha', \alpha)(A \circ I) \\ &\quad + (A \otimes A^{-1} + A^{-1} \otimes A + 2I)^{-1}(\alpha')(A \otimes I + I \otimes A)(\alpha', \alpha) = 0; \end{aligned}$$

(2) *if $(A \otimes A)^{-1}(\alpha') > 0$, then $A \circ A$ is invertible and*

$$A^{-1} \circ A + I - 2(A \circ I)(A \circ A)^{-1}(A \circ I) \geq 0,$$

and the equality holds $\Leftrightarrow$ the identity (25) holds.

Corollary 4. *Let $CH(n)$ be the set of inverse Hermitian matrices that all the diagonal elements are 1. When $R \in CH(n)$, if α is determined by (19), $\alpha' =< n^2 > -\alpha$, then*

(1) *if* $R \otimes I + I \otimes R$ *is invertible and* $(R \otimes R^{-1} + R^{-1} \otimes R + 2I)^{-1}(\alpha') > 0$, *then*

$$R \circ R - 2(R^{-1} \circ R + I)^{-1} \geq 0,$$

and the equality holds

$$\Leftrightarrow 2(R \otimes R^{-1} + R^{-1} \otimes R + 2I)^{-1}(\alpha', \alpha) + (R \otimes R^{-1} + R^{-1} \otimes R + 2I)^{-1}(\alpha')(R \otimes I + I \otimes R)(\alpha', \alpha) = 0;$$

(2) *if* $(R \otimes R)^{-1}(\alpha') > 0$, *then* $R \circ R$ *is invertible and*

$$R^{-1} \circ R + I - 2(R \circ R)^{-1} \geq 0,$$

and the equality holds $\Leftrightarrow$ *the identity* (26) *holds.*

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