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CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce a modified three-step iteration scheme with errors for asymptotically nonexpansive mappings in the framework of uniformly convex Banach spaces. Weak and strong convergence theorems are established.

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1. Introduction and preliminaries

Let E be real Banach space and C a nonempty subset of E. Throughout this paper, we always assume that N denotes the set of positive integers. Let $T : C \to C$ be a nonlinear mapping. In this paper, we use $F(T) := \{x : Tx = x\}$ to denote the set of fixed points of T.

Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, n \in N.$$

$T : C \to C$ is said to be uniformly *k-Lipschitzain* if for some $k > 0$,

$$\|T^n x - T^n y\| \leq k\|x - y\|, \quad \forall x, y \in C, n \in N.$$

It is easy to see that, if T is asymptotically nonexpansive, then it is uniformly k-*Lipschitzain*. Recently, Mann iterative algorithm, Ishikawa iterative algorithm and Noor iteration algorithm have been studied extensively by many authors. In 1995, Liu [6] introduced iterative algorithm with errors as follows:

The sequence $\{x_n\}$ defined by

$$x_1 = x \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + u_n, \quad n \geq 1, \tag{1.1}$$

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where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{u_n\}$ is a sequence in E satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$ is known as Mann iterative algorithm with errors. The sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \quad n \geq 1 \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$ is known as Ishikawa iterative algorithm with errors.

While it is clear that consideration of errors terms in iterative schemes is an important part of the theory, it is also clear that the iterative schemes with errors introduced by Liu [6], as in (1.1) and (1.2) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (1.1), (1.2) which say that they tend to zero as n tends to infinity are, therefore, unreasonable. Xu [19] introduced a more satisfactory error term in the following iterative algorithm

The sequence $\{x_n\}$ defined by

$$x_1 = x \in C, \quad , x_{n+1} = \alpha_n T x_n + \beta_n x_n + \gamma_n u_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is known as Mann iterative scheme with errors. This algorithm reduces to Mann iterative algorithm if $\{\gamma_n\} = 0$.

The sequence $\{x_n\}$ defined by $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \\ y_n = \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in C, is known as Ishikawa iterative algorithm with errors. This algorithm becomes Ishikawa iterative algorithm if $\{\gamma_n\} = \{\gamma'_n\} = 0$. Chidume and Moore [2] and Takahashi and Tamura [18] studied the above schemes in 1999.

The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{cases} z_n = \alpha''_n T x_n + \beta''_n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n T z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \quad n \geq 1 \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C, is known as Noor iterative algorithm with errors. This algorithm becomes Noor iterative algorithm if $\{\gamma_n\} = \{\gamma'_n\} = \{\gamma''_n\} = 0$.

Recently, Khan and Fukhar-ud-din [4] generalized iterative scheme (1.4) to the one with errors as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n S y_n + \beta_n x_n + \gamma_n u_n, \\ y_n = \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ with $0 < \delta \leq \alpha_n, \alpha'_n \leq 1-\delta < 1$, $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in C.

Many authors starting from Das and Debeta [3] and including Takahashi and Tamura [18] and Khan and Takahashi [5] have studied the two mappings case of iterative schemes for different types of mappings. We now suggest an iterative scheme with errors for three classes asymptotically nonexpansive mappings. It worth mentioning that our algorithm can be viewed as an extension of all above schemes.

In this paper, we generalize the algorithm (1.5) to the one with errors as following: $x_1 \in C$ and

$$\begin{cases} z_n = \alpha''_n T_1^n x_n + \beta''_n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n T_2^n z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n T_3^n y_n + \beta_n x_n + \gamma_n u_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. Weak and strong convergence theorems are established in the framework of Banach spaces. The results presented in this paper improve and extend the corresponding results in [4,5,9-14,17,18].

In order to prove our main results, we need the following definitions and results.

Definition 1.1 [8]. A normed space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Definition 1.2 [8]. Let C be a nonempty subset of a normed space E. A mapping $T : C \to E$ is said to be demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 1.1 (Schu [15]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$$

hold for some $r \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.2 (Tan and Xu [17]). *Let* $\{r_n\}$, $\{s_n\}$ *and* $\{t_n\}$ *be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1+s_n)r_n + t_n, \quad \textit{for all } n \in N.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ *and* $\sum_{n=1}^{\infty} t_n < \infty$, *then* $\lim_{n\to\infty} r_n$ *exists.*

Lemma 1.3 (Browder [1]). *Let* E *be a uniformly convex Banach space satisfying Opial's condition and let* C *be a nonempty closed convex subset of* E. *Let* T *be asymptotically nonexpansive mapping of* C *into itself. Then* I-T *is demiclosed with respect to zero.*

Recall that a mapping $T : C \to C$ is said to satisfy the condition (A) [10] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x-p\| : p \in F(T)\}$.

Senter and Dotson [16] approximated fixed points of a nonexpansive mapping T by Mann iterative algorithm, Later on, Maiti and Ghosh [7] and Tan and Xu [17] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterative algorithm under the same condition (A) which is weaker than the requirement that T is demi-compact. We modify this condition for three mappings T_1, T_2 and $T_3 : C \to C$ as follows:

Three mappings T_1, T_2 and $T_3 : C \to C$, are said to satisfy the condition (A$'$) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$a\|x - T_1x\| + b\|x - T_2x\| + c\|x - T_3x\| \geq f(d(x, F(T)))$$

for all $x \in C$ where $d(x, F(T)) = \inf\{\|x-p\| : p \in F(T_1) \cap F(T_2) \cap F(T_3)\}$ and a, b and c are three nonnegative real numbers such that $a + b + c = 1$.

Note that condition (A$'$) reduces to condition (A) when $T_1 = T_2 = T_3$. we shall use the condition (A$'$) instead of compactness of C to study the strong convergence of $\{x_n\}$ defined by (1.6).

2. Main results

In this section, we shall prove the weak and strong convergence of the iterative algorithm (1.6) for asymptotically nonexpansive mappings T_1, T_2 and T_3. We first prove the following lemmas.

Lemma 2.1. *Let* E *be a normed space and* C *a nonempty convex subset. Let* $T_1, T_2, T_3 : C \to C$ *be asymptotically nonexpansive mappings with the sequence* $\{k_n\}$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be the sequence as defined in (1.6)* *with* $\sum_{n=1}^{\infty}\gamma_n'' < \infty$ $\sum_{n=1}^{\infty}\gamma_n' < \infty$ $\sum_{n=1}^{\infty}\gamma_n < \infty$. *If* $F(T_1)\cap F(T_2)\cap F(T_3) \neq \emptyset$, *then* $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in F(T_1)\cap F(T_2)\cap F(T_3)$.

Proof. Let $p \in F(T_1)\cap F(T_2)\cap F(T_3)$. Since w_n, v_n and u_n are bounded sequences in C, we set
$M_1 = \sup\{\|u_n - p\| : n \geq 1\}$, $M_2 = \sup\{\|v_n - p\| : n \geq 1\}$,
$M_3 = \sup\{\|w_n - p\| : n \geq 1\}$, $M = \max\{M_i : i = 1, 2, 3\}$.
It follows from (1.6) that

$$\begin{aligned}\|z_n - p\| &= \|\alpha_n'' T_1^n x_n + \beta_n'' x_n + \gamma_n'' w_n - p\| \\ &\leq \alpha_n''\|T_1^n x_n - p\| + \beta_n''\|x_n - p\| + \gamma_n''\|w_n - p\| \\ &\leq \alpha_n'' k_n\|x_n - p\| + \beta_n''\|x_n - p\| + \gamma_n''\|w_n - p\| \\ &\leq \alpha_n'' k_n\|x_n - p\| + (1-\alpha_n'')\|x_n - p\| + \gamma_n''\|w_n - p\| \\ &\leq k_n\|x_n - p\| + \gamma_n'' M.\end{aligned} \tag{2.1}$$

From (1.6) and (2.1), we get

$$\begin{aligned}\|y_n - p\| &= \|\alpha_n' T_2^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\ &\leq \alpha_n'\|T_2^n z_n - p\| + \beta_n'\|x_n - p\| + \gamma_n'\|v_n - p\| \\ &\leq \alpha_n' k_n\|z_n - p\| + \beta_n'\|x_n - p\| + \gamma_n'\|v_n - p\| \\ &\leq \alpha_n' k_n\|z_n - p\| + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|v_n - p\| \\ &\leq \alpha_n' k_n(k_n\|x_n - p\| + \gamma_n'' M) + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|v_n - p\| \\ &\leq k_n^2\|x_n - p\| + k_n\gamma_n'' M + \gamma_n' M.\end{aligned} \tag{2.2}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n T_3^n y_n + \beta_n x_n + \gamma_n u_n - p\| \\ &\leq \alpha_n\|T_3^n y_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|u_n - p\| \\ &\leq \alpha_n k_n\|y_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|u_n - p\| \\ &\leq \alpha_n k_n\|y_n - p\| + (1-\alpha_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\ &\leq \alpha_n k_n(k_n^2\|x_n - p\| + k_n\gamma_n'' M + \gamma_n' M) + (1-\alpha_n')\|x_n - p\| + \gamma_n M \\ &\leq k_n^3\|x_n - p\| + k_n^2\gamma_n'' M + k_n\gamma_n' M + \gamma_n M.\end{aligned} \tag{2.3}$$

Note that $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. By Lemma 1.2, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T_1)\cap F(T_2)\cap F(T_3)$. This completes the proof. □

Notation. From Lemma 2.1, we obtain that $\lim_{n\to\infty}\|x_n - p\|$ exists. That is, $\{x_n\}$ is bounded. Next, we set
$r_1 = \sup\{\|u_n - x_n\| : n \geq 1\}$, $r_2 = \sup\{\|v_n - x_n\| : n \geq 1\}$,
$r_3 = \sup\{\|w_n - x_n\| : n \geq 1\}$, $r = \max\{r_i : i = 1, 2, 3\}$.

Lemma 2.2. *Let* E *be a normed space and* C *a nonempty convex closed subset. Let* T_1, T_2 *and* $T_3 : C \to C$ *be uniformly* k*-Lipschitzian mappings with the* *constant k. Define a sequence $\{x_n\}$ as in (1.6) with $\{w_n\}$, $\{v_n\}$ and $\{u_n\}$ sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=\alpha''_n+\beta''_n+\gamma''_n=1$ set $d''_n=\|x_n-T_1^nx_n\|$, $d'_n=\|x_n-T_2^nx_n\|$ and $d_n=\|x_n-T_3^nx_n\|$. Then*

$$\|x_{n+1}-T_1x_{n+1}\|\le d''_{n+1}+kd''_n+k(k+1)(d_n+\gamma_nr+k(d'_n+\gamma'_nr+kd''_n+k\gamma''_nr)),$$

$$\|x_{n+1}-T_2x_{n+1}\|\le d'_{n+1}+kd'_n+k(k+1)(d_n+\gamma_nr+k(d'_n+\gamma'_nr+kd''_n+k\gamma''_nr))$$

and

$$\|x_{n+1}-T_3x_{n+1}\|\le d_{n+1}+kd_n+k(k+1)(d_n+\gamma_nr+k(d'_n+\gamma'_nr+kd''_n+k\gamma''_nr)).$$

Proof. From (1.6), we obtain

$$\begin{aligned}\|x_n-z_n\|&=\|x_n-(\alpha''_nT_1^nx_n+\beta''_nx_n+\gamma''_nw_n)\|\\&=\|\alpha''_n(x_n-T_1^nx_n)+\gamma''_n(x_n-w_n)\|\\&\le\|x_n-T_1^nx_n\|+\gamma''_n\|x_n-w_n\|\\&=d''_n+\gamma''_nr.\end{aligned}\tag{2.4}$$

Similarly, it follows from (1.6) and (2.4) that

$$\begin{aligned}\|x_n-y_n\|&=\|x_n-(\alpha'_nT_2^nz_n+\beta'_nx_n+\gamma'_nv_n)\|\\&=\|\alpha'_n(x_n-T_2^nz_n)+\gamma'_n(x_n-v_n)\|\\&\le\|x_n-T_2^nz_n\|+\gamma'_n\|x_n-v_n\|\\&\le\|x_n-T_2^nx_n\|+\|T_2^nx_n-T_2^nz_n\|+\gamma'_n\|x_n-v_n\|\\&\le d'_n+\gamma'_nr+k\|x_n-z_n\|\\&\le d'_n+\gamma'_nr+k(d''_n+\gamma''_nr)\\&=d'_n+\gamma'_nr+kd''_n+k\gamma''_nr.\end{aligned}\tag{2.5}$$

It follows that

$$\begin{aligned}\|x_n-x_{n+1}\|&=\|(\alpha_n+\beta_n+\gamma_n)x_n-(\alpha_nT_3^ny_n+\beta_nx_n+\gamma_nu_n)\|\\&\le\|\alpha_n(x_n-T_3^ny_n)+\gamma_n(x_n-u_n)\|\\&\le\|x_n-T_3^ny_n\|+\gamma_nr\\&\le\|x_n-T_3^nx_n\|+\|T_3^nx_n-x_nT_3^ny_n\|+\gamma_nr\\&\le d_n+\gamma_nr+k\|x_n-y_n\|\\&\le d_n+\gamma_nr+k(d'_n+\gamma'_nr+kd''_n+k\gamma''_nr).\end{aligned}\tag{2.6}$$

Next, we consider

$$\begin{aligned}\|x_{n+1} - T_1x_{n+1}\| &\le \|x_{n+1} - T_1^{n+1}x_{n+1}\| + \|T_1^{n+1}x_{n+1} - T_1x_{n+1}\| \\ &\le d''_{n+1} + k\|T_1^nx_{n+1} - x_{n+1}\| \\ &\le d''_{n+1} + k\|(x_{n+1} - x_n)+(x_n - T_1^nx_n)+(T_1^nx_n - T_1^nx_{n+1})\| \\ &\le d''_{n+1} + k(k+1)\|x_{n+1} - x_n\| + kd''_n.\end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7), we obtain that

$$\|x_{n+1} - T_1x_{n+1}\| \le d''_{n+1} + kd''_n + k(k+1)(d_n + \gamma_n r + k(d'_n + \gamma'_n r + kd''_n + k\gamma''_n r)). \tag{2.8}$$

In a similar way, we can prove that

$$\|x_{n+1} - T_2x_{n+1}\| \le d'_{n+1} + kd'_n + k(k+1)(d_n + \gamma_n r + k(d'_n + \gamma'_n r + kd''_n + k\gamma''_n r)) \tag{2.9}$$

and

$$\|x_{n+1} - T_3x_{n+1}\| \le d_{n+1} + kd_n + k(k+1)(d_n + \gamma_n r + k(d'_n + \gamma'_n r + kd''_n + k\gamma''_n r)) \tag{2.10}$$

This completes the proof of the Lemma 2.2. □

Lemma 2.3. *Let E be a uniformly convex Banach space and C a nonempty convex closed subset. Let T_1, T_2 and $T_3 : C \to C$ be asymptotically nonexpansive mappings with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence as defined in (1.6) with $\sum_{n=1}^{\infty}\gamma''_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\epsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1-\epsilon$ for all $n \in N$ and some $\epsilon > 0$. If $F(T_1) \cap F(T_2) \cap F(T_3) \ne \emptyset$, then*

$$\lim_{n\to\infty}\|T_1x_n - x_n\| = \lim_{n\to\infty}\|T_2x_n - x_n\| = \lim_{n\to\infty}\|T_3x_n - x_n\| = 0.$$

Proof. By Lemma 2.1, we suppose $\lim_{n\to\infty}\|x_n - p\| = c$. Now, Taking limsup on both the sides in the inequality (2.1), we have

$$\limsup_{n\to\infty}\|z_n - p\| \le c. \tag{2.11}$$

Similarly, taking limsup on both the sides in the inequality (2.2), we have

$$\limsup_{n\to\infty}\|y_n - p\| \le c. \tag{2.12}$$

Next, we consider

$$\begin{aligned}\|T_3^ny_n - p + \gamma_n(u_n - x_n)\| &\le \|T_3^ny_n - p\| + \gamma_n\|u_n - x_n\| \\ &\le k_n\|y_n - p\| + \gamma_n M_2.\end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.14), we get that

$$\limsup_{n\to\infty}\|T_3^ny_n - p + \gamma_n(u_n - x_n)\| \le c$$

and

$$\begin{aligned}\|x_n - p + \gamma_n(u_n - x_n)\| &\le \|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\le \|x_n - p\| + \gamma_n M_2,\end{aligned}$$

which imply that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(u_n - x_n)\| \le c.$$

Again, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\lim_{n\to\infty} \|\alpha_n(T_3^n y_n - p + \gamma_n(u_n - x_n)) + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| = c.$$

Hence applying Lemma 1.1, we have

$$\lim_{n\to\infty} \|T_3^n y_n - x_n\| = 0. \tag{2.13}$$

Notice that

$$\begin{aligned}\|x_n - p\| &\le \|T_3^n y_n - x_n\| + \|T_3^n y_n - p\| \\ &\le \|T_3^n y_n - x_n\| + k_n\|y_n - p\|,\end{aligned}$$

which yields that

$$c \le \liminf_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le c.$$

That is,

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Again, $\lim_{n\to\infty} \|y_n - p\| = c$ is expressible as

$$\lim_{n\to\infty} \|\alpha_n'(T_2^n z_n - p + \gamma_n'(v_n - x_n)) + (1-\alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| = c. \tag{2.14}$$

On the other hand, we have

$$\begin{aligned}\|T_2^n z_n - p + \gamma_n'(v_n - x_n)\| &\le \|T_2^n z_n - p\| + \gamma_n'\|v_n - x_n\| \\ &\le k_n\|z_n - p\| + \gamma_n' M_2'.\end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.11), we have

$$\limsup_{n\to\infty} \|T_2^n z_n - p + \gamma_n'(v_n - x_n)\| \le c \tag{2.15}$$

and

$$\begin{aligned}\|x_n - p + \gamma_n'(v_n - x_n)\| &\le \|x_n - p\| + \gamma_n'\|v_n - x_n\| \\ &\le \|x_n - p\| + \gamma_n' M_2',\end{aligned}$$

which yield that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n'(v_n - x_n)\| \le c. \tag{2.16}$$

Applying Lemma 1.1, it follows from (2.14), (2.15) and (2.16) that

$$\lim_{n\to\infty} \|T_2^n z_n - x_n\| = 0. \tag{2.17}$$

Again, notice that

$$\begin{aligned}\|x_n - p\| &\le \|T_2^n z_n - x_n\| + \|T_2^n z_n - p\| \\ &\le \|T_2^n z_n - x_n\| + k_n\|z_n - p\|,\end{aligned}$$

which yields that

$$c \leq \liminf_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq c.$$

That is,

$$\lim_{n\to\infty} \|z_n - p\| = c.$$

Again, $\lim_{n\to\infty} \|z_n - p\| = c$ is expressible as

$$\lim_{n\to\infty} \|\alpha_n''(T_1^n x_n - p + \gamma_n''(w_n - x_n)) + (1-\alpha_n'')(x_n - p + \gamma_n''(w_n - x_n)) - p\| = c. \tag{2.18}$$

Moreover, we have

$$\begin{aligned} \|T_1^n x_n - p + \gamma_n''(w_n - x_n)\| &\leq \|T_1^n x_n - p\| + \gamma_n''\|w_n - x_n\| \\ &\leq k_n\|x_n - p\| + \gamma_n'' M_2'', \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|T_1^n x_n - p + \gamma_n''(w_n - x_n)\| \leq c. \tag{2.19}$$

Similarly, it follows from

$$\begin{aligned} \|x_n - p + \gamma_n''(w_n - x_n)\| &\leq \|x_n - p\| + \gamma_n''\|w_n - x_n\| \\ &\leq \|x_n - p\| + \gamma_n'' M_2'' \end{aligned}$$

that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n''(w_n - x_n)\| \leq c. \tag{2.20}$$

Combine (2.18), (2.19) with (2.20) yields

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0. \tag{2.21}$$

Observe that

$$\begin{aligned} \|x_n - T_2^n x_n\| &\leq \|T_2^n x_n - T_2^n z_n\| + \|T_2^n z_n - x_n\| \\ &\leq k_n\|x_n - z_n\| + \|T_2^n z_n - x_n\| \\ &\leq k_n\|x_n - (\alpha_n'' T_1^n x_n + \beta_n'' x_n + \gamma_n'' w_n)\| + \|T_2^n z_n - x_n\| \\ &\leq k_n\|\alpha_n''(T_1^n x_n - x_n) + \gamma_n''(w_n - x_n)\| + \|T_2^n z_n - x_n\| \\ &\leq k_n\|T_1^n x_n - x_n\| + k_n\gamma_n'' M_2'' + \|T_2^n z_n - x_n\|. \end{aligned}$$

From (2.17) and (2.21), we have

$$\lim_{n\to\infty} \|T_2^n x_n - x_n\| = 0. \tag{2.22}$$

By the same method, we get

$$\begin{aligned} \|x_n - T_3^n x_n\| &\leq \|T_3^n x_n - T_3^n y_n\| + \|T_3^n y_n - x_n\| \\ &\leq k_n\|x_n - y_n\| + \|T_3^n y_n - x_n\| \\ &\leq k_n\|x_n - (\alpha_n' T_2^n z_n + \beta_n' x_n + \gamma_n' v_n)\| + \|T_3^n y_n - x_n\| \\ &\leq k_n\|\alpha_n'(T_2^n z_n - x_n) + \gamma_n'(v_n - x_n)\| + \|T_3^n y_n - x_n\| \\ &\leq k_n\|T_2^n z_n - x_n\| + k_n\gamma_n' M_2' + \|T_3^n y_n - x_n\|. \end{aligned}$$

From (2.13) and (2.17), we arrive at

$$\lim_{n\to\infty} \|T_3^n x_n - x_n\| = 0. \tag{2.23}$$

Hence, we have

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = \lim_{n\to\infty} \|T_2^n x_n - x_n\| = \lim_{n\to\infty} \|T_3^n x_n - x_n\| = 0.$$

Lemma 2.2 reveals that

$$\lim_{n\to\infty} \|T_1 x_n - x_n\| = \lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0.$$

This completes the proof of the lemma 2.3. □

Next, we prove the weak and strong convergence theorems of iterative algorithm (1.6) for the three classes of asymptotically nonexpansive mappings.

Theorem 2.4. *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, T_1, T_2 and T_3 and $\{x_n\}$ be as taken in Lemma 2.1. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.*

Proof. Take $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. As proved in Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T_1) \cap F(T_2) \cap F(T_3)$. Towards this end , let δ_1, δ_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 2.2, and $I - T_1$, $I - T_2$ and $I - T_3$ are demiclosed with respect to zero by Lemma 1.3, therefore, we obtain $T_1\delta_1 = \delta_1$, $T_2\delta_1 = \delta_1$ and $T_3\delta_1 = \delta_1$. Similarly we can prove that $\delta_2 \in p \in F(T_1) \cap F(T_2) \cap F(T_3)$. If $\delta_1 \neq \delta_2$, by *Opial's* condition

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \delta_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - \delta_1\| < \lim_{n_i\to\infty} \|x_{n_i} - \delta_2\| \\ &= \lim_{n\to\infty} \|x_n - \delta_2\| = \lim_{n_j\to\infty} \|x_{n_j} - \delta_2\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - \delta_1\| = \lim_{n\to\infty} \|x_n - \delta_1\|.\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. This completes the proof. □

Theorem 2.5. Let E be a uniformly convex Banach space and C, T_1, T_2, T_3 and $\{x_n\}$ be as in Lemma 2.1. Further T_1, T_2 and T_3 satisfy condition (A$'$). If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F = F(T_1) \cap F(T_2) \cap F(T_3)$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Lemma 2.3, $\lim_{n\to\infty} \|T_1 x_n - x_n\| = \lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0$, and (2.3) gives that

$$\inf_{p\in F} \|x_{n+1} - p\| \leq [1 + (k_n^3 - 1)] \inf_{p\in F} \|x_n - p\| + k_n^2 \gamma_n'' M + k_n \gamma_n' M + \gamma_n' M.$$

That is,

$$d(x_{n+1}, F) \leq [1 + (k_n^3 - 1)]d(x_n, F) + k_n^2\gamma_n''M + k_n\gamma_n'M + \gamma_n'M$$

gives that $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 1.2. Now by condition (A′), $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_j}\}$ of x_n and sequence $y_j \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the method proof of Tan and Xu [17], we get that y_j is a Cauchy sequence in F and so it converges. Let $\{y_j\} \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. As $\lim_{n\to\infty} \|x_n - p\|$ exists, $x_n \to y \in F = F(T_1) \cap F(T_2) \cap F(T_3)$ thereby completing the proof. □

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