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PERIODIC SOLUTIONS FOR DISCRETE ONE-PREDATOR TWO-PREY SYSTEM WITH THE MODIFIED LESLIE-GOWER FUNCTIONAL RESPONSE

XIANGYUN SHI, XUEYONG ZHOU* AND XINYU SONG

ABSTRACT. In this paper, we study a discrete Leslie-Gower one-predator two-prey model. By using the method of coincidence degree and some techniques, we obtain the existence of at least one positive periodic solution of the system. By linalization of the model at positive periodic solution and construction of Lyapunov function, sufficient conditions are obtained to ensure the global stability of the positive periodic solution. Numerical simulations are carried out to explain the analytical findings..

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1. Introduction

Predator-prey phenomena have many important applications in many different field, such as biology, economic, ecology and others sciences. The study of predator-prey phenomena is now a dominant problem in many ecological sciences. There is a growing explicit biological and physiological evidence that in many situations, especially when predators have to search for food, a more suitable general predator-prey theory should be based on the Holling II function response. But, recently, a major trend in theoretical work on prey-predator dynamics has been to derive more realistic models, trying to keep to maximum the unavoidable increase in complexity of their mathematics. These models incorporates a modified version of Leslie-Gower functional response as well as that of the Holling-type II [1-4].

They consider the following model

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$$\begin{cases} \dot{u}(t) = u(\bar{a}_1 - \bar{b}u - \dfrac{\bar{c}_1 v}{u + \bar{k}_1}), \\ \dot{v}(t) = v(\bar{a}_2 - \dfrac{\bar{c}_2 v}{u + \bar{k}_2}), \end{cases}$$

with the initial conditions $u(0) > 0$ and $v(0) > 0$. This two species food chain model describes a prey population u which serves as food for a predator v. The model parameters $\bar{a}_1$, $\bar{a}_2$, $\bar{b}$, $\bar{c}_1$, $\bar{c}_2$, $\bar{k}_1$, $\bar{k}_2$ are assuming only positive values. These parameters are defined as follows: $\bar{a}_1$ is the growth rate of prey u, $\bar{b}$ measures the strength of competition among individuals of species u, $\bar{c}_1$ is the maximum value of the per capita reduction rate of u due to v, $\bar{k}_1$ (respectively, $\bar{k}_2$) measures the extent to which environment provides protection to prey u (respectively, to the predator v), $\bar{a}_2$ describes the growth rate of v, and $\bar{c}_2$ has a similar meaning to $\bar{c}_1$.

The Leslie-Gower formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Indeed, Leslie [5] introduced a predator prey model where the carrying capacity of the predator environment is proportional to the number of prey. He stresses the fact that there are upper limits to the rates of increase of both prey u and predator v, which are not recognized in the Lotka-Volterra model. In case of continuous time, the considerations lead to the following:

$$\frac{dv}{dt} = \bar{a}_2 v(1 - \frac{v}{mu}),$$

where $\bar{a}_2$ describes the growth rate of v and m measures the capacity set by the environmental resources. In this formulation, the growth of the predator population is taken as logistic type, i.e., $\frac{dv}{dt} = \bar{a}_2 v(1 - \frac{v}{K})$. Where the measures of the environmental carrying capacity K is assumed to be proportional to the prey abundance that is, $K = mu$. Thus, the logistic equation becomes $\frac{dv}{dt} = \bar{a}_2 v(1 - \frac{v}{m_1 + mu})$, the additional constant m_1 normalized the residual reduction in the predator population v because of severe scarcity of the favorite food, simplifying, we obtain

$$\frac{dv}{dt} = \bar{a}_2 v - \frac{\bar{a}_2}{m} \frac{v^2}{\frac{m_1}{m} + u} = \bar{a}_2 v - \frac{\bar{c}_2 v^2}{u + \bar{k}_2},$$

where $\bar{k}_2 = \frac{m_1}{m}$ and $\bar{c}_2 = \frac{\bar{a}_2}{m}$.

Food webs are common in nature. Many investigations have been carried out on multi-species ecological systems comprising of food chains of variable lengths. The food web models are more complex and intractable as compared to food chains as more complex multi level interactions are possible in food webs. Relatively less attention has been given to the study of food webs and their rich complex dynamical behavior. One can get the following non-autonomous one-predator two-prey model with the modified Leslie-Gower functional response for the predator:

$$\begin{cases} \dot{x}(t) = x(a_1(t) - b_1(t)x - \frac{c_1(t)z}{x+y+d_1(t)}), \end{cases}$$

$$\begin{cases} \dot{y}(t) = y(a_2(t) - b_2(t)y - \frac{c_2(t)z}{x+y+d_2(t)}), \\ \dot{z}(t) = z(a_3(t) - \frac{c_3(t)z}{x+y+d_3(t)}), \end{cases} \tag{1.1}$$

where x and y are the prey, z is the predator. And $a_i(t)$ $(i = 1,2,3)$, $b_i(t)$ $(i = 1,2)$, $c_i(t)$ $(i = 1,2,3)$, $d_i(t)$ $(i = 1,2,3) \in C(R, R^+)$, $R^+ = (0, +\infty)$ are $\omega-$periodic function. These parameters are defined as follows: $a_1(t)$, $a_2(t)$ and $a_3(t)$ describe the growth rate of x, y and z, respectively; $b_1(t)$, $b_2(t)$ describe measures the strength of competition among individuals of species x and y, respectively; $c_1(t)$ is the maximum value of the per capita reduction rate of x and y due to z; $d_1(t)$ and $d_2(t)$ (respectively, $d_3(t)$) measure the extent to which environment provides protection to prey x and y (respectively, to predator z); $c_2(t)$ and $c_3(t)$ have the similar meanings to $c_1(t)$.

The model (1.1) does not consider any direct competition between the two prey populations, but they are in apparent competition through the shared predation. Indeed, this apparent competition appears, as both prey types are included in predators diet. In the model, the third equation is written according to the Leslie Gower scheme in which the conventional carrying capacity term is being replaced by the renewable resources for the predator as $x + y$. The additional constant d_3 normalizes the residual reductions in the predator population in case of severe scarcity of food.

However, many authors have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations [6-8]. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. By considering the above factors, we extend (1.1) to the following discrete model by the way of deriving discrete time version of dynamical systems corresponding to continuous time formulations:

$$\begin{cases} x(k+1) = x(k)\exp(a_1(k) - b_1(k)x(k) - \frac{c_1(k)z(k)}{x(k)+y(k)+d_1(k)}), \\ y(k+1) = y(k)\exp(a_2(k) - b_2(k)y(k) - \frac{c_2(k)z(k)}{y(k)+x(k)+d_2(k)}), \\ z(k+1) = z(k)\exp(a_3(k) - \frac{c_3(k)z(k)}{x(k)+y(k)+d_3(k)}), \end{cases} \tag{1.2}$$

which can be looked as a discrete analogue of system (1.1), where $a_i(k)$ $(i = 1,2,3)$, $b_i(k)$ $(i = 1,2)$, $c_i(k)$ $(i = 1,2,3)$, $d_i(k)$ $(i = 1,2,3) : Z \to R^+$ are ω periodic, i.e., $a_i(k+\omega) = a_i(k)$, $b_i(k+\omega) = b_i(k)$, $c_i(k+\omega) = c_i(k)$, $d_i(k+\omega) = d_i(k)$, for any $k \in Z$, where Z, R^+ denote the sets of all integers, and nonnegative real numbers, respectively.

In this paper, we derive a set of sufficient conditions for existence of positive periodic solutions for the discrete two prey one predator model (1.2). Such an existence problem is highly non-trivial and to the best of our knowledge, no work has been done for the discrete model (1.2) of modified version of Leslie-Gower functional response as well as that of the Holling-type II.

2. Preliminaries

Let Z, Z^+, R, R^+ and R^3 denote the sets of all integers, nonnegative integers, real numbers, nonnegative real numbers, and three-dimensional Euclidean vector space, respectively. Let $I_w = 0, 1, 2, \cdots, \omega - 1$, $f = \frac{1}{\omega}\sum_{k=0}^{\omega-1} f(k)$, $f^L = \max_{k\in I_\omega} f(k)$, $f^U = \min_{k\in I_\omega} f(k)$, where $f(k)$ is an ω-periodic sequence of nonnegative real numbers defined for $k \in Z$. Let X, Y be normed vector spaces, $L : DomL \cap X \to Y$ be a linear mapping, $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = codimImL < +\infty$ and ImL is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL$, $ImL = KerQ = Im(I - Q)$, it follows that $L|DomL \cap KerP : (I - P)X \to ImL$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN\bar{\Omega}$ is bounded and $KP(I-Q)N : \bar{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \to KerL$.

Lemma 2.1(Continuation Theorem [9]). *Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose:*

(*a*) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(*b*) *$QNx \neq 0$ for each $x \in \partial\Omega \cap KerL$ and the Brouwer degree* $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$. *Then the operator equation $Lx = Nx$ has at least one solution lying in $DomL \cap \bar{\Omega}$.*

Lemma 2.2([10, Lemma 3.2]). *Let $f : Z \to R$ be ω periodic function, i.e., $f(k + w) = f(k)$, then for any fixed $k_1, k_2 \in I_w$, and any $k \in Z$, one has*

$$f(k) \leq f(k_1) + \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|,$$

$$f(k) \geq f(k_2) - \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|.$$

Define $l_3 = \{u = u(k) : u(k) \in R^3, k \in Z^+\}$. Let $l^w \subset l_3$ denote the subspace of all w periodic sequences with the usual supremum norm $\|.\|$, i.e., $\|u\| = \max_{k\in I_w} \|u(k)\|$, for any $u = \{u(k) : k \in Z^+\} \in l_w$. It is not difficult to show l^w is a finite-dimensional Banach space. Let

$$l_0^w = \{u = \{u(k)\} \in l^w : \sum_{s=0}^{\omega-1} u(k) = 0\},$$

$$l_c^w = \{u = \{u(k)\} \in l^w : u(k) = h \in R^3, k \in Z^+\}.$$

Then it follows that l_0^w and l_c^w are both closed linear subspaces of l^w and $l^w = l_0^w \bigoplus l_c^w$, $diml_c^w = 3$.

Lemma 2.3. *If condition* (H): $\min\{\bar{a}_1 - \bar{b}_1, \bar{a}_2\} > \bar{a}_3$, $\max\{c_1^U, c_2^U\} > c_3^L$,

$\max\{\frac{c_1^U}{d_1^L}, \frac{c_2^U}{d_2^L}\} < \frac{c_3^L}{d_3^U}$ *holds, then the algebraic equation*

$$\begin{cases} \bar{a}_1 = \bar{b}_1 v_1 + \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_1(k)v_3}{v_1+v_2+d_1(k)}, \\ \bar{a}_2 = \bar{b}_2 v_2 + \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_2(k)v_3}{v_1+v_2+d_2(k)}, \\ \bar{a}_3 = \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_3(k)v_3}{v_1+v_2+d_3(k)} \end{cases} \tag{2.1}$$

has at least one positive solution (v_1^*, v_2^*, v_3^*).

Proof. From the third equation of (2.1), we have

$$v_3 = \frac{\bar{a}_3\omega}{\sum\limits_{k=0}^{\omega-1} \frac{c_3(k)}{v_1+v_2+d_3(k)}}. \tag{2.2}$$

From the first two equations of (2.1) and the equation (2.2), we can easily derive

$$\bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 = \bar{b}_1\bar{b}_2(v_1+v_2) + \bar{b}_2\bar{a}_3\frac{\sum\limits_{k=0}^{\omega-1}\frac{c_1(k)}{v_1+v_2+d_1(k)}}{\sum\limits_{k=0}^{\omega-1}\frac{c_3(k)}{v_1+v_2+d_3(k)}} + \bar{b}_1\bar{a}_3\frac{\sum\limits_{k=0}^{\omega-1}\frac{c_2(k)}{v_1+v_2+d_2(k)}}{\sum\limits_{k=0}^{\omega-1}\frac{c_3(k)}{v_1+v_2+d_3(k)}}.$$

Define

$$f(u) = \bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 - \bar{b}_1\bar{b}_2 u - \bar{b}_2\bar{a}_3\frac{\sum\limits_{k=0}^{\omega-1}\frac{c_1(k)}{u+d_1(k)}}{\sum\limits_{k=0}^{\omega-1}\frac{c_3(k)}{u+d_3(k)}} - \bar{b}_1\bar{a}_3\frac{\sum\limits_{k=0}^{\omega-1}\frac{c_2(k)}{u+d_2(k)}}{\sum\limits_{k=0}^{\omega-1}\frac{c_3(k)}{u+d_3(k)}}.$$

One can easily see that $\lim\limits_{u\to\infty} f(u) < 0$, and

$$\begin{aligned} f(0) &= \bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 - \bar{b}_1\bar{b}_2 - \bar{b}_2\bar{a}_3\frac{\sum\limits_{k=0}^{\omega-1}\frac{c_1(k)}{d_1(k)}}{\sum\limits_{k=0}^{\omega-1}\frac{c_3(k)}{d_3(k)}} - \bar{b}_1\bar{a}_3\frac{\sum\limits_{k=0}^{\omega-1}\frac{c_2(k)}{d_2(k)}}{\sum\limits_{k=0}^{\omega-1}\frac{c_3(k)}{d_3(k)}} \\ &> \bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 - \bar{b}_1\bar{b}_2 - \bar{b}_2\bar{a}_3\frac{\frac{c_1^U}{d_1^L}}{\frac{c_3^L}{d_3^U}} - \bar{b}_1\bar{a}_3\frac{\frac{c_2^U}{d_2^L}}{\frac{c_3^L}{d_3^U}} \\ &> \bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 - \bar{b}_1\bar{b}_2 - \bar{b}_2\bar{a}_3 - \bar{b}_1\bar{a}_3 > 0. \end{aligned}$$

Then from the zero-point theorem, it follows that there exists a $u^* > 0$ such that $f(u^*) = 0$.

Now we shall claim that $v_i^* > 0 (i = 1, 2, 3)$. Obviously, $v_1^* + v_2^* > 0$ and $v_3^* > 0$. From the first equation of (2.1) and substituting (2.2) into it, we have

$$\bar{a}_1 = \bar{b}_1 v_1 + \bar{a}_3 \frac{\sum_{k=0}^{\omega-1} \frac{c_1(k)}{v_1+v_2+d_1(k)}}{\sum_{k=0}^{\omega-1} \frac{c_3(k)}{v_1+v_2+d_3(k)}}.$$

Then

$$v_1^* = \frac{1}{\bar{b}_1}(\bar{a}_1 - \bar{a}_3 \frac{\sum_{k=0}^{\omega-1} \frac{c_1(k)}{v_1^*+v_2^*+d_1(k)}}{\sum_{k=0}^{\omega-1} \frac{c_3(k)}{v_1^*+v_2^*+d_3(k)}}) > \frac{1}{\bar{b}_1}(\bar{a}_1 - \bar{a}_3 \frac{\sum_{k=0}^{\omega-1} \frac{c_1^U}{v_1^*+v_2^*+d_1^L}}{\sum_{k=0}^{\omega-1} \frac{c_3^L}{v_1^*+v_2^*+d_3^U}}) > 0.$$

Similarly, $v_2^* > 0$. The proof is completed.

3. Main results

Theorem 3.1. *If condition (H) holds, system (1.2) has at least one positive $\omega-$periodic solution.*

Proof. Make the change of variables

$$x_1(k) = \ln\{x(k)\}, \quad x_2(k) = \ln\{y(k)\}, \quad x_3(k) = \ln\{z(k)\}. \tag{3.1}$$

Substituting (3.1) into (1.2), we have

$$\begin{cases} x_1(k+1) - x_1(k) = a_1(k) - b_1(k)e^{x_1(k)} - \frac{c_1(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_1(k)}, \\ x_2(k+1) - x_2(k) = a_2(k) - b_2(k)e^{x_2(k)} - \frac{c_2(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_2(k)}, \\ x_3(k+1) - x_3(k) = a_3(k) - \frac{c_3(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_3(k)}. \end{cases} \tag{3.2}$$

It is easy to say that (3.2) has an $\omega-$periodic solution $\{(x_1^*(k), x_2^*(k), x_3^*(k))\}$, then $\{(x(k),$
$y(k), z(k))\} = \{(e^{x_1^*(k)}, e^{x_2^*(k)}, e^{x_3^*(k)})\}$ is a positive $\omega-$periodic solution of system (1.2). Therefore, to complete the proof, it is only to show that system (3.2) has at least one $\omega-$periodic solution. Define $X = Y = l^w$, the difference operator $L : X \to X$ given by $Lx = \{(Lx)(k)\}$ with $(Lx)(k) = x(k+1) - x(k)$, for $x \in X$ and $k \in Z$, and $N : X \to X$ as follows

$$N \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1(k) - b_1(k)e^{x_1(k)} - \frac{c_1(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_1(k)} \\ a_2(k) - b_2(k)e^{x_2(k)} - \frac{c_2(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_2(k)} \\ a_3(k) - \frac{c_3(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_3(k)} \end{bmatrix}, \tag{3.3}$$

for any $(x_1, x_2, x_3)^T \in X$ and $k \in Z^+$. It is trivially easy to see that L is a bounded linear operator and $KerL = l_c^w$, $ImL = l_0^w$, as well as that $dimKerL = 3 = codimImL$. Since ImL is closed in Y , it follows that L is a Fredholm mapping of index zero. Define

$$P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Q \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\sum\limits_{s=0}^{\omega-1} x_1(s) \\ \frac{1}{\omega}\sum\limits_{s=0}^{\omega-1} x_2(s) \\ \frac{1}{\omega}\sum\limits_{s=0}^{\omega-1} x_3(s) \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in X = Y.$$

It is easy to show that P and Q are continuous projections such that $ImP = KerL$ and $ImL = KerQ = Im(I - Q)$. Then, the generalized inverse $K_P : ImL \to KerP \cap DomL$ exists and is given by

$$K_p(x) = \sum_{k=0}^{k-1} x(s) - \frac{1}{\omega}\sum_{s=0}^{k-1}(\omega - s)x(s).$$

Then $QN : X \to Y$ and $K_p(I - Q)N : X \to X$ are given by

$$QNx = \frac{1}{\omega}\sum_{s=0}^{k-1} Nx(s)$$

and

$$K_p(I - Q)Nx = \sum_{s=0}^{k-1} Nx(s) - \frac{1}{\omega}\sum_{s=0}^{k-1}(\omega - s)Nx(s) - (\frac{k}{\omega} - \frac{l+\omega}{2\omega})\sum_{s=0}^{k-1} Nx(s).$$

It is trivial to show that N is $L-$compact on $\bar{\Omega} \subset X$. In order to apply Lemma 2.2, we need to search for an open bounded subset Ω.

Corresponding to three operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we get

$$\begin{cases} x_1(k+1) - x_1(k) = \lambda[a_1(k) - b_1(k)e^{x_1(k)} - \frac{c_1(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_1(k)}], \\ x_2(k+1) - x_2(k) = \lambda[a_2(k) - b_2(k)e^{x_2(k)} - \frac{c_2(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_2(k)}], \\ x_3(k+1) - x_3(k) = \lambda[a_3(k) - \frac{c_3(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_3(k)}]. \end{cases} \tag{3.4}$$

Suppose that $x = \{x(k)\} = \{(x_1(k), x_2(k), x_3(k))\} \in X$ is a solution of (3.4) for a certain $\lambda \in (0, 1)$. Summing on both sides of (3.4) form 0 to $\omega - 1$ with respect to k, we can derive

$$\begin{cases} \bar{a}_1\omega = \sum\limits_{k=0}^{\omega-1} b_1(k)e^{x_1(k)} + \sum\limits_{k=0}^{\omega-1} \frac{c_1(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_1(k)}, \\ \bar{a}_2\omega = \sum\limits_{k=0}^{\omega-1} b_2(k)e^{x_2(k)} + \sum\limits_{k=0}^{\omega-1} \frac{c_2(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_2(k)}, \\ \bar{a}_3\omega = \sum\limits_{k=0}^{\omega-1} \frac{c_3(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_3(k)}. \end{cases} \tag{3.5}$$

From (3.2) and (3.5), we have

$$\sum_{k=0}^{\omega-1} |x_1(k+1)-x_1(k)| \le \sum_{k=0}^{\omega-1}(a_1(k)+b_1(k)e^{x_1(k)}+\frac{c_1(k)e^{x_3(k)}}{e^{x_1(k)} + e^{x_2(k)} + d_1(k)}) \le 2\bar{a}_1\omega, \tag{3.6}$$

$$\sum_{k=0}^{\omega-1}|x_2(k+1)-x_2(k)| \le \sum_{k=0}^{\omega-1}(a_2(k)+b_2(k)e^{x_2(k)}+\frac{c_2(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_2(k)}) \le 2\bar{a}_2\omega, \tag{3.7}$$

and

$$\sum_{k=0}^{\omega-1}|x_3(k+1)-x_3(k)| \le \sum_{k=0}^{\omega-1}(a_3(k)+\frac{c_3(k)e^{x_3(k)}}{e^{x_1(k)}+e^{x_2(k)}+d_3(k)}) \le 2\bar{a}_3\omega. \tag{3.8}$$

It follows from (3.5) that we have

$$b_1^L\sum_{k=0}^{\omega-1}e^{x_1(k)} < a_1^U\omega, \tag{3.9}$$

$$b_2^L\sum_{k=0}^{\omega-1}e^{x_2(k)} < a_2^U\omega, \tag{3.10}$$

and

$$\frac{c_3^U}{d_3^L}\sum_{k=0}^{\omega-1}e^{x_3(k)} > a_3^L\omega. \tag{3.11}$$

Since $x=\{x(k)\}\in X$, there exist $\xi_i,\eta_i\in I_\omega(i=1,2,3)$ such that $x_i(\xi_i)=\min_{k\in I_\omega}\{x_i(k)\}$, $x_i(\eta_i)=\max_{k\in I_\omega}\{x_i(k)\}$. By (3.9), (3.10) and (3.11) we have

$$x_1(\xi_1) < \ln\frac{a_1^U}{b_1^L}, \tag{3.12}$$

$$x_2(\xi_2) < \ln\frac{a_2^U}{b_2^L}, \tag{3.13}$$

and

$$x_3(\eta_3) > \ln\frac{a_3^L d_3^L}{c_3^U}. \tag{3.14}$$

(3.12) and (3.13) together with (3.6) (3.7) and Lemma 2.2 lead to

$$x_1(k) < x_1(\xi_1)+\sum_{s=0}^{\omega-1}|x_1(s+1)-x_1(s)| < \ln\frac{a_1^U}{b_1^L}+2a_1^U\omega \triangleq H_1, \tag{3.15}$$

and

$$x_2(k) < x_2(\xi_2)+\sum_{s=0}^{\omega-1}|x_2(s+1)-x_2(s)| < \ln\frac{a_2^U}{b_2^L}+2a_2^U\omega \triangleq H_2. \tag{3.16}$$

By using the third equation of system (3.5), one can deduce that

$$a_3^U\omega > \frac{c_3^L e^{x_3(\xi_3)}\omega}{e^{H_1}+e^{H_2}+d_3^U}.$$

From above, we have

$$x_3(\xi_3) < \ln \frac{a_3^U(e^{H_1} + e^{H_2} + d_3^U)}{c_3^L}. \tag{3.17}$$

Also by Lemma 2.2 we have

$$x_3(k) < x_3(\xi_3) + \sum_{s=0}^{\omega-1} |x_3(s+1) - x_3(s)| < \ln \frac{a_3^U(e^{H_1} + e^{H_2} + d_3^U)}{c_3^L} + 2a_3^U\omega \triangleq H_3. \tag{3.18}$$

From (3.5) and (3.18) we have

$$\sum_{k=0}^{\omega-1} b_1(k)e^{x_1(k)} > \bar{a}_1\omega - \sum_{k=0}^{\omega-1} \frac{c_1(k)e^{H_3}}{d_1(k)}.$$

Therefore

$$\omega b_1^U e^{x_1(\eta_1)} > a_1^L\omega - \frac{c_1^U}{d_1^L}e^{H_3}\omega,$$

i.e.,

$$x_1(\eta_1) > \ln \frac{a_1^L d_1^L - c_1^U e^{H_3}}{b_1^U d_1^L}. \tag{3.19}$$

Similarly, we can get

$$x_2(\eta_2) > \ln \frac{a_2^L d_2^L - c_2^U e^{H_3}}{b_2^U d_2^L}. \tag{3.20}$$

It follows from (3.6) (3.7) (3.19) and (3.20) and Lemma 2, we have

$$x_1(k) > x_1(\eta_1) - \sum_{s=0}^{\omega-1} |x_1(s+1) - x_1(s)| > \ln \frac{a_1^L d_1^L - c_1^U e^{H_3}}{b_1^U d_1^L} - 2a_1^U\omega \triangleq S_1, \tag{3.21}$$

$$x_2(k) > x_2(\eta_1) - \sum_{s=0}^{\omega-1} |x_2(s+1) - x_2(s)| > \ln \frac{a_2^L d_2^L - c_2^U e^{H_3}}{b_2^U d_2^L} - 2a_2^U\omega \triangleq S_2. \tag{3.22}$$

Similarly, we have

$$x_3(k) > x_3(\eta_1) - \sum_{s=0}^{\omega-1} |x_3(s+1) - x_3(s)| > \ln \frac{a_3^L d_3^L}{c_3^U} - 2a_3^U\omega \triangleq S_3. \tag{3.23}$$

Denote

$$|x_i(k)| \le \max\{|H_i|, |S_i|\} \triangleq M_i, i = 1, 2, 3.$$

Consider the following algebraic equation

$$\begin{cases} \bar{a}_1 = \bar{b}_1 e^{x_1} + \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_1(k)e^{x_3}}{e^{x_1}+e^{x_2}+d_1(k)}, \\ \bar{a}_2 = \bar{b}_2 e^{x_2} + \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_2(k)e^{x_3}}{e^{x_1}+e^{x_2}+d_2(k)}, \\ \bar{a}_3 = \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_3(k)e^{x_3}}{e^{x_1}+e^{x_2}+d_3(k)}. \end{cases} \tag{3.24}$$

It is easy to see that if equation (2.1) has a positive solution $(v_1^*, v_2^*, v_3^*)^T$, then (3.24) have a positive solution $(x_1^*, x_2^*, x_3^*)^T = (\ln v_1^*, \ln v_2^*, \ln v_3^*)^T$. From Lemma 2.3, we know that the algebraic equations (2.1) has at least one positive solution if condition (H) holds.

Take $M = M_1 + M_2 + M_3 + M_0$, where M_0 is taken sufficiently large such that $\|(x_1^*, x_2^*, x_3^*)^T\|$ $= \max\{|x_1^*|, |x_2^*|, |x_3^*|\} < M_0$. We now take $\Omega = \{x = \{x(k)\} \in X : \|x\| < M\}$. It is clear that Ω is an open bounded set in X and verifies condition (a) in Lemma 2.1. When $x \in \partial\Omega \cap KerL, x = \{(x_1, x_2, x_3)^T\}$, and $(x_1, x_2, x_3)^T$ is a constant vector in R^3 with $\|x\| = \max\{|x_1|, |x_2|, |x_3|\} = M$. Thus, we have

$$QN\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \bar{a}_1 - \bar{b}_1 e^{x_1} - \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_1(k)e^{x_3}}{e^{x_1}+e^{x_2}+d_1(k)}, \\ \bar{a}_2 - \bar{b}_2 e^{x_2} - \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_2(k)e^{x_3}}{e^{x_1}+e^{x_2}+d_2(k)}, \\ \bar{a}_3 - \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} \frac{c_3(k)e^{x_3}}{e^{x_1}+e^{x_2}+d_3(k)}. \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This proves that condition (b) in Lemma 2.1 is satisfied. Next, we show that condition (c) in Lemma 2.1 holds. Let $J = I : ImQ \to KerL, (x_1, x_2, x_3)^T \to (x_1, x_2, x_3)^T$, then

$$\deg\{JQN(x_1, x_2, x_3)^T, \Omega \cap KerL, (0,0,0)^T\} = sign\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$$

where

$$a_{11} = \bar{b}_1 e^{x_1^*} - \frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_1(k)e^{x_1^*}e^{x_3^*}}{(e^{x_1^*}+e^{x_2^*}+d_1(k))^2}, \quad a_{12} = -\frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_1(k)e^{x_2^*}e^{x_3^*}}{(e^{x_1^*}+e^{x_2^*}+d_1(k))^2},$$
$$a_{13} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_3(k)e^{x_3^*}}{e^{x_1^*}+e^{x_2^*}+d_1(k)}, \quad a_{21} = -\frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_2(k)e^{x_1^*}e^{x_3^*}}{(e^{x_1^*}+e^{x_2^*}+d_2(k))^2},$$
$$a_{22} = \bar{b}_2 e^{x_2^*} - \frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_2(k)e^{x_2^*}e^{x_3^*}}{(e^{x_1^*}+e^{x_2^*}+d_2(k))^2}, \quad a_{23} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_3(k)e^{x_3^*}}{e^{x_1^*}+e^{x_2^*}+d_2(k)},$$
$$a_{31} = -\frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_3(k)e^{x_1^*}e^{x_3^*}}{(e^{x_1^*}+e^{x_2^*}+d_3(k))^2}, \quad a_{32} = -\frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_3(k)e^{x_2^*}e^{x_3^*}}{(e^{x_1^*}+e^{x_2^*}+d_3(k))^2},$$
$$a_{33} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} \frac{c_3(k)e^{x_3^*}}{e^{x_1^*}+e^{x_2^*}+d_3(k)}.$$

By now we have proved that Ω satisfies all the requirements in Lemma2.1. Hence, (3.2) has at least one ω-periodic solution. Therefore, (1.2) has at least one positive periodic solution. This completes the proof.

Remark 3.1. By using the method of coincidence degree, we can also obtain the existence positive periodic solution of the continuous system (1.1).

Theorem 3.2. *In addition to the assumptions made in Theorem 3.1, assume further that*

$$\begin{aligned}
(i)\quad & n_1(-b_1(k)+\tfrac{c_1(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_1(k))^2})+n_2\tfrac{c_2(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2}\\
&+n_3\tfrac{c_3(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_3(k))^2})<-\sigma;\\
& n_2(-b_2(k)+\tfrac{c_2(k)e^{H_2}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2})+n_1\tfrac{c_2(k)e^{H_2}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2}\\
&+n_3\tfrac{c_3(k)e^{H_2}e^{H_3}}{(e^{S_1}+e^{S_2}+d_3(k))^2}<-\sigma;\\
(ii)\quad & -b_1(k)+\tfrac{c_1(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_1(k))^2}<0;\\
& -b_2(k)+\tfrac{c_2(k)e^{H_2}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2}<0.
\end{aligned}$$

Then, the positive periodic solution of (1.2) is globally stable.

Proof. Let $\{(x^*(k),y^*(k),z^*(k))^T\}$ be a positive periodic solution of (1.2), we prove below that it is uniformly asymptotically stable. To this end, we introduce the change of variables $u_1(k)=x(k)-x^*(k)$, $u_2(k)=y(k)-y^*(k)$, $u_3(k)=z(k)-z^*(k)$. System (1.2) is then transformed into

$$\begin{aligned}
u_1(k+1)= &\ \exp[a_1(k)-b_1(k)x^*(k)-\tfrac{c_1(k)z^*(k)}{x^*(k)+y^*(k)+d_1(k)}][(1-b_1(k)x^*(k)\\
&+\tfrac{c_1(k)x^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_1(k))^2})u_1(k)+\tfrac{c_1(k)x^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_1(k))^2}u_2(k)\\
&-\tfrac{c_1(k)x^*(k)}{x^*(k)+y^*(k)+d_1(k)}u_3(k)+f_1(k,u(k))]\\
u_2(k+1)= &\ \exp[a_2(k)-b_2(k)y^*(k)-\tfrac{c_2(k)z^*(k)}{x^*(k)+y^*(k)+d_2(k)}][\tfrac{c_2(k)y^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_2(k))^2}u_1(k)\\
&+(1-b_2(k)y^*(k)+\tfrac{c_2(k)y^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_2(k))^2})u_2(k)\\
&-\tfrac{c_2(k)y^*(k)}{x^*(k)+y^*(k)+d_2(k)}u_3(k)+f_2(k,u(k))],\\
u_3(k+1)= &\ \exp[a_3(k)-\tfrac{c_3(k)z^*(k)}{x^*(k)+y^*(k)+d_3(k)}][\tfrac{c_3(k)z^{*2}(k)}{(x^*(k)+y^*(k)+d_3(k))^2}u_1(k)\\
&+\tfrac{c_3(k)z^{*2}(k)}{(x^*(k)+y^*(k)+d_3(k))^2}u_2(k)+(1-\tfrac{c_3(k)z^*(k)}{x^*(k)+y^*(k)+d_3(k)})u_3(k)\\
&+f_3(k,u(k))].
\end{aligned}\tag{3.27}$$

where $\frac{|f_i(k,u)|}{\|u\|}$ converges uniformly with respect to $k\in N$ to zero as $\|u\|\to 0$. In view of system (1.2), it follows from (3.27) that

$$\begin{aligned}
u_1(k+1)= &\ x^*(k+1)[(1-b_1(k)x^*(k)+\tfrac{c_1(k)x^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_1(k))^2})\tfrac{u_1(k)}{x^*(k)}\\
&+\tfrac{c_1(k)z^*(k)}{(x^*(k)+y^*(k)+d_1(k))^2}u_2(k)-\tfrac{c_1(k)}{x^*(k)+y^*(k)+d_1(k)}u_3(k)+\tfrac{f_1(k,u(k))}{x^*(k)}],\\
u_2(k+1)= &\ y^*(k+1)[\tfrac{c_2(k)z^*(k))}{(x^*(k)+y^*(k)+d_2(k))^2}u_1(k)+(1-b_2(k)y^*(k)\\
&+\tfrac{c_2(k)y^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_2(k))^2})\tfrac{u_2(k)}{y^*(k)}-\tfrac{c_2(k)}{x^*(k)+y^*(k)+d_2(k)}u_3(k)+\tfrac{f_2(k,u(k))}{y^*(k)}],\\
u_3(k+1)= &\ z^*(k+1)[\tfrac{c_3(k)z^*(k)}{(x^*(k)+y^*(k)+d_3(k))^2}u_1(k)+\tfrac{c_3(k)z^*(k)}{(x^*(k)+y^*(k)+d_3(k))^2}u_2(k)\\
&+(1-\tfrac{c_3(k)z^*}{x^*(k)+y^*(k)+d_3(k)})\tfrac{u_3(k)}{z^*(k)}+\tfrac{f_3(k,u(k))}{z^*(k)}].
\end{aligned}\tag{3.28}$$

Let us define the function V by

$$V(u(k))=n_1|\frac{u_1(k)}{x^*(k)}|+n_2|\frac{u_2(k)}{y^*(k)}|+n_3|\frac{u_3(k)}{z^*(k)}|,$$

where n_j $(j = 1,2,3)$ are positive constants given in (i). Calculating the deference of V along the solution of system (1.2) and using (3.28), we obtain

$$\begin{aligned}\Delta V = & n_1|(1-b_1(k)x^*(k)+\tfrac{c_1(k)x^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_1(k))^2})\tfrac{u_1(k)}{x^*(k)}+\tfrac{c_1(k)z^*(k)}{(x^*(k)+y^*(k)+d_1(k))^2}u_2(k)\\ & -\tfrac{c_1(k)}{x^*(k)+y^*(k)+d_1(k)}u_3(k)+\tfrac{f_1(k,u(k))}{x^*(k)}|+n_2|\tfrac{c_2(k)z^*(k)}{(x^*(k)+y^*(k)+d_2(k))^2}u_1(k)\\ & +(1-b_2(k)y^*(k)+\tfrac{c_2(k)y^*(k)z^*(k)}{(x^*(k)+y^*(k)+d_2(k))^2})\tfrac{u_2(k)}{y^*(k)}\\ & -\tfrac{c_2(k)}{x^*(k)+y^*(k)+d_2(k)}u_3(k)+\tfrac{f_2(k,u(k))}{y^*(k)}|+n_3|\tfrac{c_3(k)z^*(k)}{(x^*(k)+y^*(k)+d_3(k))^2}u_1(k)\\ & +\tfrac{c_3(k)z^*(k)}{(x^*(k)+y^*(k)+d_3(k))^2}u_2(k)+(1-\tfrac{c_3(k)z^*(k)}{x^*(k)+y^*(k)+d_3(k)})\tfrac{u_3(k)}{z^*(k)}+\tfrac{f_3(k,u(k))}{z^*(k)}|\\ & -n_1|\tfrac{u_1(k)}{x^*(k)}|-n_2|\tfrac{u_2(k)}{y^*(k)}|-n_3|\tfrac{u_3(k)}{z^*(k)}|.\end{aligned} \tag{3.29}$$

Form the proof of Theorem 3.1, we can get $x^*(k) < e^{H_1}$, $y^*(k) < e^{H_2}$, $z^*(k) < e^{H_3}$ and $x^*(k) > e^{S_1}$, $y^*(k) > e^{S_2}$, $z^*(k) > e^{S_3}$.

Therefore,

$$\begin{aligned}\Delta V \le & \{n_1(-b_1(k)+\tfrac{c_1(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_1(k))^2})+n_2(\tfrac{c_2(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2})\\ & +n_3(\tfrac{c_3(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_3(k))^2})\}|\tfrac{u_1(k)}{x^*(k)}|+\{n_2(-b_2(k)+\tfrac{c_2(k)e^{H_2}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2})\\ & +n_1(\tfrac{c_1(k)e^{H_1}e^{H_3}}{(e^{S_1}+e^{S_2}+d_2(k))^2})+n_3(\tfrac{c_3(k)e^{H_2}e^{H_3}}{(e^{S_1}+e^{S_2}+d_3(k))^2})\}|\tfrac{u_2(k)}{y^*(k)}|\\ & +n_1|\tfrac{f_1(k,u(k))}{x^*(k)}|+n_2|\tfrac{f_2(k,u(k))}{y^*(k)}|+n_3|\tfrac{f_3(k,u(k))}{z^*(k)}|,\end{aligned}$$

for large k. Since $\frac{|f_i(k,u)|}{\|u\|}$ converges uniformly with respect to $k \in N$ to zero as $\|u\| \to 0$. It follows from Conditions (i) that there is a positive constant σ such that if k is sufficiently large and $\| u(k) \| < \sigma$, then

$$\Delta V \le -\frac{\gamma \| u(k) \|}{2}$$

By [11], we see that the trivial solution of equation (3.28) is uniformly asymptotically stable, and so is the solution $(x^*(k), y^*(k), z^*(k))$ of equation (1.2). Note that the positive solution $(x(k), y(k), z(k))$ is chosen in an arbitrary way. We conclude that the positive periodic solution $(x^*(k), y^*(k), z^*(k))$ of (1.2) is globally stable. This completes the proof.

Remark 3.2. There are still many interesting and challenging mathematical question need to be studied for system (1.2). For example, we do not discuss the bifurcations that occur when conditions of stability are violated, we will leave this for future work.

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Xiangyun Shi is a lecturer of Xinyang Normal University. Her research interests focus on the stability and periodicity of the functional differential equations and difference equations. So far she has published more 10 articles with respect to functional differential equations.

College of Mathematics and Information Science, Xinyang Normal University, Xinyang 464000, Henan, PR China
e-mail: xiangyunshi@126.com

Xueyong Zhou
College of Mathematics and Information Science, Xinyang Normal University, Xinyang 464000, Henan, PR China
e-mail: xueyongzhou@126.com

Xinyu Song
College of Mathematics and Information Science, Xinyang Normal University, Xinyang 464000, Henan, PR China
e-mail: xysong88@163.com