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POLYNOMIAL COMPLEXITY OF PRIMAL-DUAL INTERIOR-POINT METHODS FOR CONVEX QUADRATIC PROGRAMMING

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ABSTRACT. Recently, Peng et al. proposed a primal-dual interior-point method with new search direction and self-regular proximity for LP. This new large-update method has the currently best theoretical performance with polynomial complexity of $O(n^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon})$. In this paper we use this search direction to propose a primal-dual interior-point method for convex quadratic programming (QP). We overcome the difficulty in analyzing the complexity of the primal-dual interior-point methods for convex quadratic programming, and obtain the same polynomial complexity of $O(n^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon})$ for convex quadratic programming.

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1. Introduction

In this paper we consider the following convex quadratic programming problem

$$\text{(P)} \qquad \min_{x}\{c^T x + \frac{1}{2}x^T Qx : Ax = b,\ x \geq 0\}$$

and its associated (Wolfe's) dual form

$$\text{(D)} \quad \max_{x,y,z}\{b^T y - \frac{1}{2}x^T Qx : A^T y + z - Qx = c,\ z \geq 0\},$$

where Q is a given $n \times n$ symmetric matrix, $c \in \mathbb{R}^{\kappa}$, $b \in \mathbb{R}^{>}$, and A is a given $m \times n$ matrix. Primal-dual interior-point methods (IPMs) are among the most efficient algorithms to solve linear programming (LP), semidefinite

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programming (SDP), complementarity problems (CP) (see [7, 8]). Recently, many interior-point methods have been proposed for solving LP problems. These methods are based on new search directions and self-regular proximity with attractive theoretical properties such as polynomial time complexity and efficient practical implementations, see ([3, 4, 5, 6]). In [1], the author extends an interior-point method to QP and proves that the small-update algorithm finds an ε-solution in a polynomial time. In this paper, we continue this idea by using the search direction from Peng's work, extend it to QP, and prove its polynomial complexity. Usually, for QP problems, the property of orthogonality between the two scaled directions in the primal and dual spaces is not satisfied. This is the main difference of complexity analysis between LP and QP, and this will make the problem a little difficult.

The main aim of this paper is to extend a new class of primal-dual methods with new search direction from LP to QP. Our main effort is to deal with the case of $d_x^T d_z \neq 0$ which is different from the case in LP. Usually, it needs to solve a high-order equation if we want to find the maximal feasible step length (see (24) in Lemma 8). In this paper we succeed in transforming this problem into finding a suitable step size by applying Lemma 3. Therefore the difficult problem is solved.

This paper is organized as follows. In the next section, the state of the problem is described. Section 3 provides some technical lemmas and Section 4 estimates the maximal feasible step size. In Section 5, a damped Newton step on the proximity is presented, and its effect is evaluated. Based on the above discussions, in Section 6, the primal-dual interior-point algorithm for convex quadratic programming is built, and the polynomial complexity of $O(n^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon})$ is established. Finally, in Section 7, a brief conclusion in contained.

Here we introduce some notations. $\mathbb{R}^n$ denotes the space of real n-dimensional vectors. Given u, $v \in R^n$, $u^T v$ is the inner product, and uv denotes Hadamard-product. $\|u\|$ is the Euclidean norm, $\|u\|_\infty$ is the l_∞ norm. Given a vector $u \in \mathbb{R}^n$, U=diag(u) is an $n \times n$ diagonal matrix with $U_{ii} = u_i$, e is the all-one vector. And for a vector d_x, the componentwise expression $d_{xi} := (d_x)_i$, and so on. Finally, we define the vector $(d_x, d_z) := (d_{x1}, \dots, d_{xn}, d_{z1}, \dots, d_{zn}) \in \mathbb{R}^{2n}$ and the norm

$$\|(d_x, d_z)\| := \sqrt{\sum_{i=1}^{n} d_{xi}^2 + \sum_{i=1}^{n} d_{zi}^2},$$

and so on.

2. The state of the problem

We first assume that both (P) and (D) satisfy the following conditions:

• **Positive semidefiniteness (PSD).** The matrix Q is symmetric and positive semidefinite.

• **Interior-point condition (IPC).** There exists (x^0, y^0, z^0) such that $Ax^0 = b$, $A^T y^0 + z^0 - Qx^0 = c$, $x^0 > 0$, $z^0 > 0$.

- **The full rank condition (Full-Rank).** The $m \times n$ matrix A is of rank m.

Finding an optimal solution of (P) and (D) is equivalent to solve the following system:

$$\begin{aligned} Ax &= b,\ x \geq 0, \\ A^T y + z - Qx &= c,\ z \geq 0, \\ xz &= 0. \end{aligned} \tag{1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (1) by the parameterized equation $xz = \mu e$, $(\mu > 0)$. Thus we consider the system

$$\begin{aligned} Ax &= b,\ x > 0, \\ A^T y + z - Qx &= c,\ z > 0, \\ xz &= \mu e. \end{aligned} \tag{2}$$

It is shown, under our assumptions, that there exists one unique solution $(x(\mu), y(\mu), z(\mu))$ called as μ-center. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of (P) and (D).

IPMs follow the central path approximately. Without loss of generality, we assume that $(x(\mu), y(\mu), z(\mu))$ is known for some positive μ. Then μ is updated and reduced to $\mu_+ = (1-\theta)\mu$ for some $\theta \in (0,1)$ and with Newton's method one constructs a new triplet (x, y, z) that is 'close' to $(x(\mu_+), y(\mu_+), z(\mu_+))$. This process is repeated until the point (x, y, z) is in a certain neighborhood of the central path.

Usually, if θ is a constant independent of n, for instance $\theta = \frac{1}{2}$, then we call the algorithm a large-update (or long-step) method. If θ depends on n such as $\theta = \frac{1}{\sqrt{n}}$, the algorithm is named a small-update (or short-step) method. It is known that small-update methods for (LP) have an $O(\sqrt{n} \log \frac{n}{\varepsilon})$ iteration bound and large-update methods have the worst case iteration bound as $O(n \log \frac{n}{\varepsilon})$.

In the analysis of IPMs we need to keep control of the 'distance' from the current iterates to the current μ-centers. In other words, we need to quantify the 'distance' from the vector xz to the vector μe in some proximity measure. Usually the proximity measure is defined as follows:

$$\delta(xz, \mu) = \left\| \sqrt{\frac{xz}{\mu}} - \sqrt{\frac{\mu}{xz}} \right\|.$$

In the algorithm we use a threshold value τ for the proximity and we assume that we are given a triple (x^0, y^0, z^0) such that $\delta(x^0, y^0, z^0) \leq \tau$ for $\mu^0 = 1$. This can be done without loss of generality [7]. Let Δx, Δy, Δz denote the solutions of the following Newton equations for the parameterized system (2):

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta z - Q\Delta x &= 0, \\ x\Delta z + z\Delta x &= \mu e - xz. \end{aligned} \tag{3}$$

For convenience, let us introduce the vector v defined as

$$v = \sqrt{\frac{xz}{\mu}},$$

$$d_x = X^{-1}V\Delta x,\ d_z = Z^{-1}V\Delta z,\ d_y = \Delta y,$$

and the matrix

$$\bar{A} = \frac{1}{\mu}AV^{-1}X,\ \bar{Q} = \frac{1}{\mu}V^{-1}XQV^{-1}X,$$

then the system (3) reduces to

$$\begin{aligned} \bar{A}d_x &= 0, \\ -\bar{A}^T d_y - d_z + \bar{Q}d_x &= 0, \\ d_x + d_z &= v^{-1} - v, \end{aligned} \tag{4}$$

which leads to the system

$$\begin{pmatrix} I+\bar{Q} & -\bar{A}^T \\ -\bar{A} & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_y \end{pmatrix} = \begin{pmatrix} v^{-1} - v \\ 0 \end{pmatrix},$$

which is nonsingular since $\bar{Q} \succeq 0$ and $\bar{A}$ is of full rank.

The third equation in (4) is

$$d_x + d_z = v^{-1} - v.$$

When dealing with the squared proximity, i.e., $\|v^{-1} - v\|^2$, the search direction is

$$d_x + d_z = -\nabla(\frac{1}{2}\|v^{-1} - v\|^2) = v^{-3} - v.$$

According to [4], the Newton equation is modified as follows:

$$d_x + d_z = v^{-q} - v$$

with $q \geq 1$.

3. Technical results

As we know, a key issue in the analysis of an interior-point method, particularly for a large-update IPM, is the decreasing property of a positive sequence of values of the proximity measure. This is crucial for the complexity of the algorithm. In this section we consider a general positive decreasing sequence. First, several available lemmas are required.

Lemma 1. ([4, Lemma 2.1]) *Suppose that* $\alpha \geq 1$. *Then*

$$\begin{aligned} 1 - \alpha t &\leq (1-t)^\alpha,\ t \in [0,1], \\ \alpha(t-1) &\leq t^\alpha - 1,\ t \geq 0. \end{aligned}$$

If $\alpha_1 \geq \alpha_2 > 0$, *then*

$$|t - t^{-\alpha_1}| \geq |t - t^{-\alpha_2}|,\ t > 0. \tag{5}$$

Lemma 2. ([6, lemma 1.3.1]) *Suppose that* $\alpha \in [0,1]$. *Then*

$$(1+t)^{\alpha} \leq 1 + \alpha t, \qquad \forall t \geq -1.$$

Lemma 3. ([2, Lemma 3]) *Suppose that* $f(x) = f_1(x) + f_2(x)$, *both* $f_1(x)$ *and* $f_2(x)$ *are strictly monotonically increasing in the interval* $[a, b]$. *The roots of* $f_1(x) = 0$ *and* $f_2(x) = 0$ *are* x_1, x_2 *respectively. Then the root* x^* *of* $f(x) = 0$ *satisfies that*

$$x^* \geq \min\{x_1, x_2\}.$$

The following proposition is important for our discussion.

Proposition 1. ([6, Proposition 1.3.2]) *Suppose that* $t_k > 0$ $(k = 0, 1, 2, \dots, \bar{k})$ *is a given sequence satisfying the inequalities*

$$t_{k+1} \leq t_k - \beta t_k^{\gamma}, \quad \beta > 0, \quad k = 0, 1, \dots, \bar{k}$$

with $\gamma \in [0, 1)$. *Then*

$$\bar{k} \leq \left\lceil \frac{t_0^{1-\gamma}}{\beta(1-\gamma)} \right\rceil.$$

Moreover, for any fixed $\rho \geq 0$, $t_k > \rho$ *implies*

$$k \leq \left\lceil \frac{t_0^{1-\gamma} - \rho^{1-\gamma}}{\beta(1-\gamma)} \right\rceil.$$

4. Bounds for v and the step size

In the analysis of the algorithm, we use a well-known norm-based proximity measure $\delta(v)$ according to

$$\delta := \delta(v) := \|v^{-1} - v\|. \tag{6}$$

In addition, we also define

$$\sigma(v) := \|d_x + d_z\| = \|v^{-q} - v\|. \tag{7}$$

Letting Δx, Δy, Δz denote the displacements in the original space, the result of a damped Newton step with damping factor α is denoted as

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad z_+ = z + \alpha\Delta z.$$

Lemma 4. ([3, Lemma 3.1]) *Let* $\sigma := \sigma(v)$ *as defined by (7). And we write* $v_{\min} = \min_i v_i$ *and* $v_{\max} = \max_i v_i$. *Then*

$$v_{\min} \geq (1+\sigma)^{-\frac{1}{q}}, \quad v_{\max} \leq 1 + \sigma.$$

In the sequel, we use the following notations:

$$\bar{d}_x = \frac{d_x}{v} = \frac{\Delta x}{x}, \quad \bar{d}_z = \frac{d_z}{v} = \frac{\Delta z}{z} \tag{8}$$

and

$$\bar{d}_{xi} = \frac{d_{xi}}{v_i} = \frac{\Delta x_i}{x_i}, \quad \bar{d}_{zi} = \frac{d_{zi}}{v_i} = \frac{\Delta z_i}{z_i}.$$

We then may write

$$x_+ = x(e + \alpha \bar{d}_x), \quad z_+ = z(e + \alpha \bar{d}_z). \tag{9}$$

Hence the maximum step size is determined by the vector $(\bar{d}_x, \bar{d}_z)$: the step size α is feasible if and only if $e + \alpha \bar{d}_x \geq 0$ and $e + \alpha \bar{d}_z \geq 0$, and this will certainly hold if

$$1 - \alpha \|(\bar{d}_x, \bar{d}_z)\| \geq 0. \tag{10}$$

Since $\mu d_x d_z = \Delta x \Delta z$ and

$$d_x^T d_z = d_x^T(\bar{Q} d_x - \bar{A}^T d_y) = d_x^T \bar{Q} d_x \geq 0. \tag{11}$$

This enables us, by use of (7), to write

$$\sigma(v) = \|d_x + d_z\| \geq \|(d_x, d_z)\|. \tag{12}$$

Now we are ready to give the following lemmas.

Lemma 5. *One has*

$$\|(\bar{d}_x, \bar{d}_z)\| \leq \sigma(1+\sigma)^{\frac{1}{q}}. \tag{13}$$

Consequently, the maximal feasible step size, $\alpha_{\max}$, *satisfies*

$$\alpha_{\max} \geq \frac{1}{\sigma(1+\sigma)^{\frac{1}{q}}}.$$

Proof. Using Lemma 4 and (7), we may write

$$\|(\bar{d}_x, \bar{d}_z)\| = \|(\frac{d_x}{v}, \frac{d_z}{v})\| \leq \frac{\|(d_x, d_z)\|}{v_{\min}} \leq \frac{\sigma}{v_{\min}} \leq \sigma(1+\sigma)^{\frac{1}{q}}.$$

From (10) we derive that

$$\alpha_{\max} \geq \frac{1}{\|(\bar{d}_x, \bar{d}_z)\|}.$$

Hence the lemma follows. □

Lemma 6. *One has*

$$0 \leq d_x^T d_z \leq \frac{1}{2}\sigma^2. \tag{14}$$

Proof. Using (11), the left-hand side of (14) is obvious. Also,

$$d_x^T d_z = \sum_{i=1}^{n} d_{xi} d_{zi} \leq \frac{1}{2}\sum_{i=1}^{n}(d_{xi}^2 + d_{zi}^2) = \frac{1}{2}\|(d_x, d_z)\|^2 \leq \frac{1}{2}\|d_x + d_z\|^2 = \frac{1}{2}\sigma^2,$$

where the first inequality is due to

$$\sum_{i=1}^{n}(d_{xi} - d_{zi})^2 = \sum_{i=1}^{n}(d_{xi}^2 + d_{zi}^2 - 2d_{xi}d_{zi}) \geq 0,$$

and the last inequality due to (12). □

Lemma 7. *One has*

$$\sigma \geq \delta. \tag{15}$$

Proof. It follows from (5) in Lemma 1 that

$$\sigma^2 = \sum_{i=1}^{n}(v_i - v_i^{-q})^2 \geq \sum_{i=1}^{n}(v_i - v_i^{-1}) = \delta^2.$$

The lemma follows. □

5. Effect of a damped Newton step on the proximity

Recall from (6) that

$$\delta^2 := \delta^2(xz, \mu) = e^T(v^2 + v^{-2} - 2e), \tag{16}$$

and

$$\delta_+^2 := \delta^2(x_+z_+, \mu) = e^T(v_+^2 + v_+^{-2} - 2e), \quad v_+ = \sqrt{\frac{x_+z_+}{\mu}}. \tag{17}$$

Using (8) and (9) we find

$$\begin{aligned} v_+^2 = \frac{x_+z_+}{\mu} &= \frac{xz(e+\alpha\bar{d}_x)(e+\alpha\bar{d}_z)}{\mu} \\ &= v^2(e+\alpha\bar{d}_x)(e+\alpha\bar{d}_z) \\ &= (v+\alpha d_x)(v+\alpha d_z). \end{aligned}$$

Hence we obtain

$$e^Tv_+^2 = e^T(v^2 + \alpha v(d_x + d_z) + \alpha^2 d_x d_z) = e^Tv^2 + \alpha v^T(v^{-q} - v) + \alpha^2 d_x^T d_z.$$

Furthermore,

$$\delta_+^2 = e^T(v^2 + \alpha(v^{1-q} - v^2) + \alpha^2 d_x d_z + \frac{e}{v^2 + \alpha(v^{1-q} - v^2) + \alpha^2 d_x d_z} - 2e).$$

We define the difference between the proximities of two neighboring steps as a function of α, i.e., $f(\alpha) = \delta_+^2 - \delta^2$. From (16) and (17), we get

$$\begin{aligned} f(\alpha) &= \sum_{i=1}^{n}\left(\alpha(v_i^{1-q} - v_i^2) + \alpha^2 d_{xi}d_{zi} + \frac{1}{v_i^2 + \alpha(v_i^{1-q} - v_i^2) + \alpha^2 d_{xi}d_{zi}} - \frac{1}{v_i^2}\right) \\ &= \alpha^2 d_x^T d_z + \sum_{i=1}^{n}\left(\alpha(v_i^{1-q} - v_i^2) + \frac{1}{v_i^2}\left(\frac{1}{(1+\alpha\bar{d}_{xi})(1+\alpha\bar{d}_{zi})} - 1\right)\right). \end{aligned} \tag{18}$$

Obviously, $f(\alpha)$ is a twice continuously differentiable function of α if the step is feasible. The next result shows that in the interval $[0, \alpha_{\max})$ the function $f(\alpha)$ is a convex function of α, where $\alpha_{\max}$ is the maximal feasible step size.

Lemma 8. *Let the function $f(\alpha)$ be defined by (18), $\alpha \in [0, \alpha_{\max})$. Then $f(\alpha)$ is convex. Furthermore,*

$$\sum_{i=1}^{n}\frac{1}{v_i^2}\left(\frac{\bar{d}_{xi}^2}{(1+\alpha\bar{d}_{xi})^3(1+\alpha\bar{d}_{zi})} + \frac{\bar{d}_{zi}^2}{(1+\alpha\bar{d}_{zi})^3(1+\alpha\bar{d}_{xi})}\right) \leq f''(\alpha) \tag{19}$$

and

$$f''(\alpha) \le \sigma^2 + 3\sum_{i=1}^{n} \frac{1}{v_i^2}\left(\frac{\bar{d}_{xi}^2}{(1+\alpha\bar{d}_{xi})^3(1+\alpha\bar{d}_{zi})} + \frac{\bar{d}_{zi}^2}{(1+\alpha\bar{d}_{zi})^3(1+\alpha\bar{d}_{xi})}\right) \quad (20)$$

hold.

Proof. By direct calculations, we have

$$\begin{aligned} f''(\alpha) &= 2d_x^T d_z + \sum_{i=1}^{n} \frac{1}{v_i^2}\left(\frac{2\bar{d}_{xi}^2}{(1+\alpha\bar{d}_{xi})^3(1+\alpha\bar{d}_{zi})} + \frac{2\bar{d}_{xi}\bar{d}_{zi}}{(1+\alpha\bar{d}_{xi})^2(1+\alpha\bar{d}_{zi})^2}\right. \\ &\quad \left. + \frac{2\bar{d}_{zi}^2}{(1+\alpha\bar{d}_{zi})^3(1+\alpha\bar{d}_{xi})}\right). \end{aligned}$$

By using the well-known inequality $|2t_1t_2| \le t_1^2 + t_2^2$, we have

$$\left|\frac{2\bar{d}_{xi}\bar{d}_{zi}}{(1+\alpha\bar{d}_{xi})^2(1+\alpha\bar{d}_{zi})^2}\right| \le \frac{\bar{d}_{xi}^2}{(1+\alpha\bar{d}_{xi})^3(1+\alpha\bar{d}_{zi})} + \frac{\bar{d}_{zi}^2}{(1+\alpha\bar{d}_{zi})^3(1+\alpha\bar{d}_{xi})}.$$

By using Lemma 6, the result follows. □

Denote $\omega_i = \sqrt{\bar{d}_{xi}^2 + \bar{d}_{zi}^2}$ and $\omega = \|(\omega_1, \dots, \omega_n)\|$. Obviously, we have $\omega = \|(\bar{d}_x, \bar{d}_z)\|$. From Lemma 5, it follows that

$$\omega \le \frac{\sigma}{v_{\min}} \le \sigma(1+\sigma)^{\frac{1}{q}}. \quad (21)$$

Now recalling (19) and (20) in Lemma 8 we can conclude that for any $\alpha \in [0, \alpha_{\max})$,

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{v_i^2}\frac{\omega_i^2}{(1+\alpha\omega)^4} &\le f''(\alpha) \le \sigma^2 + 3\sum_{i=1}^{n}\frac{1}{v_i^2}\frac{\omega_i^2}{(1-\alpha\omega)^4} && (22) \\ &\le \sigma^2 + \frac{3\omega^2}{v_{\min}^2(1-\alpha\omega)^4}. && (23) \end{aligned}$$

A direct calculation gives $f'(0) = -\sigma^2$. It follows from (22) and the convexity of $f(\alpha)$ that

$$\begin{aligned} f(\alpha) &= f(0) + \int_0^\alpha f'(\xi)d\xi \\ &= \int_0^\alpha \left(f'(0) + \int_0^\xi f''(\zeta)d\zeta\right) d\xi \\ &= f'(0)\alpha + \int_0^\alpha \int_0^\xi f''(\zeta)d\zeta d\xi \\ &\le f_1(\alpha) := f'(0)\alpha + \int_0^\alpha \int_0^\xi \sigma^2 + \frac{3\omega^2}{v_{\min}^2(1-\zeta\omega)^4} d\zeta d\xi. \end{aligned}$$

The second equality is due to $f(0) = 0$. Obviously, $f_1''(\alpha) = \sigma^2 + \frac{3\omega^2}{v_{\min}^2(1-\alpha\omega)^4} > 0$. Hence $f_1(\alpha)$ is convex and twice differentiable with respect to α. Since $f_1(0) = 0, f_1'(0) = f'(0) < 0$, and $f_1''(\alpha)$ goes to infinite if α approaches $1/\omega$, the function $f_1(\alpha)$ attains its minimal value at some positive value $\tilde{\alpha}$ of α, and $\tilde{\alpha}$ is the stationary point of $f_1(\alpha)$. Since $f_1'(\tilde{\alpha}) = 0$, we get

$$f_1'(0) + \int_0^{\tilde{\alpha}} \sigma^2 + \frac{3\omega^2}{v_{\min}^2(1-\xi\omega)^4} d\xi = 0,$$

which is equivalent to

$$-\sigma^2 + \sigma^2\tilde{\alpha} + \frac{\omega}{v_{\min}^2}((1-\tilde{\alpha}\omega)^{-3} - 1) = 0. \tag{24}$$

Note that in the case of LP, where $d_x^T d_z \geq 0$ holds, the equation (24) reduces to

$$-\sigma^2 + \frac{\omega}{v_{\min}^2}((1-\tilde{\alpha}\omega)^{-3} - 1) = 0.$$

And we can get the root $\tilde{\alpha}$ exactly. But in the case of QP, it is not easy because the equation is of fourth order. Luckily, we can estimate a lower bound of $\tilde{\alpha}$.

Let us define

$$\omega_1(\alpha) = -\frac{\sigma^2}{2} + \sigma^2\alpha$$

and

$$\omega_2(\alpha) = -\frac{\sigma^2}{2} + \frac{\omega}{v_{\min}^2}((1-\alpha\omega)^{-3} - 1).$$

Note that both functions $\omega_1(\alpha)$ and $\omega_2(\alpha)$ are strictly monotonically increasing for $\alpha \in [0, \bar{\alpha})$, $\bar{\alpha} := 1/\omega$. And the root α_1^* of $\omega_1(\alpha) = 0$ is $\alpha_1^* = 1/2$. By simple calculations, we find that the root α_2^* of $\omega_2(\alpha) = 0$ can be represented as

$$\alpha_2^* = \frac{1}{\omega}\left(1 - \left(\frac{1}{1 + \frac{\sigma^2 v_{\min}^2}{2\omega}}\right)^{\frac{1}{3}}\right). \tag{25}$$

Since the right-hand side of (25) is monotonically increasing with respect to $v_{\min}$ and monotonically decreasing with respect to ω, respectively. By Lemma 4 and (21) one can get

$$\alpha_2^* \geq \frac{1}{\sigma(1+\sigma)^{\frac{1}{q}}}\left(1 - \left(\frac{1}{1 + \frac{\frac{\sigma^2}{(1+\sigma)^{\frac{2}{q}}}}{2\sigma(1+\sigma)^{\frac{1}{q}}}}\right)^{\frac{1}{3}}\right).$$

By use of Lemma 2 we can write

$$\left(\frac{1}{1+\frac{\frac{\sigma^2}{(1+\sigma)^{\frac{2}{q}}}}{2\sigma(1+\sigma)^{\frac{1}{q}}}}\right)^{\frac{1}{3}} = \left(\frac{1}{1+\frac{\sigma}{2(1+\sigma)^{\frac{3}{q}}}}\right)^{\frac{1}{3}} = \left(1-\frac{\frac{\sigma}{2(1+\sigma)^{\frac{3}{q}}}}{1+\frac{\sigma}{2(1+\sigma)^{\frac{3}{q}}}}\right)^{\frac{1}{3}}$$

$$\leq 1-\frac{1}{3}\cdot\frac{\frac{\sigma}{2(1+\sigma)^{\frac{3}{q}}}}{1+\frac{\sigma}{2(1+\sigma)^{\frac{3}{q}}}}.$$

Hence we have

$$\alpha_2^* \geq \frac{1}{3}\cdot\frac{1}{(1+\sigma)^{\frac{1}{q}}}\cdot\frac{1}{2(1+\sigma)^{\frac{3}{q}}+\sigma}.$$

By using Lemma 3, one can get

$$\tilde{\alpha} \geq \min\{\alpha_1^*, \alpha_2^*\} \geq \frac{1}{3}\cdot\frac{1}{(1+\sigma)^{\frac{1}{q}}}\cdot\frac{1}{2(1+\sigma)^{\frac{3}{q}}+\sigma} \geq \frac{1}{6\sigma^{\frac{1}{q}}}\cdot\frac{1}{16\sigma^{\frac{3}{q}}+\sigma},$$

where the last inequality is due to $\sigma \geq 1$. So, We can use

$$\alpha^* = \frac{1}{6\sigma^{\frac{1}{q}}}\cdot\frac{1}{16\sigma^{\frac{3}{q}}+\sigma} \tag{26}$$

as a default step size.

Lemma 9. ([6, Lemma 1.3.3]) *Let $h(t)$ be a twice differentiable convex function with $h(0)=0$ and $h'(0)<0$, and let $h(t)$ attain its (global) minimum at $t^*>0$. If $h''(t)$ is increasing for $t\in[0,t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2}, \quad \textit{for } 0\leq t\leq t^*.$$

We apply this lemma with $h=f_1$ and $t=\alpha$. First, we verify that the hypotheses of the lemma are satisfied: we have $f_1(0)=0$ and $f_1'(0)<0$. Furthermore, $f_1''(\alpha)>0$, and

$$f_1'''(\alpha) = h'(\alpha) = \frac{12\omega^3}{v_{\min}^2}\frac{1}{(1-\alpha\omega)^5} > 0,$$

where the inequality is due to $\alpha\in[0,\frac{1}{\omega})$. Hence the lemma applies. Using also (5), we obtain

$$f(\alpha^*) \leq f_1(\alpha^*) \leq \frac{\alpha^* f_1'(0)}{2} = -\frac{\alpha^*\sigma^2}{2}.$$

Finally, by substitution of (26), we immediately have

$$f(\alpha^*) \leq -\frac{1}{6\sigma^{\frac{1}{q}}}\cdot\frac{\sigma^2}{16\sigma^{\frac{3}{q}}+\sigma}.$$

When $q \geq 3$, noting that $\sigma \geq 1$, we have $\sigma \geq \sigma^{\frac{3}{q}}$, and furthermore

$$f(\alpha^*) \leq -\frac{\sigma^{1-\frac{1}{q}}}{102}. \tag{27}$$

6. Algorithm and complexity analysis

Lemma 10. ([4, Lemma 3.11]) *Let (x, y, z) be strictly feasible and $\mu > 0$. If $\mu_+ = (1-\theta)\mu$, then*

$$\delta(xz, \mu_+) \leq \frac{\delta(xz, \mu) + \theta\sqrt{n}}{\sqrt{1-\theta}}.$$

In the algorithm we monitor the progress of inner iterations by the proximity $\delta(xz, \mu)$. The algorithm can be stated as follows.

Primal-dual algorithm

Input:
A proximity parameter τ;
an accuracy parameter $\varepsilon > 0$;
a variable damping factor α;
a fixed barrier update parameter θ, $0 < \theta < 1$;
a strictly interior point (x^0, y^0, z^0)
and $\mu^0 = 1$ such that $\delta(x^0 z^0, \mu^0) \leq \tau$.
begin
    $x := x^0$; $y := y^0$; $z := z^0$; $\mu := \mu^0$
    while $n\mu \geq \varepsilon$ **do**
    begin
      $\mu := (1-\theta)\mu$;
      while $\delta(xz; \mu) \geq \tau$ **do**
          solve the system (3);
          begin
            $x := x + \alpha\Delta x$;
            $y := y + \alpha\Delta y$;
            $z := z + \alpha\Delta z$;
          end
    end
end

Lemma 11. *Let $\delta(xz, \mu) \leq \tau$ and $\tau \geq 1$. Then, after an update of the barrier parameter, no more than*

$$\left\lceil \frac{204q}{q+1}\left(\frac{\tau^2 + 2\tau\theta\sqrt{n} + \theta^2 n}{1-\theta}\right)^{\frac{q+1}{2q}} \right\rceil \tag{28}$$

iterations are needed, where $q \geq 3$.

Proof. By Lemma 10, after an update,

$$\delta(xz, \mu_+)^2 \leq \frac{(2\delta(xz, \mu) + \theta\sqrt{n})^2}{4(1-\theta)}.$$

Each damped Newton step decreases δ^2 by at least $\dfrac{\delta^{1-\frac{1}{q}}}{102}$. Hence, by Proposition 1 with $\gamma = \frac{1}{2} - \frac{1}{2q}$, $\beta = 1/102$, and $t_0 = (\frac{\tau+\theta\sqrt{n}}{\sqrt{1-\theta}})^2$, after at most

$$\left\lceil \frac{204q}{q+1} \left(\frac{\tau^2 + 2\tau\theta\sqrt{n} + \theta^2 n}{1-\theta} \right)^{\frac{q+1}{2q}} \right\rceil$$

iterations, the proximity will have passed the threshold τ. This implies the lemma holds. □

Theorem 1. *If $\tau \geq 1$, the total number of iterations required by the algorithm is no more than*

$$\left\lceil \frac{204q}{q+1} \left(\frac{\tau^2 + 2\tau\theta\sqrt{n} + \theta^2 n}{1-\theta} \right)^{\frac{q+1}{2q}} \right\rceil \left\lceil \frac{1}{\theta} \log \frac{n}{\varepsilon} \right\rceil. \tag{29}$$

Furthermore, $O(n^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon})$ $(q \geq 3)$ can be as an upper bound of iteration number.

Proof. It can be shown that the number of outer iterations is given by

$$\left\lceil \frac{1}{\theta} \log \frac{n}{\varepsilon} \right\rceil$$

(see [7, Lemma II.17]). Multiplying this number by the bound in Lemma 11 yields (29).

For large-update IPMs, omitting the part of the round-off brackets in (29) does not change the order of magnitude of the iteration bound. Hence we may consider the following expression as an upper bound for the number of iterations:

$$O\left(\frac{204q}{q+1} \left(\frac{\tau^2 + 2\tau\theta\sqrt{n} + \theta^2 n}{1-\theta} \right)^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon} \right) = O(n^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon}), \quad q \geq 3,$$

which is as the currently best iteration bound for interior-point methods of LP with large-update. □

7. Conclusion

We have extended the primal-dual interior-point methods with new search direction from LP to QP. In the case of LP, the scaled Newton directions d_x and d_z are orthogonal, but it is not true in the case of QP. There exists a little difficulty to analyze the complexity of QP when evaluating the step size. We observed that the step size is bounded below. Hence, a suitable lower bound which can be regarded as a suitable step size is obtained. The iteration bound obtained is $O(n^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon})$ for $q \geq 3$ which is the same as that in the case of LP. Note that in this paper, we only prove that this result holds for $q \geq 3$. We expect that the further research by ourselves will deal with the case of $q \geq 1$. In addition, we will, in the future, deal with the numerical implementation of this algorithm for convex quadratic programming problems.

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