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# THE RIESZ THEOREM IN FUZZY $n$-NORMED LINEAR SPACES

J. KAVIKUMAR*, YOUNG BAE JUN AND AZME KHAMIS

ABSTRACT. The primary purpose of this paper is to prove the fuzzy version of Riesz theorem in $n$-normed linear space as a generalization of linear $n$-normed space. Also we study some properties of fuzzy $n$-norm and introduce a concept of fuzzy anti $n$-norm.

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## 1. Introduction

Gahler [4] introduced the theory of $n$-norm on a linear space. Following Gunawan and Mashadi [5], Kim and Cho [6], Malčeski [8] and Misiak [9] developed the theory of $n$-normed space. A detailed theory of fuzzy normed linear space can be found in [1, 2, 3, 7, 12]. Narayanan and Vijayabalaji [10] introduced the concept of fuzzy $n$-norm on a linear space and Also, Vijayabalaji and Thillaigovindan [14] introduced the concept of complete fuzzy $n$-normed linear space. Riesz [13] obtained the *Riesz theorem* in a normed space. Park and Chu [11] have extended the *Riesz theorem* in a normed space to $n$-normed linear space. In this paper, we extend the *Riesz theorem* in $n$-normed linear space to the case of fuzzy $n$-normed linear space and establish some results on it.

## 2. Preliminaries

Riesz [13] obtained the following theorem in a normed space

**Theorem 1.** *Let $Y$ and $Z$ be subspaces of a normed space $X$ and $Y$ a closed proper subset of $Z$. For each $\theta \in (0,1)$, there exists an element $z \in Z$ such that*

$$\| z \| = 1, \qquad \| z - y \| \geq \theta$$

*for all $y \in Y$.*

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**Definition 1.** [5]. Let $n \in N$ (natural numbers) and $X$ be a real linear space of dimension $d \geq n$. (Here we allow $d$ to be infinite). A real valued function $\| \bullet, \bullet, \cdots, \bullet \|$ on $X \times X \times ... \times X$ (n times)$=X^n$ satisfying the following four properties:

(N1) $\| x_1, x_2, ..., x_n \| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(N2) $\| x_1, x_2, ..., x_n \|$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(N3) $\| x_1, x_2, ..., cx_n \| = | c | \, \| x_1, x_2, ..., x_n \|$, for any real c.

(N4) $\| x_1, x_2, ..., x_{n-1}, y + z \| \leq \| x_1, x_2, ..., x_{n-1}, y \| + \| x_1, x_2, ..., x_{n-1}, z \|$

is called an $n$-norm on X and the pair $(X, \| \bullet, ..., \bullet \|)$ is called an $n$-normed linear space.

**Definition 2.** [5]. A sequence $\{x_n\}$ in a linear $n$-normed space $(X, \| \bullet, ..., \bullet \|)$ is said to $n$-convergent to $x \in X$ and denoted by $x_k \to x$ as $k \to \infty$ if

$$\lim_{k \to \infty} \| x_1, x_2, ..., x_{n-1}, x_n - x \| = 0$$

From the above definitions, Park and Chu [11] obtained the following theorem in a $n$-normed spaces.

**Theorem 2.** *Let $Y$ and $Z$ be subspaces of a linear n-normed space $X$ and $Y$ an n-compact proper subset of $Z$ with codimension greater than n - 1. For each $\theta \in (0, 1)$, there exists an element $(z_1, z_2, ..., z_n) \in Z^n$ such that*

$$\| z_1, z_2, ..., z_n \| = 1, \qquad \| z_1 - y, z_2 - y, ..., z_n - y \| \geq \theta$$

*for all $y \in Y$.*

**Definition 3.** [14]. A binary operation *: $[0,1] \times [0,1] \to [0,1]$ is called a continuous $t$-norm if * satisfies the following conditions:

: (1) * is commutative and associative

: (2) * is continuous

**Definition 4.** [15]. Let $X$ be a linear space over a real field $F$. A fuzzy subset $N$ of $X^n \times [0, \infty)$ is called a fuzzy $n$-norm on $X$ if and only if:

(FN1) $N(x_1, x_2, ..., x_n, t) > 0$.

(FN2) $N(x_1, x_2, ..., x_n, t) = 1 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.

(FN3) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(FN4) $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$(field)

(FN5) $N(x_1, x_2, ..., x_n + x_n', s + t) \geq N(x_1, x_2, ..., x_n, t) * N(x_1, x_2, ..., x_n', t)$ for all $s, t \in \mathbf{R}$

(FN6) $N(x_1, x_2, ..., x_n, \cdot)$ is left continuous and non-decreasing function of $\mathbf{R}$ such that

$$\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$$

Then $(X, N)$ is called a fuzzy n-normed linear space.

**Definition 5.** [15]. A sequence $\{x_n\}$ in a fuzzy n-normed space $(X, N)$ is said to converge to x if given $r > 0$, $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > 1 - r$ for all $n \geq n_0$.

## 3. Fuzzy anti $n$-normed spaces and $\alpha$-$n$-normed spaces

**Theorem 3.** *Let $(X, N)$ be a fuzzy n-normed space. Assume the condition that (FN7) $N(x_1, x_2, ..., x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent. Define $\| x_1, x_2, ..., x_n \|_\alpha = inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\}, \alpha \in (0, 1)$. Then $\{\| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on $X$. These n-norms are called $\alpha - n -$norms on $X$ corresponding to the fuzzy n-norm on $X$.*

*Proof.* (1) $\| x_1, x_2, ..., x_n \|_\alpha = 0$. This (i) implies $\inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\} = 0$,

(ii) implies, for all $t \in \mathbf{R}$, $t > 0$, $N(x_1, x_2, ..., x_n, t) \geq \alpha > 0, \alpha \in (0, 1)$,

(iii) implies, by (FN7), $x_1, ..., x_n$ are linearly dependent. Conversely, assume that $x_1, x_2, ..., x_n$ are linearly dependent. This

(i) implies, by (FN2), $N(x_1, x_2, ..., x_n, t) = 1$ for all $t > 0$.

(ii) implies, for all $\alpha \in (0, 1)$, $\inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\} = 0$

(iii) implies $\| x_1, x_2, ..., x_n \|_\alpha = 0$

(2) As $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation, it follows that $\| x_1, x_2, ..., x_n \|_\alpha$ is invariant under any permutation.

(3) If $c \neq 0$, then

$$\begin{aligned} \| x_1, x_2, ..., cx_n \|_\alpha &= \inf\{s : N(x_1, x_2, ..., cx_n, s) \geq \alpha\} \\ &= \inf\{s : N(x_1, x_2, ..., x_n, \frac{s}{\mid c \mid}) \geq \alpha\} \end{aligned}$$

Let $t = s / \mid c \mid$, then

$$\begin{aligned} \| x_1, x_2, ..., cx_n \|_\alpha &= \inf\{\mid c \mid t : N(x_1, x_2, ..., x_n, t) \geq \alpha\} \\ &= \mid c \mid \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\} \\ &= \mid c \mid \inf \| x_1, x_2, ..., x_n \|_\alpha \end{aligned}$$

If $c = 0$, then

$$\begin{aligned} \| x_1, x_2, ..., cx_n \|_\alpha &= \| x_1, x_2, ..., 0 \|_\alpha \\ &= 0 = 0 \| x_1, x_2, ..., x_n \|_\alpha \\ &= \mid c \mid \inf \| x_1, x_2, ..., x_n \|_\alpha, \forall c \in F \end{aligned}$$

(4) $\| x_1, x_2, ..., x_n \|_\alpha + \| x_1, x_2, ..., x_n^{'} \|_\alpha$

$$\begin{aligned} &= \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\} + \inf\{s : N(x_1, x_2, ..., x_n^{'}, s) \geq \alpha\} \\ &= \inf\{t + s : N(x_1, x_2, ..., x_n, t) \geq \alpha, N(x_1, x_2, ..., x_n^{'}, s) \geq \alpha\} \\ &= \inf\{t + s : N(x_1, x_2, ..., x_n + x_n^{'}, t + s) \geq \alpha\} \\ &= \inf\{r : N(x_1, x_2, ..., x_n + x_n^{'}, r) \geq \alpha\}, r = t + s \\ &= \| x_1, x_2, ..., x_n + x_n^{'} \|_\alpha . \end{aligned}$$

Therefore, $\| x_1, x_2, ..., x_n + x_n^{'} \|_\alpha \geq \| x_1, x_2, ..., x_n \|_\alpha + \| x_1, x_2, ..., x_n^{'} \|_\alpha$. Thus,

$\{\| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0,1)\}$ is an $\alpha - n-$norm on $X$.

(5) Let $0 < \alpha_1 < \alpha_2$. Then

$$\| x_1, x_2, ..., x_n \|_{\alpha_1} = \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_1\}$$

$$\| x_1, x_2, ..., x_n \|_{\alpha_2} = \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_2\}$$

As $\alpha_1 < \alpha_2$,

$$\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_2\} \subset \{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_1\}$$

$$\inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_2\} \geq \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_1\}$$

which implies

$$\| x_1, x_2, ..., x_n \|_{\alpha_2} \geq \| x_1, x_2, ..., x_n \|_{\alpha_1}$$

Hence, $\{\| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0,1)\}$ is an ascending family of $\alpha - n-$norms on $X$ corresponding to the fuzzy $n-$norm on $X$. □

**Theorem 4.** *Let* $\{\| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0,1)\}$ *be an ascending family of* $\alpha - n-$*norms on* $X$ *corresponding to the fuzzy n-norm on* $X$*. Define a function* $N'$ $: X^n \times \boldsymbol{R} \to [0,1]$ *as*

$$N'(x_1, x_2, ..., x_n, t) = \begin{cases} \sup\{\alpha \in (0,1) : \| x_1, ..., x_n \|_\alpha \leq t\} \\ \textit{when } x_1, x_2, ..., x_n \textit{ are linearly} \\ \textit{independent and } t \neq 0 \\ 0 \qquad \textit{otherwise} \end{cases}$$

*for* $i = 1, 2, ..., n$*. Then* $(X, N')$ *is a fuzzy n-normed linear space.*

*Proof.* (FN1) For all $t \in \mathbf{R}$ with $t < 0$ we have $N(x_1, x_2, ..., x_n, t) = \sup\{\alpha \in (0,1) : \| x_1, x_2, \cdots, x_n \|_\alpha \leq t\} = 0 \forall x \in X$. Similarly for $t = 0$ and $x \neq \underline{0}$, $N(x_1, x_2, ..., x_n, t) = 0$. When $x = \underline{0}$ and $t = 0$ then from definition $N(x_1, x_2, ..., x_n, t) = 0$. Thus $\forall t \in \mathbf{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$ $\forall x \in X$. So (FN1) holds.

(FN2) Let $\forall t \in \mathbf{R}$ with $t > 0$, we have $N(x_1, x_2, ..., x_n, t) = 1$. Choose $\varepsilon \in (0,1)$. Then for any $t > 0$, $\exists \alpha_t \in (\varepsilon, 1)$ such that $\| x_1, x_2, \cdots, x_n \|_\alpha \leq t$, and hence $\| x_1, x_2, \cdots, x_n \|_\varepsilon \leq t$. Since $t > 0$ is arbitrary, this implies that $\| x_1, x_2, \cdots, x_n \|_\varepsilon = 0$. Hence $x_1, x_2, ..., x_n$ are linearly dependent. Conversely, if $x_1, x_2, ..., x_n$ are linearly dependent, $\forall t \in \mathbf{R}$ with $t > 0$, $N'(x_1, x_2, ..., x_n, t)$ $=\sup\ \{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t\} = \sup\ \{\alpha : \alpha \in (0,1)\}$=1. Thus for all $t > 0$, $N'(x_1, x_2, ..., x_n, t)$=1 if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(FN3) As $\| x_1, ..., x_n \|_\alpha$ is invariant under any permutation of $x_1, .., x_n$, so we have $N'(x_1, ..., x_n, t)$ is invariant under any permutation of $x_1, ..., x_n$.

(FN4) For all $t \in \mathbf{R}$ with $t > 0$, $c \in F$,

$$\begin{aligned} N'(x_1, x_2, ..., cx_n, t) &= sup\{\alpha : \| x_1, x_2, ..., cx_n \|_\alpha \leq t\} \\ &= sup\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq \frac{t}{\mid c \mid}\} \\ &= N'(x_1, x_2, ..., x_n, \frac{t}{\mid c \mid}). \end{aligned}$$

(FN5) We have to show that for all $s, t \in \mathbf{R}$,
$N'(x_1, x_2, ..., x_n + x_n', s+t) \geq N'(x_1, x_2, ..., x_n, t) * N'(x_1, x_2, ..., x_n', t)$.
If $(a)$ $s+t < 0$ $(b)$ $s = t = 0; s > 0, t < 0; s < 0, t > 0$, then in these cases the relation is obvious. If $(d)$ $s > 0$, $t > 0$, let $p = N(x_1, x_2, ..., x_n, s)$, $q = N'(x_1, x_2, ..., x_n', t)$ and $p \leq q$. If $p = 0$ and $q = 0$ then obviously (FN5) holds.

Let $0 < r < p \leq q$. Then there exists $\alpha > r$ such that $\| x_1, x_2, ..., x_n \|_\alpha \leq s$ and there exists $\beta > r$ such that $\| x_1, x_2, ..., x_n' \|_\alpha \leq t$

Let $\gamma = \alpha * \beta = \min\{\alpha, \beta\} > r$. Thus $\| x_1, x_2, ..., x_n \|_\gamma \leq \| x_1, x_2, ..., x_n \|_\alpha \leq s$ and $\| x_1, x_2, ..., x_n' \|_\gamma \leq \| x_1, x_2, ..., x_n' \|_\alpha \leq s$ .
Now $\| x_1, x_2, ..., x_n + x_n' \|_\gamma \leq \| x_1, x_2, ..., x_n \|_\alpha + \| x_1, x_2, ..., x_n \|_\alpha \leq s + t$.
Therefore $N'(x_1, x_2, ..., x_n + x_n', s+t) \geq \gamma > r$. Since $0 < r < \gamma$ is arbitrary,

$$N'(x_1, x_2, ..., x_n + x_n', s+t) \geq p = min\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, ..., x_n', t)\}$$

$= N'(x_1, x_2, ..., x_n, s) * N'(x_1, x_2, ..., x_n', t)$.

Similarly if $p \geq q$, then also the relation holds. Thus

$$N'(x_1, x_2, ..., x_n + x_n', s+t) \geq N'(x_1, x_2, ..., x_n, s) * N'(x_1, x_2, ..., x_n', t)$$

(FN6) Let $(x_1, x_2, .., x_n) \in X^n$ and $\alpha \in (0,1)$. Now $t > \| x_1, x_2, ..., x_n \|_\alpha$ which implies that

$$N'(x_1, x_2, ..., x_n, t) = \sup\{\beta : \| x_1, x_2, ..., x_n \|_\beta \leq t\} \geq \alpha$$

So, $\lim_{t \to \infty} N'(x_1, x_2, ..., x_n, t) = 1$.

If $t_1 < t_2 \leq 0$, then $N'(x_1, x_2, ..., x_n, t_1) = N'(x_1, x_2, ..., x_n, t_2) = 0$ for all $(x_1, x_2, ..., x_n) \in X^n$.

If $t_2 > t_1 > 0$, then

$$\begin{aligned} &\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_1\} \subset \{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_2\} \\ &\Rightarrow \sup\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_1\} \leq \sup\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_2\} \\ &\Rightarrow N'(x_1, x_2, ..., x_n, t_1) \leq N'(x_1, x_2, ..., x_n, t_2). \end{aligned}$$

Thus $N'(x_1, x_2, ..., x_n, t)$ is a non decreasing function of $t \in \mathbf{R}$. Hence $(X, N')$ is a fuzzy $n$-normed linear space. □

**Remark 1.** In theorem 5, given below, we show that if the index set $(0,1)$ of the family of crisp $n$-norms $\{\| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0,1)\}$ of theorem 4 is extended to $(0,1]$ then a fuzzy $n$-norm $N$ is generated, satisfying an additional property that $N(x_1, x_2, ..., x_n, .)$ attains the value 1 at some finite value $t$.

**Theorem 5.** *Let* $\{\| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0,1]\}$ *be an ascending family of* $\alpha-$ $n-$*norms on* $X$ *corresponding to the fuzzy* $n$*-norm on* $X$. *Define a function*

$N : X^n \times \boldsymbol{R} \to [0,1]$ *as*

$$N^{'}(x_1, x_2, ..., x_n, t) = \begin{cases} \sup\{\alpha \in (0,1] : \| x_1, ..., x_n \|_\alpha \leq t\} \\ \textit{when } x_1, x_2, ..., x_n \textit{ are linearly} \\ \textit{independent and } t \neq 0 \\ 0 \qquad \textit{otherwise} \end{cases}$$

*for* $i = 1, 2, ..., n$. *Then* (*a*) $(X, N^{'})$ *is a fuzzy n-normed linear space.*

(*b*) *for each* $(x_1, x_2, ..., x_n) \in X^n$, *there exists* $t > 0$ *such that* $N(x_1, x_2, ..., x_n, s) = 1$, *for all* $s \geq t$.

*Proof.* (a) First we prove that $N$ is a fuzzy $n$-norm on $X$.

(FN1) For all $t \in \mathbf{R}$ with $t < 0$ we have $N(x_1, x_2, ..., x_n, t)$=sup$\{\alpha \in (0,1] : \| x_1, x_2, \cdots, x_n \|_\alpha \leq t\} = 0 \forall x \in X$.
Similarly for $t = 0$ and $x \neq \underset{\sim}{0}$, $N(x_1, x_2, ..., x_n, t) = 0$. When $x = \underset{\sim}{0}$ and $t = 0$ then from definition $N(x_1, x_2, ..., x_n, t) = 0$. Thus $\forall t \in \mathbf{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0 \ \forall x \in X$. So (FN1) holds.

(FN2) Let $\forall t \in \mathbf{R}$ with $t > 0$, we have $N(x_1, x_2, ..., x_n, t) = 1$. Choose $\varepsilon \in (0, 1]$. Then for any $t > 0$, $\exists \alpha_t \in (\varepsilon, 1)$ such that $\| x_1, x_2, \cdots, x_n \|_\alpha \leq t$, and hence $\| x_1, x_2, \cdots, x_n \|_\varepsilon \leq t$. Since $t > 0$ is arbitrary, this implies that $\| x_1, x_2, \cdots, x_n \|_\varepsilon = 0$. Hence $x_1, x_2, ..., x_n$ are linearly dependent. Conversely, if $x_1, x_2, ..., x_n$ are linearly dependent, $\forall t \in \mathbf{R}$ with $t > 0$, $N^{'}(x_1, x_2, ..., x_n, t)$ =sup $\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t\}$ = sup $\{\alpha : \alpha \in (0, 1]\}$=1. Thus for all $t > 0$, $N^{'}(x_1, x_2, ..., x_n, t)$=1 if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(FN3) As $\| x_1, ..., x_n \|_\alpha$ is invariant under any permutation of $x_1, .., x_n$, so we have $N^{'}(x_1, ..., x_n, t)$ is invariant under any permutation of $x_1, ..., x_n$.

(FN4) For all $t \in \mathbf{R}$ with $t > 0$, $c \in F$,

$$\begin{aligned} N^{'}(x_1, x_2, ..., cx_n, t) &= \sup\{\alpha : \| x_1, x_2, ..., cx_n \|_\alpha \leq t\} \\ &= \sup\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq \frac{t}{\mid c \mid}\} \\ &= N^{'}(x_1, x_2, ..., x_n, \frac{t}{\mid c \mid}). \end{aligned}$$

(FN5) We have to show that for all $s, t \in \mathbf{R}$,

$$N^{'}(x_1, x_2, ..., x_n + x_n^{'}, s + t) \geq N^{'}(x_1, x_2, ..., x_n, t) * N^{'}(x_1, x_2, ..., x_n^{'}, t)$$

If (*a*) $s + t < 0$ (*b*) $s = t = 0; s > 0, t < 0; s < 0, t > 0$, then in these cases the relation is obvious.
If (*d*) $s > 0, t > 0$, let $p = N(x_1, x_2, ..., x_n, s)$, $q = N^{'}(x_1, x_2, ..., x_n^{'}, t)$ and $p \leq q$. If $p = 0$ and $q = 0$ then obviously (FN5) holds.
Let $0 < r < p \leq q$. Then there exists $\alpha > r$ such that $\| x_1, x_2, ..., x_n \|_\alpha \leq s$ and there exists $\beta > r$ such that $\| x_1, x_2, ..., x_n^{'} \|_\alpha \leq t$ Let $\gamma = \alpha * \beta$=min$\{\alpha, \beta\} > r$. Thus $\| x_1, x_2, ..., x_n \|_\gamma \leq \| x_1, x_2, ..., x_n \|_\alpha \leq s$ and $\| x_1, x_2, ..., x_n^{'} \|_\gamma \leq \| x_1, x_2, ..., x_n^{'} \|_\alpha \leq t$ . Now $\| x_1, x_2, ..., x_n + x_n^{'} \|_\gamma \leq \| x_1, x_2, ..., x_n \|_\alpha + \| x_1, x_2, ..., x_n \|_\alpha \leq s + t$. Therefore $N^{'}(x_1, x_2, ..., x_n + x_n^{'}, s + t) \geq \gamma > r$.

Since $0 < r < \gamma$ is arbitrary,
$N'(x_1, x_2, ..., x_n+x'_n, s+t) \geq p = \min\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, ..., x'_n, t)\} = N'(x_1, x_2, ..., x_n, s) * N'(x_1, x_2, ..., x'_n, t)$. Similarly if $p \geq q$, then also the relation holds. Thus $N'(x_1, x_2, ..., x_n+x'_n, s+t) \geq N'(x_1, x_2, ..., x_n, s) * N'(x_1, x_2, ..., x'_n, t)$

(FN6) Let $(x_1, x_2, .., x_n) \in X^n$ and $\alpha \in (0, 1]$. Now $t > \| x_1, x_2, ..., x_n \|_\alpha$ which implies that $N'(x_1, x_2, ..., x_n, t) = \sup\{\beta : \| x_1, x_2, ..., x_n \|_\beta \leq t\} \geq \alpha$. So, $\lim_{t\to\infty} N'(x_1, x_2, ..., x_n, t) = 1$. If $t_1 < t_2 \leq 0$, then $N'(x_1, x_2, ..., x_n, t_1) = N'(x_1, x_2, ..., x_n, t_2) = 0$ for all $(x_1, x_2, ..., x_n) \in X^n$. If $t_2 > t_1 > 0$, then

$$\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_1\} \subset \{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_2\}$$
$$\Rightarrow \sup\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_1\} \leq \sup\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_2\}$$
$$\Rightarrow N'(x_1, x_2, ..., x_n, t_1) \leq N'(x_1, x_2, ..., x_n, t_2).$$

Thus $N'(x_1, x_2, ..., x_n, t)$ is a non decreasing function of $t \in \mathbf{R}$. Hence $(X, N')$ is a fuzzy $n$-normed linear space.

($b$) For $(x_1, x_2, ..., x_n) \in X$, Define $\| x_1, x_2, ..., x_n \|_1$ and hence there exists $t > 0$ such that $\| x_1, x_2, ..., x_n \|_1 \leq t$.
So, $N_1(x_1, x_2, ..., x_n, t) = \sup\{\alpha \in (0, 1] : \| x_1, x_2, ..., x_n \|_1 \leq t\} = 1$. □

**Remark 2.** Assume further that for $x_1, x_2, ..., x_n$ are linearly independent, (FN8) $N(x_1, x_2, ..., x_n, t)$ is a continuous function of $t \in \mathbf{R}$ ($\mathbf{R}$-set of real numbers) and strictly increasing in the subset $\{t : 0 < N(x_1, x_2, ..., x_n, t) < 1\}$ of $\mathbf{R}$.

**Definition 6.** Let $X$ be a linear space over a real field $F$. A fuzzy subset $N^*$ of $X^n \times [0, \infty)$ is called a fuzzy anti $n$-norm on $X$ if and only if:

(FN*1) for all $t \in \mathbf{R}$ with $t \leq 0$, $N^*(x_1, x_2, ..., x_n, t) = 1$.

(FN*2) for all $t \in \mathbf{R}$ with $t > 0$, $N^*(x_1, x_2, ..., x_n, t) = 0 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.

(FN*3) $N^*(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(FN*4) for all $t \in \mathbf{R}$ with $t > 0$, $N^*(x_1, x_2, ..., cx_n, t) = N^*(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$(field)

(FN*5) for all $s, t \in \mathbf{R}$,

$$N^*(x_1, x_2, ..., x_n + x'_n, s + t) \leq \max\{N^*(x_1, x_2, ..., x_n, s), N^*(x_1, x_2, ..., x'_n, t)\}$$

(FN*6) $N^*(x_1, x_2, ..., x_n, \cdot)$ is right continuous and non-increasing function of $\mathbf{R}$ such that

$$\lim_{t\to\infty} N^*(x_1, x_2, ..., x_n) = 0.$$

Then $(X, N^*)$ is called a fuzzy anti $n$-normed linear space.

To strengthen the above definition, we present the following example.

**Example 1.** Let $(X, \| \bullet, \bullet, \cdots, \bullet \|)$ be a $n$-normed linear space
Define,

$$N^*(x_1, x_2, ..., x_n, t) = \begin{cases} 1 - \frac{t}{t+\| x_1, x_2, \cdots, x_n \|} & \text{when } t(> 0) \in R, \forall x \in X \\ 1 & \text{when } t(\leq 0) \in R, \forall x \in X \end{cases}$$

Then $(X, N^*)$ is a fuzzy anti $n$-normed linear space.

**Theorem 6.** *$N^*$ is a fuzzy anti $n$-norm on $X$ if and only if $(1 - N^*)$ is a fuzzy $n$-norm on $X$.*

*Proof.* Let $(X, N^*)$ is a fuzzy anti $n$-norm on $X$.

$$\begin{aligned}(\mathrm{FN}^*1) &\Leftrightarrow \forall t \in \mathrm{R} \text{ with } t \leq 0, N^*(x_1, x_2, ..., x_n, t) = 1\\ &\Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, t) = 1 - 1\\ &\Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, t) = 0 \Leftrightarrow (\mathrm{FN1}) \text{ holds.}\end{aligned}$$

(FN*2)$\Leftrightarrow$ If $x_1, x_2, ..., x_n$ are linearly dependent on $(X, N^*)$ $\Leftrightarrow$for all $t \in \mathbf{R}$ with $t > 0$, $N^*(x_1, x_2, ..., x_n, t) = 0 \Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, t) = 1 - 0 \Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, t) = 1 \Leftrightarrow$ if $x_1, x_2, ..., x_n$ are linearly dependent on $(X, 1 - N^*)$ $\Leftrightarrow$(FN2) holds.

(FN*3)$\Leftrightarrow$ $N^*(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$ $\Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$ $\Leftrightarrow$ (FN3) holds. Clearly (FN*4 )$\Leftrightarrow$ (FN4).

(FN*5)$\Leftrightarrow$ for all $s, t \in \mathbf{R}$,

$$N^*(x_1, x_2, ..., x_n + x_n', s + t) \leq \max\{N^*(x_1, x_2, ..., x_n, s), N^*(x_1, x_2, ..., x_n', t)\}$$

$\Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n + x_n', s + t)$
$\geq 1 - \max\{N^*(x_1, x_2, ..., x_n, s), N^*(x_1, x_2, ..., x_n', t)\}$
$= \min\{1 - N^*(x_1, x_2, ..., x_n, s), 1 - N^*(x_1, x_2, ..., x_n', t)\} \Leftrightarrow$ (FN*5) holds.

(FN*6) $N^*(x_1, x_2, ..., x_n, \cdot)$ is a non-increasing function of $\mathbf{R}\Leftrightarrow$ if $t_2 < t_1 \leq 1$ then, $N^*(x_1, x_2, ..., x_n, t_1) \geq N^*(x_1, x_2, ..., x_n, t_2) \Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, t_1) \leq 1 - N^*(x_1, x_2, ..., x_n, t_2)$ which implies that $t_2 > t_1 \geq 0 \Leftrightarrow 1 - N^*(x_1, x_2, ..., x_n, \cdot)$ is a non-decreasing function of $\mathbf{R}$. □

**Theorem 7.** *Let $(X, N^*)$ be a fuzzy anti $n$-normed space. Assume the condition that (FN*7) $N^*(x_1, x_2, ..., x_n, t) < 1$ for all $t > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent. Define $\| x_1, x_2, ..., x_n \|_\alpha^* = inf\ \{t > 0 : N(x_1, x_2, ..., x_n, t) < \alpha\}, \alpha \in (0, 1]$. Then $\{\| \bullet, \bullet, ..., \bullet \|_\alpha^*: \alpha \in (0, 1)\}$ is a descending family of $n$-norms on $X$. These $n$-norms are called $\alpha - n-$norms on $X$ corresponding to the fuzzy anti $n$-norm on $X$.*

*Proof.* (1) $\| x_1, x_2, ..., x_n \|_\alpha^* \geq 0$, for all $\alpha \in (0, 1]$ and $(x_1, x_2, ..., x_n) \in X^n$.

(2) $\| x_1, x_2, ..., x_n \|_\alpha^* = 0$. This
(i) implies inf $\{t > 0 : N^*(x_1, x_2, ..., x_n, t) < \alpha\} = 0$,
(ii) implies, for all $t \in \mathbf{R}$, $t > 0$, $N^*(x_1, x_2, ..., x_n, t) < \alpha \leq 1, \alpha \in (0, 1]$,
(iii) implies, by (FN*7), $x_1, ..., x_n$ are linearly dependent.

Conversely, assume that $x_1, x_2, ..., x_n$ are linearly dependent. This
(i) implies, by (FN*2), $N^*(x_1, x_2, ..., x_n, t) = 0\ \forall t > 0$.
(ii) implies, for all $\alpha \in (0, 1]$, $\inf\{t > 0 : N^*(x_1, x_2, ..., x_n, t) < \alpha\} = 0$
(iii) implies $\| x_1, x_2, ..., x_n \|_\alpha^* = 0$.

As $N^*(x_1, x_2, ..., x_n, t)$ is invariant under any permutation, it follows that $\| x_1, x_2, ..., x_n \|_\alpha^*$ is invariant under any permutation.

(3) If $c \neq 0$, then

$$\| x_1, x_2, ..., cx_n \|_\alpha^* = \inf\{t > 0 : N(x_1, x_2, ..., cx_n, s) < \alpha\}$$
$$= \inf\{t > 0 : N(x_1, x_2, ..., x_n, \frac{s}{|c|}) < \alpha\}$$

Let $t = s/ \mid c \mid$, then

$$\| x_1, x_2, ..., cx_n \|_\alpha^* = \inf\{\mid c \mid t : N(x_1, x_2, ..., x_n, t) < \alpha\}$$
$$= \mid c \mid \inf\{t > 0 : N(x_1, x_2, ..., x_n, t) < \alpha\}$$
$$= \mid c \mid \inf \| x_1, x_2, ..., x_n \|_\alpha^*$$

If $c = 0$, then

$$\| x_1, x_2, ..., cx_n \|_\alpha^* = \| x_1, x_2, ..., 0 \|_\alpha^*$$
$$= 0 = 0 \| x_1, x_2, ..., x_n \|_\alpha^*$$
$$= \mid c \mid \inf \| x_1, x_2, ..., x_n \|_\alpha^*, \forall c \in F$$

(4) We have to show that

$$\| x_1, x_2, ..., x_n + x_n^{'} \|_\alpha^* \leq \| x_1, x_2, ..., x_n \|_\alpha^* + \| x_1, x_2, ..., x_n^{'} \|_\alpha^* \ \forall \alpha \in (0, 1]$$

Now,
$\| x_1, x_2, ..., x_n \|_\alpha^* + \| x_1, x_2, ..., x_n^{'} \|_\alpha^* = \inf\{t > 0 : N^*(x_1, x_2, ..., x_n, t) < \alpha\} + \inf\{s > 0 : N^*(x_1, x_2, ..., x_n^{'}, s) < \alpha\}$
$= \inf\{t + s > 0 : N(x_1, x_2, ..., x_n, t) < \alpha, N(x_1, x_2, ..., x_n^{'}, s) < \alpha\}$
$= \inf\{t + s > 0 : N^*(x_1, x_2, ..., x_n + x_n^{'}, t + s) < \alpha\}$
$= \inf\{r > 0 : N^*(x_1, x_2, ..., x_n + x_n^{'}, r) < \alpha\}, r = t + s$
$= \| x_1, x_2, ..., x_n + x_n^{'} \|_\alpha^*$.
Therefore, $\| x_1, x_2, ..., x_n + x_n^{'} \|_\alpha^* \leq \| x_1, x_2, ..., x_n \|_\alpha^* + \| x_1, x_2, ..., x_n^{'} \|_\alpha^*$. Thus, $\{\| \bullet, \bullet, ..., \bullet \|_\alpha^* : \alpha \in (0, 1]\}$ is an $\alpha - n-$norm on $X$. Obviously,

$$\| x_1, x_2, ..., x_n \|_{\alpha_1}^* \geq \| x_1, x_2, ..., x_n \|_{\alpha_2}^*$$

for $\alpha_2 > \alpha_1 \geq 0$. Thus, $\{\| \bullet, \bullet, ..., \bullet \|_\alpha^* : \alpha \in (0, 1]\}$ is a descending family of $\alpha - n-$norms on $X$ corresponding to the fuzzy anti $n-$norm on $X$. □

**Theorem 8.** *Let $\{\| \bullet, \bullet, ..., \bullet \|_\alpha^* : \alpha \in (0, 1]\}$ be a descending family of $\alpha - n-$norms on $X$ corresponding to the fuzzy anti n-norm on $X$. Define a function $N^{'} : X^n \times \boldsymbol{R} \rightarrow [0, 1]$ as*

$$N^{'}(x_1, x_2, ..., x_n, t) = \begin{cases} \inf\{\alpha \in (0, 1] : \| x_1, ..., x_n \|_\alpha^* \leq t\} \\ \text{when } x_1, x_2, ..., x_n \text{ are linearly independent and} \\ t \neq 0 \\ 1 \qquad \text{otherwise} \end{cases}$$

*Then $(X, N^{'})$ is a fuzzy anti n-normed linear space.*

*Proof.* (FN1) For all $t \in \mathbf{R}$ with $t < 0$ we have $N'(x_1, x_2, ..., x_n, t)$=inf$\{\alpha \in (0,1] :\| x_1, x_2, ..., x_n \|_\alpha^* \leq t\} = 1\ \forall x \in X$. Similarly for $t = 0$ and $x \neq \underline{0}$, $N'(x_1, x_2, ..., x_n, t) = 1$. When $x = 0$ and $t = 0$ then from definition
$N(x_1, x_2, ..., x_n, t) = 1$. Thus $\forall t \in \mathbf{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 1\ \forall x \in X$. So (FN*1) holds.

(FN2) Let $\forall t \in \mathbf{R}$ with $t > 0$, we have $N(x_1, x_2, ..., x_n, t) = 0$. Choose $\varepsilon \in (0,1]$. Then for any $t > 0$, $\exists \alpha_t \in (\varepsilon, 1]$ such that $\| x_1, x_2, ..., x_n \|_\alpha^* \leq t$, and hence $\| x_1, x_2, ..., x_n \|_\varepsilon^* \leq t$. Since $t > 0$ is arbitrary, this implies that $\| x_1, x_2, ..., x_n \|_\varepsilon^* = 1$. Hence $x_1, x_2, ..., x_n$ are linearly dependent. Conversely, if $x_1, x_2, ..., x_n$ are linearly dependent, $\forall t \in \mathbf{R}$ with $t > 0$, $N'(x_1, x_2, ..., x_n, t)$ =inf $\{\alpha :\| x_1, x_2, ..., x_n \|_\alpha^* \leq t\}$ = inf $\{\alpha : \alpha \in (0,1]\}$=0. Thus for all $t > 0$, $N'(x_1, x_2, ..., x_n, t)$=0 if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(FN3) As $\| x_1, ..., x_n \|_\alpha^*$ is invariant under any permutation of $x_1, .., x_n$, so we have $N'(x_1, ..., x_n, t)$ is invariant under any permutation of $x_1, ..., x_n$.

(FN4) For all $t \in \mathbf{R}$ with $t > 0$, $c \in F$,

$$\begin{aligned} N'(x_1, x_2, ..., cx_n, t) &= \inf\{\alpha :\| x_1, x_2, ..., cx_n \|_\alpha^* \leq t\} \\ &= \inf\{\alpha :\| x_1, x_2, ..., x_n \|_\alpha^* \leq \frac{t}{|c|}\} \\ &= N'(x_1, x_2, ..., x_n, \frac{t}{|c|}). \end{aligned}$$

(FN5) We have to show that for all $s, t \in \mathbf{R}$,
$N'(x_1, x_2, ..., x_n + x_n', s + t) \leq \max\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, ..., x_n', t)\}$.
If possible, suppose that

$$N'(x_1, x_2, ..., x_n + x_n', s + t) > \max\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, ..., x_n', t)\}.$$

Choose $k$ such that

$$N'(x_1, x_2, ..., x_n + x_n', s + t) > k > \max\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, ..., x_n', t)\}.$$

Now $N'(x_1, x_2, ..., x_n + x_n', s + t) > k$
$\Rightarrow$ inf $\{\alpha \in (0,1] :\| x_1, x_2, ..., x_n + x_n' \|_\alpha^* \leq s + t\} > k$.
$\Rightarrow \| x_1, x_2, ..., x_n + x_n' \|_k^* > s + t$.
$\Rightarrow \| x_1, x_2, ..., x_n \|_k^* + \| x_1, x_2, ..., x_n' \|_k^* > s + t$
Again $k > \max\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, ..., x_n', t)\}$
$\Rightarrow k > N'(x_1, x_2, ..., x_n, s)$ and $k > N'(x_1, x_2, ..., x_n', t)$
$\Rightarrow \| x_1, x_2, ..., x_n \|_k^* \leq s$ and $\| x_1, x_2, ..., x_n' \|_k^* \leq t$
$\Rightarrow \| x_1, x_2, ..., x_n \|_k^* + \| x_1, x_2, ..., x_n' \|_k^* \leq s + t$
Thus $s + t < \| x_1, x_2, ..., x_n \|_k^* + \| x_1, x_2, ..., x_n' \|_k^* \leq s + t$
a contradiction.
Hence $N'(x_1, x_2, ..., x_n + x_n', s + t) \leq \max\{N'(x_1, ..., x_n, s), N'(x_1, ..., x_n', t)\}$.

(FN6) Let $(x_1, x_2, .., x_n) \in X^n$ and $\alpha \in (0,1]$. Now $t > \| x_1, x_2, ..., x_n \|_\alpha^*$ which imples that $N'(x_1, x_2, ..., x_n, t)$ =inf $\{\beta :\| x_1, x_2, ..., x_n \|_\beta^* \leq t\} < \alpha$.

So, $\lim_{t\to\infty} N'(x_1, x_2, ..., x_n, t) = 0$. If $t_1 < t_2 \leq 0$ then $N'(x_1, x_2, ..., x_n, t_1) = N'(x_1, x_2, ..., x_n, t_2) = 0$ for all $(x_1, x_2, ..., x_n) \in X^n$. It $t_2 > t_1 \geq 0$ then $\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_1\} \subset \{\alpha : \| x_1, x_2, ..., x_n \|_\alpha \leq t_2\}$ which implies that $\inf\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha^* \leq t_1\} \geq \inf\{\alpha : \| x_1, x_2, ..., x_n \|_\alpha^* \leq t_2\}$ which implies that $N'(x_1, x_2, ..., x_n, t_1) \geq N'(x_1, x_2, ..., x_n, t_2)$. Thus $N'(x_1, x_2, ..., x_n, t)$ is a non-increasing function of $t \in \mathbf{R}$. Hence $(X, N')$ is a fuzzy anti $n$-normed linear space. □

## 4. Fuzzy Riesz theorem

Now we introduce the concept of *fuzzy n-compact* in a fuzzy $n$-normed linear space.

**Definition 7.** A subset $Y$ of a fuzzy $n$-normed linear space $(X, N)$ is called an *fuzzy n-compact subset* if for every sequence $\{y_n\}$ in $Y$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to an element $y \in Y$. In other words, given $t > 0, 0 < r < 1$, there exists an integer $n_0 \in N$ such that

$$N(y_1, y_2, ..., y_{n-1}, y_{n_k} - y, t/k) > 1 - r$$

for all $n, k \geq n_0$ and $n_k > n_0$.

**Lemma 1.** *Let $(X, N)$ be a fuzzy $n$-normed linear space. Assume that $x_i \in X$ for each $i \in \{1, 2, ..., n\}$ and $c \in F$ (Field). Then*

$$N(x_1, x_2, ..., x_i, ..., x_j + cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, t).$$

*Proof.* $N(x_1, x_2, ..., x_i, ..., x_j + cx_i, ..., x_n, t)$
$= N(x_1, x_2, ..., x_i, ..., x_j + cx_i, ..., x_n, \frac{t}{2} + \frac{t}{2})$
$\geq min\{N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, \frac{t}{2}), N(x_1, x_2, ..., x_i, ..., x_j, ..., cx_i, ..., x_n, \frac{t}{2}\}$
$= \min\{N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, \frac{t}{2}), N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, \frac{t}{|c|2}\}$
Since $| c |= 1$, then
$=\min\{N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, \frac{t}{2}), N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, \frac{t}{2}\}$
$\leq N(x_1, x_2, ..., x_i, ..., x_j, ..., x_n, t)$ □

**Theorem 9.** *Let $(X, N)$ be a fuzzy $n$-normed linear space. If the*

$$\inf_{y\in Y}\{t > 0 : N(x_1 - y, x_2 - y, ..., x_n - y, t)\} = 1$$

*for $(x_1, ..., x_2) \in X^n$ and $Y$ is a fuzzy $n$-compact subset of $X$, then there exists an element $y_0 \in Y$ such that*

$$\{t > 0 : N(x_1 - y_0, x_2 - y_0, ..., x_n - y_0, t)\} = 1.$$

*Proof.* Let $t > 0$ and $\varepsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1-r)*(1-r) > 1-\varepsilon$. Since $Y$ is a fuzzy $n$-compact subset of $X$, there exists an integer $n_0 \in \mathbf{N}$ such that

$$N(x_1 - y_k, x_2 - y_k, ..., x_n - y_k, ct) > 1 - r$$

for all $n, k \geq n_0$ and $c$ is a constant.
Since $\{y_k\}$ is a sequence in a fuzzy $n$-compact subset $Y$ of $X$. Without loss of

generality assume that $\{y_k\}$ is a converges to $y_0$ in $Y$, as $k \longrightarrow \infty$. Then for given $\lambda$, $0 < \lambda < 1$, there exists an integer $n_1 \in \mathbf{N}$ such that

$$N(y_k - y_0, , w_2, ..., w_n, t) > 1 - \lambda,$$

for all $w_i \in X (i = 1, 2, ..., n)$ and $n_0 > n_1$. For every $r \in (0, 1)$, we can find a $\lambda \in (0, 1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \geq 1 - r$$

By Lemma 1, if $n_0 > n_1$, then we have

$$\begin{aligned}
N(x_1 - y_0, x_2 - y_0, ..., x_n - y_0, t) &\geq N(y_k - y_0, x_2 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, x_2 - y_0, ..., x_n - y_0, \frac{(k-1)t}{k}) \\
\geq N(y_k - y_0, x_2 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, y_k - y_0, x_3 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, x_2 - y_k, x_3 - y_0, ..., x_n - y_0, \frac{(k-2)t}{k}) \\
\geq N(y_k - y_0, x_2 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, y_k - y_0, x_3 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, x_2 - y_k, y_k - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., x_n - y_0, \frac{(k-3)t}{k}) \\
\geq N(y_k - y_0, x_2 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, y_k - y_0, x_3 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, x_2 - y_k, y_k - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * \cdots \\
&\quad * N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., y_k - y_0, x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., x_{n-1} - y_k, x_n - y_0 \frac{(k-(n-1)t}{k})
\end{aligned}$$

$$\begin{aligned}
N(x_1 - y_0, x_2 - y_0, ..., x_n - y_0, t) \\
\geq N(y_k - y_0, x_2 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&\quad * N(x_1 - y_k, y_k - y_0, x_3 - y_0, ..., x_n - y_0, \frac{t}{k})
\end{aligned}$$

$$
\begin{aligned}
&* N(x_1 - y_k, x_2 - y_k, y_k - y_0, ..., x_n - y_0, \frac{t}{k}) \\
&* \cdots \\
&* N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., y_k - y_0, x_n - y_0, \frac{t}{k}) \\
&* N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., x_{n-1} - y_k, y_k - y_0, \frac{t}{k}) \\
&* N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., x_{n-1} - y_k, x_n - y_k, \frac{(k-n)t}{k})
\end{aligned}
$$

$$
\begin{aligned}
= N(y_k - y_0, x_2 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
& * N(x_1 - y_0, y_k - y_0, x_3 - y_0, ..., x_n - y_0, \frac{t}{k}) \\
& * N(x_1 - y_0, x_2 - y_0, y_k - y_0, ..., x_n - y_0, \frac{t}{k}) \\
& * \cdots \\
& * N(x_1 - y_0, x_2 - y_0, x_3 - y_0, ..., y_k - y_0, x_n - y_0, \frac{t}{k}) \\
& * N(x_1 - y_0, y_k - y_0, x_3 - y_0, ..., x_{n-1} - y_0, y_k - y_0, \frac{t}{k}) \\
& * N(x_1 - y_k, x_2 - y_k, x_3 - y_k, ..., x_{n-1} - y_k, x_n - y_k, ct) \\
& > \underbrace{(1-\lambda) * (1-\lambda) * \overset{n \ times}{\cdots} * (1-\lambda)} * (1-r) \\
& > (1-r) * (1-r) > 1 - \varepsilon.
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, $\{t > 0 : N(x_1 - y_0, x_2 - y_0, ..., x_n - y_0, t)\} = 1$. □

**Theorem 10.** *Riesz Theorem: Let $(X, N)$ be a fuzzy $n$-normed linear space satisfying conditions (FN7) and (FN8) and $\{\|\bullet, \bullet, \cdots, \bullet\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of $\alpha$-$n$-norms corresponding to $(X, N)$. Let $Y$ and $Z$ be subspaces of $X$ and $Y$ be a fuzzy $n$-compact proper subset of $Z$ with $dimZ \geq n$. For each $k_1 \in (0,1)$, there exists an element $(z_1, \cdots, z_n) \in Z^n$ such that*

$$\| z_1, z_2, \cdots, z_n \|_\alpha = 1, \qquad N(z_1 - y, \cdots, z_n - y, k_1) \leq \alpha$$

*for all $y \in Y$.*

*Proof.* Let $\alpha \in (0,1)$, $(v_1, \cdots, v_n) \in Z - Y$ with $v_1, \cdots, v_n$ are linearly independent. Let

$$\inf_{y \in Y} \|v_1 - y, \cdots, v_n - y\|_\alpha = k$$

*Case(i)*: Assume that $k = 0$. By theorem 9, there is an element $y_0 \in Y$ such that $N(v_1 - y_0, \cdots, v_n - y_0, t) = 1$.

(a) If $y_0 = 0$, then $v_1, \cdots, v_n$ are linearly dependent, which is a contradiction.

(b) If $y_0 \neq 0$, then $v_1, \cdots, v_n, y_0$ are linearly independent.

*Case(ii)* Let $k > 0$, $k = \|v_1 - y, \cdots, v_n - y\|_\alpha = \inf\{s : N(v_1 - y, \cdots, v_n - y, s) \geq \alpha\}$. Since $N(v_1 - y, \cdots, v_n - y, s)$ is continuous (by $(FN8)$), we have by theorem 4.4 in [10]

$$N(v_1 - y, \cdots, v_n - y, k) \geq \alpha$$

$\Rightarrow$ for each $k_1 \in (0,1)$, there exists an element $y_0 \in Y$ such that

$$k \leq \|v_1 - y_0, \cdots, v_n - y_0\|_\alpha \leq \frac{k}{k_1}$$

For each $j = 1, 2, \cdots, n$, let

$$z_j = \frac{v_j - y_0}{\|v_1 - y_0, v_2 - y_0, \cdots, v_n - y_0\|_\alpha^{\frac{1}{n}}}$$

Then it is obvious that $\|z_1, z_2, \cdots, z_n\|_\alpha = 1$
Now,

$$\begin{aligned}
&\|z_1 - y, \cdots, z_n - y\|_\alpha \\
&= \|\frac{v_1 - y_0}{\|v_1 - y_0, \cdots, v_n - y_0\|_\alpha^{\frac{1}{n}}} - y, \cdots, \frac{v_n - y_0}{\|v_1 - y_0, v_2 - y_0, \cdots, v_n - y_0\|_\alpha^{\frac{1}{n}}} - y\|_\alpha \\
&= \frac{1}{\|v_1 - y_0, \cdots, v_n - y_0\|_\alpha}\|v_1 - (y_0 + y\|v_1 - y_0, \cdots, v_n - y_0\|_\alpha^{\frac{1}{n}}, \cdots, \\
&\qquad\qquad v_n - (y_0 + y\|v_1 - y_0, \cdots, v_n - y_0\|_\alpha^{\frac{1}{n}}\| \\
&\geq \frac{1}{\|v_1 - y_0, \cdots, v_n - y_0\|_\alpha} k \geq \frac{k}{\frac{k}{k_1}} = k_1
\end{aligned}$$

By the condition $(FN7)$

$\Rightarrow \exists \alpha \in (0,1)$ such that $\inf\{k > 0 : N(z_1 - y, \cdots, z_n - y, k) \geq \alpha\} \geq k_1$

$\Rightarrow \exists \alpha_0 \in (0,1)$ such that $N(z_1 - y, \cdots, z_n - y, k_1) < \alpha_0 \leq \alpha$

for all $y \in Y$. □

## 5. Conclusion

In this work we have introduced the concept of fuzzy anti $n$-normed linear space and have proved some results based on $\alpha$-$n$-norm which is corresponding to fuzzy $n$-normed linear space. Also inspired by the concept of $\alpha - n$-norm, we have proved the fuzzy version of Riesz theorem in $n$-normed linear spaces.

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**Jacob Kavikumar** received his MSc and Ph.D from Annamalai University, India. Since 2006 he has been at Universiti Tun Hussein Onn Malaysia. He has published many papers in national and international journals. His research interests center on the Fuzzy algebra, Fuzzy functional analysis and Fuzzy numerical analysis.

Centre for Science Studies, Universiti Tun Hussein Onn Malaysia, 86400 Batu Pahat, Johor, Malaysia.
e-mail: kaviphd@gmail.com

**Young Bae Jun** has been an educator and research mathematician since 1982, mostly at the Gyeonsang National University; and a member of the editorial board of Far East Journal of Mathematical Science (India) since 1998, and Quasigroups and Related Systems (Moldova) since 2000. He did postdoctoral work (one year, 1989-90, supported by KOSEF) at the University of Albert in Albert, Canada; and worked for one year (1996-97) as an visiting professor at the Northwest University in Xian, China (supported by LG Yonam Foundation). His research interests focus on the structure theory of BCK/BCI-algebras, Hilbert algebras, (lattice) implication algebras, fuzzy and hyper theory of algebraic structures and fuzzy normed spaces. He is a co-author of the text *BCK-algebras* with J. Meng which is a approachable introduction to BCK/BCI-algebras.

Department of Mathematics Education (and RINS), Gyeonsang National University, Chinju 660-701, Republic of Korea.
e-mail: ybjun@nongae.gsnu.ac.kr

**Azme Khamis** received his MSc from National University of Malaysia and Ph.D in Statistics from University Technology Malaysia, Malaysia. He is now the Director of Center for Science Studies in the Universiti Tun Hussein Onn Malaysia. He has published many papers on Neural network and related topics. His research interests focus on the Neural network, Fuzzy Statistics and Fuzzy functional analysis.

Centre for Science Studies, Universiti Tun Hussein Onn Malaysia, 86400 Batu Pahat, Johor, Malaysia.
e-mail: azme@uthm.edu.my