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A STRUCTURE THEOREM AND A CLASSIFICATION OF AN INFINITE LOCALLY FINITE PLANAR GRAPH

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ABSTRACT. In this paper we first present a structure theorem for an infinite locally finite 3-connected VAP-free planar graph, and in connection with this result we study a possible classification of infinite locally finite planar graphs by reducing modulo finiteness.

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1. Introduction

Let G be an infinite connected planar graph. A path P is a *separating path* if there exist subgraphs H and K of G with $G = H \cup K$ and $H \cap K = P$. A separating path is said to be *unbounded* if each of the two endvertices of the path is incident to an unbounded face. A finite set of unbounded separating paths $\mathcal{P} = \{P_1, \ldots, P_n\}$ in G will be called a *semicycle* if there exist connected subgraphs $G_0, G_1, \ldots, G_n$ of G such that

[S1] $G = \bigcup_{i=0}^{n} G_i, \quad G_0 \cap G_i = P_i$ for all $i \in \{1, \ldots, n\}$
and $G_i \cap G_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and

[S2] G_0 is finite, but G_i $(i = 1, \ldots, n)$ are infinite.

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In this case, the finite subgraph G_0 of G is called the *center* of the semicycle $\mathcal{P}$, which will be denoted by $C(\mathcal{P})$. A semicycle $\mathcal{P}$ is *induced* if all paths in $\mathcal{P}$ are induced. Two semicycles $\mathcal{P}$ and $\mathcal{P}'$ are *disjoint* if $V(\mathcal{P}) \cap V(\mathcal{P}') = \emptyset$; for convenience, the set of vertices $V(\mathcal{P})$ (or the set of edges $E(\mathcal{P})$, respectively) of $\mathcal{P}$ will be understood to be the union of all vertices (or edges, respectively) of the paths in $\mathcal{P}$.

Let $\mathcal{P}$ and $\mathcal{P}'$ be disjoint semicycles with $\mathcal{P} \subseteq C(\mathcal{P}')$ in a connected planar graph G. A $(\mathcal{P}, \mathcal{P}')$*-semiring* in G is a subgraph of G consisting of not only the cycles in $\mathcal{P}$ and $\mathcal{P}'$ but also all vertices and edges lying between $\mathcal{P}$ and $\mathcal{P}'$. *Bridges* of a $(\mathcal{P}, \mathcal{P}')$-semiring $\mathcal{R}$ are defined by the bridges connecting P with P' in $\mathcal{R}$.

A $(\mathcal{P}, \mathcal{P}')$-semiring $\mathcal{R}$ is said to be *tight* if it satisfies following conditions:

[T1] $\mathcal{P}$ and $\mathcal{P}'$ are induced.

[T2] For each infinite component H of $G - C(\mathcal{P})$, there exists exactly one path P in $\mathcal{P}'$ such that the endvertices of P are adjacent to the endvertices of the foot of H.

[T3] $|V(B) \cap V(\mathcal{P}')| \leq 2$ for all bridges B of $\mathcal{R}$.

[T4] If B is a bridge of $\mathcal{R}$ with $V(B) \cap V(\mathcal{P}') = \{z, z'\}$, $z \neq z'$, then $zz' \in E(\mathcal{P}')$.

Our first result which was already presented in [6] is as follows:

Theorem A. *Let G be an infinite locally finite 3-connected VAP-free planar graph, and let $\mathcal{P}_0$ be an induced semicycle in G. Then there exists an infinite sequence of pairwise disjoint induced semicycles $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots)$ such that*

(1) $\mathcal{P}_j \subseteq C(\mathcal{P}_{j+1})$ *for all* $j \in \{0, 1, 2, \ldots\}$,

(2) $(\mathcal{P}_j, \mathcal{P}_{j+1})$*-semiring is tight, for all* $j \in \{0, 1, 2, \ldots\}$, *and*

(3) $G = \displaystyle\bigcup_{j=0}^{\infty} C(\mathcal{P}_j)$.

Moreover, such an infinite sequence of semicycles $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots)$ satisfying the conditions (1)–(3) is uniquely determined under the given semicycle $\mathcal{P}_0$,

In order to describe our main result, let G be an infinite planar graph. Let Δ be a partition of $V(G)$ into finite nonempty subsets such that each $T \in \Delta$ induces a connected subgraph of G, and let further G/Δ denote the graph which arises by contracting each $T \in \Delta$ onto a single vertex v_T, and letting v_T and $v_{T'}$, for distinct $T, T' \in \Delta$, be adjacent if and only if there exists an edge in G connecting vertices T and T'.

For two graphs G and G', we may say that G and G' have the same *infinite structure* if there are partitions Δ and Δ' of $V(G)$ and $V(G')$ respectively, such that G/Δ and G'/Δ' are isomorphic. This relation between graphs is clearly reflexive and symmetric, but unfortunately not transitive. In order to obtain an equivalence relation we form the transitive closure of the relation in question; we denote this relation by $\sim_\infty$. Each of the corresponding equivalence classes will be called an *infinite type* of graphs. Our main result in this paper is as follows:

Theorem B. *Every infinite locally finite connected planar graph has a countable tree in its infinity type.*

In order to get a classification of locally finite connected planar graphs, it is therefore necessary to obtain a complete set of invariants for the locally finite countable trees. A locally finite tree is called a *full ramification* (following Jung [5]) if it is a subdivision of a tree in which each vertex has degree greater than or equal to 3 (or see the references [1] and [3] which are related to these contents).

A countable tree T has uncountably many ends if and only if it contains a full ramification, and then there exists a maximal full ramification which contains all full ramifications of T as subgraphs. It is not difficult to show that all full ramifications belong to the same infinity type; it is formed by those locally finite connected infinite graphs which do not possess a free end (defined in [5]). The locally finite trees without a full ramification are those in which each end has an order in the sense of Jung [5].

A locally finite tree can either be built up by a transfinite process indicated by this concept of order, or it arises from a full ramification by attaching branches, consisting of trees of the first kind; i.e., in which each end has an order.

2. Terminology

The terminology will be that of [2]. The graphs we are considered are undirected, without loops and multiple edges. If $x \in V(G)$, the set $N_G(x) := \Big\{y \in V(G) \,|\, xy \in E(G)\Big\}$ is the *neighborhood* of x in G, and its cardinality $d_G(x)$ is the *degree* of x. A path $P = \{x_0, \ldots , _n\}$ is a graph with $V(P) = \{x_0, \ldots , x_n\}$, $x_i \neq x_j$ if $i \neq j$ and $E(P) = \Big\{x_i x_{i+1} \,|\, 0 \leq i < n\Big\}$. A *ray* or *one-way infinite path* $P := (x_0, x_1, \ldots)$ is defined similarly.

The *ends* of a graph G (this concept was introduced by Thomassen [8] and independently by Halin [4]) are the classes of the equivalence relation $\sim_G$ defined on the set of all one-way infinite paths of G by: $P \sim_G P'$ if and only if there is a one-way infinite path P'' whose intersections with P and P' are infinite; or equivalently if and only if $\mathcal{C}_{G-S}(P) = \mathcal{C}_{G-S}(P')$ for any finite subset $S \setminus V(G)$ (where $\mathcal{C}_{G-S}(P)$ denotes the component of $G - S$ containing a subpath of P).

Let G be a graph and H be a subgraph of G. Define a relation $\sim$ on $E(G) \setminus E(H)$ by the condition that $e_1 \sim e_2$ if there exists a finite path P such that

(i) the first and last edges of P are e_1 and e_2, respectively, and
(ii) P and H are edge-disjoint.

A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$ is called a *bridge* of H in G. If B is a bridge of H in G, then the elements of $V(H) \cap V(B)$ are called the *vertices of attachment* of B.

3. Structure theorem

To simplify the description of the contents in this section, we may say that an infinite locally finite planar graph is an *LV-graph* (following Jung [7]) if it is 3-connected and VAP-free. Let P be a separating path in an LV-graph G, and let H be an infinite component of $G - P$. Let further x ($\bar{x}$, respectively) be the first (the last, respectively) vertex on P adjacent to H, in the natural order. Then, clearly the subpath of P connecting x and $\bar{x}$, which will be called the *foot* of H on P, contains all neighbors of H on P. Since G is VAP-free, we easily see that the feet of distinct infinite components of $G - P$ are pairwise edge-disjoint. The proof of our main result will make use of the following proposition. For the sake of completeness a proof is included below.

Proposition 3.1. *Let G be an LV-graph and let P be an induced unbounded separating path in G with $H \cup K = G$ and $H \cap K = P$. Further let H be infinite and $H^{(1)}, \dots, H^{(r)}$ be infinite components of $H - P$. Then there exist induced unbounded separating paths $P^{(1)}, \dots, P^{(r)}$ with $P^{(i)} \subseteq H^{(i)}$ ($i = 1, \dots, r$) which satisfy the following properties:*

(1) *Each of the endvertices of $P^{(i)}$ is adjacent to an endvertex of the foot of $H^{(i)}$ on P ($i = 1, \dots, r$).*

(2) *For each bridge connecting P with $\bigcup_{i=1}^{r} P^{(i)}$ there exists an index $j \in \{1, \dots, r\}$ such that all vertices of attachment of B lie on $P \cup P^{(j)}$, and $|V(B) \cap V(P^{(j)})| \leq 2$.*

(3) *If $V(B) \cap V(P^{(i)}) = \{z, z'\}$ for a bridge B connecting P with $\bigcup_{i=1}^{r} P^{(i)}$, $z \neq z'$, it must hold $zz' \in E(P^{(i)})$.*

Moreover, for a given separating path P, the induced unbounded separating paths $P^{(1)}, \dots, P^{(r)}$ satisfying the conditions (1)–(3) is uniquely determined.

Proof. We will construct separating paths $P^{(1)}, \dots, P^{(r)}$ satisfying the conditions of this proposition. To do this, for $i = 1, \dots, r$ let $F^{(i)}$ be the foot of $H^{(i)}$ on P with the endvertices x_i and $\bar{x}_i$. We define a set of vertices $V^{(i)}$ of $H^{(i)}$ as follows: $v \in V^{(i)}$ if and only if there exists a bounded face of G incident to v and a vertex of $F^{(i)}$. Let us consider the induced subgraph $G\left[F^{(i)} \cup V^{(i)}\right]$ which may provisionally be denoted by $L^{(i)}$. It is clear that $L^{(i)}$ is connected. To show $L^{(i)}$ is 2-connected, suppose to the contrary that $L^{(i)}$ would contain a vertex u

such that $L^{(i)} - u$ is disconnected. Then, since G is 3-connected, we would have $u \in F^{(i)} \cap V(L^{(i)})$, which contradicts to the fact $H^{(i)} \cap P = \emptyset$.

Now $C^{(i)}$ denotes the outer cycle of $L^{(i)}$, and set $P^{(i)} = C^{(i)} - F^{(i)}$ whose endvertices are y_i and $\bar{y}_i$ with $x_i y_i, \bar{x}_i \bar{y}_i \in E(C^{(i)})$. Since P is an unbounded separating path and x_i is the first neighbor of $H^{(i)}$, it follows that x_i is incident to an unbounded face of G. Thus, from the choice of $V^{(i)}$ and y_i, the edge $x_i y_i$ (and therefore the vertex y_i) is also incident to the unbounded face. Similarly we can verify that $\bar{y}_i$ is incident to an unbounded face, which shows that $P^{(i)}$ is an unbounded separating path.

To show that $P^{(i)}$ is induced, suppose for contradiction that there exists a subpath $u = u_1, u_2, \dots, u_k = v$ of $P^{(i)}$ with $k \geq 3$ and $uv \in E(G)$. Then we see that

$$\tilde{P} := \Big[P^{(i)} - \{u_2, \dots, u_{k-1}\}\Big] \cup \{uv\}$$

is also an unbounded separating path of G with the same endvertices as $P^{(i)}$. Further the elements of V_0 (in particular u_2) lie in the interior of $L^{(i)}$; i.e., u_2 cannot be incident to an unbounded face of $L^{(i)}$, since $L^{(i)}$ is induced. Hence there cannot be exist a bounded face of G incident to both u_2 and a vertex of $F^{(i)}$, which contradicts the construction of $L^{(i)}$.

Next we will prove that the constructed paths $P^{(1)}, \dots, P^{(r)}$ hold the properties (1)–(3). But, since the claim (1) follows immediately from the construction of $L^{(i)}$ above ($i = 1, \dots, r$), we need only to verify the assertions (2) and (3).

To see (2), let B be a bridge which connects P with $\bigcup P^{(i)}$ and let $V(B) \cap V(P^{(j)}) \neq \emptyset$. Then, by considering the fact that $H^{(1)}, \dots, H^{(r)}$ are pairwise disjoint and that $P^{(j)} \subseteq H^{(j)}$, we see that all vertices of attachment of B are contained in $P^{(j)}$. Now assume (reductio ad absurdum) that

$$V(B) \cap V(P^{(j)}) = \Big\{y_1, \dots, y_k\Big\} \quad \text{with } k \geq 3.$$

Note that B cannot be isomorphic to K_2. Since B is a connected subgraph of G, there must exist a $y_1 y_k$-path in B, and thus the vertices $y_2, \dots, y_{k-1}$ cannot be incident to a bounded face which is incident to a vertex of P. Therefore we have a contradiction to the construction of $L^{(i)}$ (or $P^{(i)}$) as we wanted.

It remains to show that the assertion (3) is also true. For this, let B be a bridge connecting P with $\bigcup_{i=1}^{r} P^{(i)}$ satisfying the hypothesis in (3) in this proposition. Since the vertices of attachment of B are precisely z and z', they must be incident to a common bounded face. Now suppose to the contrary that $zz' \notin E(G)$. Then, from the fact that the zz'-subpath (say W) of $P^{(j)}$ has the length at least 2, there would exist a vertex (say v) of the subpath with $v \in V(W) \setminus \{z, z'\}$. By a similar argument above (more precisely, by replacing $y_2, \dots, y_{k-1}$ by v and $y_1 y_k$-path by zz'-subpath), we can also obtain a contradiction.

Now we will show the uniqueness. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the set of infinite unbounded separating paths satisfying the properties described in this proposition. First note that, for given separating path P, the foot of the infinite components of $G - P$ are uniquely determined. But, since the endvertices (say x_i and $\bar{x}_i$) of the foot of an infinite component $H^{(i)}$ must be incident to unbounded faces, we can find the unique vertices (say y_i and $\bar{y}_i$, respectively) incident to x_i and $\bar{x}_i$ such that the edges $x_i y_i$ and $\bar{x}_i \bar{y}_i$ are incident to the unbounded faces. Thus, for each infinite component $H^{(i)}$ of $G - P$, the vertices y_i and $\bar{y}_i$ are uniquely determined; i.e., the endvertices of each separating path for $H^{(i)}$ must precisely be y_i and $\bar{y}_i$. In particular the number of the paths in both $\mathcal{P}_1$ and $\mathcal{P}_2$ is the same as that of the infinite components of $G - P$.

Now let $P_1^{(i)} \in \mathcal{P}_1$ and $P_2^{(i)} \in \mathcal{P}_2$ be separating paths in $H^{(i)}$ satisfying the properties in this proposition. As observed above, the endvertices of these paths are commonly y_i and $\bar{y}_i$. Note that $y_i \bar{y}_i \notin E(G)$ because G is 3-connected, and therefore we have $|P_j^{(i)}| \geq 3$ for $j = 1, 2$.

Assume first that $V(P_1^{(i)}) \cap V(P_2^{(i)}) = \{y_i, \bar{y}_i\}$. Then clearly either all vertices of $P_2^{(i)} - \{y_i, \bar{y}_i\}$ lie in the interior of the cycle

$$P_1^{(i)} \cup F^{(i)} \cup \left\{x_i y_i, \bar{x}_i \bar{y}_i\right\}$$

or those of $P_1^{(i)} - \{y_i, \bar{y}_i\}$ lie in the interior of the cycle

$$P_2^{(i)} \cup F^{(i)} \cup \left\{x_i y_i, \bar{x}_i \bar{y}_i\right\},$$

where $F^{(i)}$ denotes the foot of $H^{(i)}$. But, because of the 3-connectedness of G, there have to exist a path between $P_1^{(i)} - \{y_i, \bar{y}_i\}$ and $P_2^{(i)} - \{y_i, \bar{y}_i\}$, and hence the bridge containing $P_2^{(i)}$ which connects P with $P_1^{(i)}$ has more than 2 vertices of attachment, which contradicts the condition (2). Thus $V(P_1^{(i)}) \cap V(P_2^{(i)}) \geq 3$.

To show $P_1^{(i)} = P_2^{(i)}$, set

$$V(P_1^{(i)}) \cap V(P_1^{(i)}) = \left\{z_1, z_2, \dots, z_{s-1}, z_s\right\}$$

with $y_i = z_1$ and $z_s = \bar{y}_i$. We will prove that $z_k z_{k+1} \in E(G)$ for all $k \in \{1, \dots, s-1\}$. Then, since both $P_1^{(i)}$ and $P_2^{(i)}$ are induced, we can get $P_1^{(i)} = P_2^{(i)}$ as desired. For this, assume to the contrary that there exists a k such that $z_k z_{k+1} \notin E(G)$. For $j = 1, 2$, let us denote the $z_k z_{k+1}$-subpath of $P_j^{(i)}$ by W_j. Then from the assumption we have $|V(W_j)| \geq 3$; i.e., there exists a vertex (say

v_j) of $W_j - \{z_k, z_{k+1}\}$. Notice that there must also exist a v_1v_2-path in G by the connectedness number. Thus, by an argument similar to one above, we obtain a desired contradiction by replacing $P_j^{(i)}$ by W_j ($j = 1, 2$) and $y_i, \bar{y}_i$ by z_k, z_{k+1}, which completes the proof. □

Corollary 3.2. *Let G be an LV-graph. Then, for a given induced semicycle $\mathcal{P} = \{P_1, \ldots, P_n\}$ in G, there exists a unique semicycle $\mathcal{P}'$ with $\mathcal{P} \subseteq C(\mathcal{P}')$ and $V(\mathcal{P}) \cap V(\mathcal{P}') = \emptyset$ such that the $(\mathcal{P}, \mathcal{P}')$-semiring is tight.*

We are now prepared to prove Theorem A.

Proof of Theorem A : Let $\mathcal{P}_0$ be an arbitrary chosen induced semicycle in G. Since $\mathcal{P}_0$ satisfies the hypothesis of Corollary 3.2 and $G - C(\mathcal{P}_j)$ contains infinite components for all $j \in \{0, 1, 2, \ldots\}$, we obtain an infinite sequence of induced semicycles $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots)$ satisfying the properties (1) and (2). To show (3), let $v \in V(G)$. If we set the metric distance between v and $V(\mathcal{P}_0)$ by d_v, we see that $v \in V(C(\mathcal{P}_{d_v}))$ by the construction of the semicycles $\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots\}$, which implies (3).

The uniqueness of such a sequence of semirings follows from Corollary 3.2 above. The proof is complete. □

4. Classification

In this section we study a possible classification of infinite planar graphs by reducing modulo finiteness; the concept seems especially useful for locally finite graphs because they can be represented by trees.

If G is an infinite planar graph and if Δ is a partition of $V(G)$ into finite nonempty subsets such that each $T \in \Delta$ induces a connected subgraph of G, then we get all members (up to isomorphisms) of the infinite type of G in the following way: Replace each vertex v of G by a connected finite graph H_v such that, for $v \neq v'$, $H_v \cap H_{v'} \neq \emptyset$ and draw edges between H_v and $H_{v'}$ if and only if v and v' are adjacent in G. Then, if the constructed graph is denoted by $\widehat{G}$, the form $\widehat{G}/\Delta$ is the same as indicated in section 1. We call a structural property Ω of graphs an *infinite property* if the following holds: If one of the members of an infinity type has Ω, then Ω is shared by all its members. To study infinity properties it is sufficient to consider an infinite planar graph G, a Δ as above and to find out which structural features are preserved by the canonical surjective mapping

$$\tau : G \to G/\Delta$$

and its inverse τ^{-1}. We shall see that infinite degree, number of components and end structure have the infinite properties. It is not hard to show the following results.

Proposition 4.1. *Let G and Δ as above be given. If $T \in \Delta$, then a vertex v_T has an infinite degree d in G/Δ if and only if there is a vertex v in T which has infinite degree d in G.*

Corollary 4.2. *Locally finiteness is an infinite property.*

Corollary 4.3. *The number of connected components is an infinite property.*

From Corollary 4.3 we can also conclude that connectedness is an infinite property.

Proposition 4.4. *Let G and Δ as above be given. If U is a one-way infinite path in G, then $\tau(U)$ is a one-way infinite path in G/Δ; and, vice versa, if there is a one-way infinite path $\tilde{U}$ in G/Δ, then there exists at least one one-way infinite path U with $\tau(U) = \tilde{U}$.*

However, not necessarily every two-way infinite path in G is mapped onto a two-way infinite path in G/Δ, namely if the two one-way infinite paths of the path belong to the same end of G. But the structure of ends is preserved.

Proposition 4.5. *Let G and Δ as above be given. If U and U' are one-way infinite paths in G which are separated by a finite subgraph F of G, then $\tau(F)$ separates $\tau(U)$ and $\tau(U')$ in G/Δ.*

Clearly Proposition 4.5 can be reversed as follows: If $\tilde{U}$ and $\tilde{U}'$ are one-way infinite paths in G/Δ which are separated by a finite $\tilde{F}$, then there are one-way infinite paths U and U' with $U \subseteq \tau^{-1}(\tilde{U})$ and $U' \subseteq \tau^{-1}(\tilde{U}')$ which are separated in G by $\tau^{-1}(\tilde{F})$.

Corollary 4.6. *Free ends of G correspond to free ends of G/Δ.*

From Corollary 4.6 it can easily be verified that the orders of ends corresponding under τ in G and G/Δ are equal. In order to get a classification of infinite graphs one would like to choose a characteristical, especially simpler member out of each infinite type.

Here one naturally thinks of trees. But not all infinity types have a tree among its members, as for instance the graphs $P_{1,\infty} \times \{v\}$ and $P_{2,\infty} \times \{v\}$ show, where v is a new vertex. Now we are prepared to show Theorem B.

Proof of Theorem B : Let H_0 be a finite connected induced subgraph of G. Then, since G is locally finite, $G - H_0$ has finitely many components, say $C_1, C_2, \ldots, C_r$. For each $i = 1, \ldots, r$, let F_i be the finite set of vertices in C_i which are adjacent to some vertices of H_0, and set H_i be a finite connected induced subgraph of C_i containing F_i. Then $C_i - H_i$ has finitely many components

$$C_{i,1},\ C_{i,2},\ \ldots,\ C_{i,r_i} \qquad (i = 1, \ldots, r)$$

In each $C_{i,j}$ choose a finite connected induced subgraph $H_{i,j}$ containing all vertices of $C_{i,j}$ from which there leads an edge into H_i. Then delete $H_{i,j}$ in $C_{i,j}$, consider the finitely many components, and so on. We can see that, by the subgraphs $H_{i,j}$, a partition Δ of $V(G)$ is defined such that G/Δ is a tree, which follows the assertion of the theorem. Our proof is complete. □

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