J. Appl. Math. & Informatics Vol. **27***(2009), No. 3 - 4, pp. 517 - 529*
Website: http://www.kcam.biz

APPROXIMATION OF DERIVATIVE TO A SINGULARLY PERTURBED REACTION-CONVECTION-DIFFUSION PROBLEM WITH TWO PARAMETERS.

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ABSTRACT. In this paper, a singularly perturbed reaction-convection-diffusion problem with two parameters is considered. A parameter -uniform error bound for the numerical derivative is derived. The numerical method considered here is a standard finite difference scheme on piecewise-uniform Shishkin mesh, which is fitted to both boundary and initial layers. Numerical results are provided to illustrate the theoretical results.

AMS Mathematics Subject Classification: 65L10, CR G1.7
Key words and phrases : Singular perturbation problems, Shishkin mesh, scaled derivative, two parameters.

1. Introduction

The theory of singular perturbation is not a settled direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving large or small parameters, become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics.

Methods for the numerical solution of problems involving singularly perturbed second order differential equations with two parameters, using special piecewise

Received December 28, 2007. Revised April 14, 2008. Accepted May 12, 2008. *Corresponding author. The first author wishes to acknowledge the Council of Scientific and Industrial Research, New Delhi (INDIA), for its financial support.

uniform meshes have been considered widely in the literature(see [1] - [7], and references therein).While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications. It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution. A few articles (see [8] - [10] and reference therein) are available in the literature for approximation to the derivative of the solution of singularly perturbed problems. So often the main objective in the investigation of heat and mass transfer processes is to determine derivatives for small values of the parameter, for example if it is necessary to find skin friction and/or heat and diffusion fluxes in problems of flow around some body for large Reynolds and Peclet numbers.

In [10], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain where as Natalia Kopteva and Martin Stynes [8] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. In [11], the authors have obtained bounds on the errors in approximations to the scaled derivative in the whole domain in the case of discontinuous source term. In [13], the authors have estimated the scaled derivative for a singularly perturbed second -order ordinary differential equation with discontinuous convection coefficient using hybrid difference scheme.

The two-parameter problem to be considered in this paper is

$$Lu(x) \equiv \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega \equiv (0,1), \tag{1}$$

$$u(0) = u_0, \quad u(1) = u_1, \tag{2}$$

where $u \in Y \equiv C^0(\bar{\Omega}) \cap C^2(\Omega)$, $0 < \varepsilon \leq 1$, $0 \leq \mu \leq 1$, the coefficients a, b, f are sufficiently smooth and $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0, \forall x \in \overline{\Omega}$, $\gamma = \min\limits_{\overline{\Omega}} \frac{b}{a}$. When the parameter ε is small and $\mu = 1$, the problem is the well-studied one-dimension convection-diffusion problem. In this case, a boundary layer of width $O(\varepsilon)$ appears in a neighbourhood of the point $x = 0$. When the parameter $\mu = 0$ and ε is small the problem is called reaction-diffusion and boundary layers of width $O(\sqrt{\varepsilon})$ appear at both the ends $x = 0$ and $x = 1$. As done in [5], we consider two cases $\alpha\mu^2 \geq \gamma\varepsilon$ and $\alpha\mu^2 \leq \gamma\varepsilon$.

Using the results available in [5, 7], we obtain an approximation to the scaled first derivative of the solution of the two-parameter singularly perturbed second order ordinary differential equation. The scaling is not carried out throughout the domain. In fact we obtain numerical approximation for scaled derivative in boundary layer regions and non-scaled derivative in the outer region separately for the above two parameter problem (1-2).

Note. Through out this paper, C denotes a generic constant that is independent of the parameters ε, μ and N, the dimension of the discrete problem. Let $y : D \longrightarrow \mathbb{R}$, $(D \subset \mathbb{R})$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\| y \|_D = \sup_{x \in D} |y(x)|$.

For the sake of completeness, we now reproduce the following analytical results, computational method and error estimates from [5] for the above problem (1-2).

In the rest of this $\| \, . \, \|$ means $\| \, . \, \|_{\overline{\Omega}}$.

Lemma 1. *If $u \in Y$ such that $u(0) \geq 0$, $u(1) \geq 0$ and $Lu(x) \leq 0$, for $x \in \Omega$ then $u(x) \geq 0$, for all $x \in \overline{\Omega}$.*

Lemma 2. *If u is the solution of the continuous problem (1-2), then*

$$\| u \| \leq C \max\{|u(0)|, |u(1)|\} + \frac{1}{\beta} \| f \| .$$

Lemma 3. *Assume that $a, b, f \in C^2(\overline{\Omega})$, the derivatives of the solution u of the continuous problem (1-2) satisfy the following bounds*

$$\| u^{(k)}(x) \| \leq \frac{C}{(\sqrt{\varepsilon})^k}(1 + (\frac{\mu}{\sqrt{\varepsilon}})^k) \max\{\| u \|, \| f \|\}, \quad k = 1, 2, \tag{3}$$

$$\| u^{(3)}(x) \| \leq \frac{C}{(\sqrt{\varepsilon})^3}(1 + (\frac{\mu}{\sqrt{\varepsilon}})^3) \max\{\| u \|, \| f \|, \| f' \|\}, \tag{4}$$

where C depends only on $\| a \|, \| a' \|, \| b \|$ and $\| b' \|$.

Lemma 4. *The solution u of the continuous problem (1-2) can be decomposed as $u = v + w_l + w_r$ on $[0, 1]$ where*

$$Lv = f, \qquad v(0),\ v(1),\ \text{suitably chosen} \tag{5}$$

$$Lw_l = 0, \qquad w_l(0) = u(0) - v(0),\ w_l(1) = 0 \tag{6}$$

$$Lw_r = 0, \qquad w_r(0) = 0,\ w_r(1) = u(1) - v(1). \tag{7}$$

The regular component v and its derivatives satisfy

$$\| v^{(k)}(x) \| \leq C, \quad k = 0, 1, 2, \tag{8}$$

$$\| v^{(3)}(x) \| \leq \frac{C}{\varepsilon}. \tag{9}$$

The singular components w_l and w_r and their derivatives satisfy similar bounds stated in Lemma 3.

Further we can obtain the following sharper bounds on the derivatives of the singular components.

Lemma 5. *The singular components w_l and w_r and their derivatives satisfy*

$$|w_l^{(k)}(x)| \leq \begin{cases} C(\frac{\mu}{\varepsilon})^k e^{-\theta_1 x}, & \text{if } \alpha\mu^2 \geq \gamma\varepsilon \\ C\varepsilon^{-k/2} e^{-\theta_1 x}, & \text{if } \alpha\mu^2 \leq \gamma\varepsilon \end{cases} \tag{10}$$

$$|w_r^{(k)}(x)| \leq \begin{cases} C\mu^{-k} e^{-\theta_2(1-x)}, & \text{if } \alpha\mu^2 \geq \gamma\varepsilon \\ C\varepsilon^{-k/2} e^{-\theta_2(1-x)}, & \text{if } \alpha\mu^2 \leq \gamma\varepsilon, \end{cases} \tag{11}$$

for $k = 0, 1, 2, 3$, where

$$\theta_1 = \begin{cases} \frac{\alpha\mu}{\varepsilon}, & \text{if } \alpha\mu^2 \geq \gamma\varepsilon \\ \frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}, & \text{if } \alpha\mu^2 \leq \gamma\varepsilon \end{cases}, \quad \theta_2 = \begin{cases} \frac{\gamma}{\mu}, & \text{if } \alpha\mu^2 \geq \gamma\varepsilon \\ \frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}, & \text{if } \alpha\mu^2 \leq \gamma\varepsilon. \end{cases}$$

Discrete Problem:

A fitted mesh method for the continuous problem (1-2) is now introduced. On Ω a piecewise uniform mesh of N mesh interval is constructed. The domain $\overline{\Omega}$ is subdivided into three subintervals $[0, \sigma_1] \cup [\sigma_1, 1-\sigma_2] \cup [1-\sigma_2, 1]$ where the transition parameters are given by $\sigma_1 = \min\{\frac{1}{4}, \frac{2}{\theta_1} \ln N\}$ and $\sigma_2 = \min\{\frac{1}{4}, \frac{2}{\theta_2} \ln N\}$. We denote the step size in each interval by $h_1 = 4\sigma_1/N$, $h_2 = 2(1-\sigma_1-\sigma_2)/N$ and $h_3 = 4\sigma_2/N$. The mesh points are given by

$$x_i = \begin{cases} ih_1, & 0 \leq i \leq \frac{N}{4} \\ \sigma_1 + (i - \frac{N}{4})h_2, & \frac{N}{4} \leq i \leq \frac{3N}{4} \\ 1 - \sigma_2 + (i - \frac{3N}{4})h_3, & \frac{3N}{4} \leq i \leq N. \end{cases}$$

On the piecewise-uniform mesh $\overline{\Omega}^N = \{x_i\}_N^0$ a standard upwind finite difference scheme is used. Then the discrete problem corresponding to the continuous problem (1-2) is

$$\begin{aligned} &L^N U(x_i) = \varepsilon\delta^2 U(x_i) + \mu a(x_i) D^+ U(x_i) - b(x_i)U(x_i) = f(x_i), \quad x_i \in \Omega^N \\ &U(x_0) = u_0, \quad U(x_N) = u_1 \end{aligned} \tag{12}$$

where $\delta^2 U(x_i) = \frac{(D^+ - D^-)U(x_i)}{(x_{i+1} - x_{i-1})/2}$, where

$$D^+ U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i} \quad \text{and} \quad D^- U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}.$$

Then L^N satisfies the discrete minimum principle on Ω^N. Results stated in the following lemmas and theorem are available in [5].

Lemma 6. *If $Z(x_i)$ is any mesh function and $Z(x_0) \geq 0$, $Z(x_N) \geq 0$ and $L^N Z(x_i) \leq 0$ for $i = 1, ..., N-1$, then $Z(x_i) \geq 0$ for all i, $0 \leq i \leq N$.*

Analogous to the continuous case we decompose the discrete solution as $U = V + W_L + W_R$, where

$$\begin{aligned} L^N V &= f(x_i), & V(0) &= v(0),\ V(1) = v(1), \\ L^N W_L &= 0, & W_L(0) &= w_l(0),\ W_L(1) = 0, \\ L^N W_R &= 0, & W_R(0) &= 0,\ W_R(1) = w_r(1). \end{aligned}$$

Lemma 7. *At each mesh point $x_i \in \overline{\Omega}^N$, the regular component of the error satisfies the estimate*

$$|(V - v)(x_i)| \le C(2 - x_i)N^{-1}.$$

Lemma 8. *At each mesh point $x_i \in \overline{\Omega}^N$, the singular component $W = W_L + W_R$ of the error satisfies the estimate*

$$|(W - w)(x_i)| \le CN^{-1}(\ln N)^2.$$

Theorem 1. *Let u be the solution of the continuous problem (1-2) and U be the solution of the corresponding discrete problem (12). Then at each mesh point $x_i \in \overline{\Omega}^N$,we have*

$$|(U - u)(x_i)| \le CN^{-1}(\ln N)^2.$$

2. Analysis on derivative estimate

In this section error estimates for the scaled derivatives of the solutions of (1-2) are given.

In the rest of the paper, ρ stands for

$$\rho = \begin{cases} \varepsilon/\mu, & \text{for, } \ 0 \le i \le N/4 - 1, \\ 1, & \text{for, } \ N/4 \le i \le 3N/4, \\ \mu, & \text{for, } \ 3N/4 + 1 \le i \le N, \end{cases} \quad \text{if } \ \alpha\mu^2 \ge \gamma\varepsilon$$

and

$$\rho = \begin{cases} \sqrt{\varepsilon}, & \text{for, } \ 0 \le i \le N/4 - 1, \\ 1, & \text{for, } \ N/4 \le i \le 3N/4, \\ \sqrt{\varepsilon}, & \text{for, } \ 3N/4 + 1 \le i \le N, \end{cases} \quad \text{if } \ \alpha\mu^2 \le \gamma\varepsilon.$$

Lemma 9. *At each mesh point $x_i \in \Omega^N$ and for all $x \in \bar{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$|\rho(D^+u(x_i) - u'(x))| \ \le \ CN^{-1}\ln N.$$

where $u(x)$ is the solution of (1-2).

Proof. Any function $\phi \in C^2(\Omega_i)$ satisfies the identity

$$D^+\phi(x_i) - \phi'(x) \le \frac{1}{x_{i+1} - x_i}\int_{s=x_i}^{x_{i+1}}\int_{t=x_i}^{s}\phi''(t)dtds - \int_{t=x+i}^{x}\phi''(t)dt. \tag{13}$$

From which it follows that

$$|D^+\phi(x_i) - \phi'(x)| \le \frac{3}{2}(x_{i+1} - x_i) \parallel \phi^{(2)} \parallel . \tag{14}$$

For all $x_i \in \Omega^N$, we have

$$|\rho(D^+u(x_i) - u'(x))| \leq |\rho(D^+v(x_i) - v'(x))| + |\rho(D^+w(x_i) - w'(x))|.$$

Now, we have

$$|(D^+v(x_i) - v'(x))| \leq C(x_{i+1} - x_i) \parallel v^{(2)} \parallel \leq CN^{-1} \tag{15}$$

which gives the required bound for the first term. To bound the second term, we consider the following cases.

Case 1: $\alpha\mu^2 \geq \gamma\varepsilon$.

We have

$$\begin{aligned} |\rho(D^+w(x_i) - w'(x))| &\leq C(x_{i+1} - x_i)(\parallel \rho w_l^{(2)} \parallel + \parallel \rho w_r^{(2)} \parallel) \\ &\leq \begin{cases} C(x_{i+1} - x_i)\frac{\mu}{\varepsilon} & \text{for}, 0 \leq i \leq N/4 - 1, \\ C(x_{i+1} - x_i) \parallel \rho w^{(2)} \parallel & \text{for}, N/4 \leq i \leq 3N/4, \\ C(x_{i+1} - x_i)\mu^{-1} & \text{for}, 3N/4 + 1 \leq i < N. \end{cases} \end{aligned}$$

When $\sigma_1 = 1/4$ and $\sigma_2 = 1/4$, we have $(x_{i+1} - x_i) = N^{-1}$, $\frac{\mu}{\varepsilon} \leq C \ln N$ and $\mu^{-1} \leq C \ln N$. Therefore, we have

$$|\rho(D^+w(x_i) - w'(x))| \leq CN^{-1} \ln N.$$

When $\sigma_1 = \frac{2\varepsilon}{\alpha\mu} \ln N$, $\sigma_2 = \frac{2\mu}{\gamma} \ln N$, for $x_i \in (0, \sigma_1)$ and $x_i \in (1 - \sigma_2, 1)$, we have

$$|\rho(D^+w(x_i) - w'(x))| \leq CN^{-1} \ln N.$$

For $x_i \in [\sigma_1, 1 - \sigma_2]$, using triangle inequality we have

$$|(D^+w(x_i) - w'(x))| \leq CN^{-1} \ln N.$$

Case 2: $\alpha\mu^2 \leq \gamma\varepsilon$.

Using the technique and procedure adopted in Case 1, one can easily obtain

$$|\rho(D^+W(x_i) - w'(x))| \leq CN^{-1} \ln N. \quad \square$$

Lemma 10. *Let v and V be the exact and discrete regular components of the solutions of (1-2) and (12) respectively. Then we have*

$$|\rho D^+(V(x_i) - v(x_i))| \leq CN^{-1}, \quad \textit{for all} \quad x_i \in \Omega^N.$$

Proof. For convenience we introduce the notation

$$e(x_i) = V(x_i) - v(x_i) \quad \text{and} \quad \tau(x_i) = L^N e(x_i).$$

We want to prove that for all i, $0 \leq i \leq N - 1$, $|\rho D^+e(x_i)| \leq CN^{-1}$. Using the result from Lemma 7, we have

$$|\rho D^+e(x_{N/4-1})| = \frac{|\rho(e(x_{N/4}) - e(x_{N/4-1}))|}{x_{N/4} - x_{N/4-1}} \leq C\rho N^{-1}. \tag{16}$$

To prove the result for $0 \le i \le N/4-2$, we rewrite the relation $\tau(x_i) = L^N e(x_i)$, in the form,

$$\begin{aligned} &\varepsilon D^+ e(x_j) - \varepsilon D^+ e(x_{j-1}) + \frac{\mu}{2}(x_{j+1} - x_{j-1}) a(x_j) D^+ e(x_j) \\ &= \frac{1}{2}(x_{j+1} - x_{j-1})[\tau(x_j) + b(x_j) e(x_j)]. \end{aligned} \tag{17}$$

Case 1: $\alpha\mu^2 \ge \gamma\varepsilon$.

Multiplying the above equation by μ^{-1}, summing and rearranging, we obtain

$$\begin{aligned} |\rho D^+ e(x_i)| \le |\rho D^+ e(x_{N/4-1})| + \frac{1}{2\mu}\Big|\sum_{j=i+1}^{N/4-1} (x_{j+1} - x_{j-1})[\tau(x_j) + b(x_j)e(x_j)]\Big| \\ + \frac{1}{2}\Big|\sum_{j=i+1}^{N/4-1} (x_{j+1} - x_{j-1}) a(x_j) D^+ e(x_j)\Big|. \end{aligned}$$

Using the telescoping effect for the last term, (16), $|\tau(x_j)| \le CN^{-1}$ and $|e(x_j)| \le CN^{-1}$ and $|a(x_j) - a(x_{j-1})| \le \| a' \| (x_j - x_{j-1})$, we get for all i, $0 \le i \le N/4-1$,

$$|\rho D^+ e(x_i)| \le CN^{-1}.$$

we rewrite the relation (17) in the form,

$$(1 + \varrho_j) D^+ e(x_j) = D^+ e(x_{j-1}) + \frac{\varrho_j}{a(x_j)\mu}(\tau(x_j) + b(x_j)e(x_j)), \tag{18}$$

where $\varrho_j = \dfrac{a(x_j)\mu(x_{j+1} - x_{j-1})}{2\varepsilon}$. Summing these equations from $j = 1$ to $N/4$, we get

$$\begin{aligned} |D^+ e(x_{N/4})| &\le |D^+ e(x_0)| \frac{(1+\varrho)^{-(N/4-1)}}{1 + \varrho_{N/4}} + CN^{-1}, \text{ where } \quad \varrho = \frac{\alpha h_1 \mu}{\varepsilon} \\ &\le \mu C N^{-1} \varepsilon^{-1} \frac{CN^{-2}}{1 + 4\alpha\sigma_1\mu/(N\varepsilon)} + CN^{-1}, \text{ since } (1+\varrho)^{-N/4} \le CN^{-2} \\ &\le CN^{-1}. \end{aligned}$$

Summing the equations in (18) from $j = N/4$ to $j = i < 3N/4$, we get

$$\begin{aligned} |D^+ e(x_i)| &\le |D^+ e(x_{N/4})| \frac{(1+\bar\varrho)^{-(i-N/2-1)}}{1 + \varrho_i} + CN^{-1}, \text{ where } \quad \bar\varrho = \frac{\alpha h_2 \mu}{\varepsilon} \\ &\le CN^{-1}. \end{aligned}$$

For $i = 3N/4$,

$$\begin{aligned} |D^+ e(x_{3N/4})| &\le |D^+ e(x_{N/4})| \frac{(1+\bar\varrho)^{-(N/4-1)}}{1 + \varrho_{3N/4}} + CN^{-1}, \\ &\le CN^{-1}. \end{aligned}$$

Multiplying throughout by μ and summing the equations in (18) from $j = 3N/4$ to $j = i < N$, we get

$$\begin{aligned} |\mu D^+ e(x_i)| &\leq |\mu D^+ e(x_{3N/4})| \frac{(1+\hat{\varrho})^{-(i-N/4-1)}}{1+\varrho_i} + CN^{-1}, \text{ where } \hat{\varrho} = \frac{\gamma h_3}{\mu} \\ &\leq CN^{-1}. \end{aligned}$$

Similarly we can prove the result for the case $\alpha\mu^2 \leq \gamma\varepsilon$. □

Lemma 11. *Let w and W be the exact and discrete singular components of the solutions of (1)-(2) and (12) respectively. Then for all $x_i \in \Omega^N$, we have*

$$|\rho D^+(W(x_i) - w(x_i))| \leq CN^{-1}(\ln N)^2.$$

Proof. Consider the following cases.

Case 1: $\alpha\mu^2 \geq \gamma\varepsilon$.

From the particular choice of transition point and from [5, Theorem 4.1] we have
$|W(x_i)| \leq CN^{-2}$, $|w(x_i)| \leq CN^{-2}$, for all $x_i \in \Omega^N \cap [\sigma_1, 1-\sigma_2]$. This implies that

$$|D^+(W - w)(x_i)| \leq CN^{-1}, \quad \text{for} \quad x_i \in \Omega^N \cap [\sigma_1, 1-\sigma_2].$$

For $x_i = \sigma_1$, we write $L^N W(\sigma_1) = 0$ in the form

$$|\rho D^+ W(\sigma_1 - h_1)| = |(1 - \frac{h_1\mu}{2\varepsilon} a(\sigma_1))(\varepsilon/\mu) D^+ W(\sigma_1) - \frac{1}{2} b(\sigma_1) h_1 W(\sigma_1)| \leq CN^{-1}.$$

Also we note that $|\rho D^+ w(\sigma_1 - h_1)| \leq CN^{-1}$. Thus $|\rho D^+(W - w)(x_{N/4-1})| \leq CN^{-1}$.

Now consider $x_i \in [0, \sigma_1)$. For convenience we introduce the notation

$$\hat{e}(x_i) = (W - w)(x_i) \quad \text{and} \quad \hat{\tau}(x_i) = L^N \hat{e}(x_i).$$

We have already established that $|\hat{e}(x_i)| \leq CN^{-1}(\ln N)^2$ and

$$|\hat{\tau}(x_i)| \leq \begin{cases} C\sigma_1 \mu^3 \varepsilon^{-2} N^{-1} e^{-\alpha\mu x_{i-1}/\varepsilon}, & 0 \leq i \leq N/4 \\ C\sigma_2 \mu^{-1} N^{-1} e^{-\gamma x_{i-1}/\mu}, & 3N/4 \leq i \leq N. \end{cases} \tag{19}$$

We write the equation $\hat{\tau}(x_i) = L^N \hat{e}(x_i)$ in the form

$$\begin{aligned} &\varepsilon D^+(\hat{e}(x_j) - \hat{e}(x_{j-1})) + \frac{\mu}{2} a(x_j)(x_{j+1} - x_{j-1}) D^+ \hat{e}(x_j) \\ &-\frac{1}{2} b(x_j)(x_{j+1} - x_{j-1}) \hat{e}(x_j) = \frac{1}{2}(x_{j+1} - x_{j-1}) \hat{\tau}(x_j). \end{aligned}$$

Multiplying the above equation by μ^{-1}, summing and rearranging, we obtain

$$\begin{aligned} \rho D^+ \hat{e}(x_i) &= \rho D^+ \hat{e}(x_{N/4-1}) + (a(x_{N/4-1}) \hat{e}(x_{N/4}) - a(x_{i-1}) \hat{e}(x_i)) \\ &\quad - \sum_{j=i+1}^{N/4-1} (a(x_j) - a(x_{j-1})) \hat{e}(x_j) - \frac{1}{\mu} \sum_{j=i+1}^{N/4-1} [b(x_j) h_1 \hat{e}(x_j) + h_1 \hat{\tau}(x_j)]. \end{aligned}$$

Hence using the result at the point $x_{N/4-1}$ and (19), we have

$$|\rho D^+\hat{e}(x_i)| \;\leq\; CN^{-1}((\ln N)^2 + \frac{\mu\sigma_1}{\varepsilon}\frac{\alpha\mu h_1/\varepsilon}{1-e^{-\alpha\mu h_1/\varepsilon}}).$$

Let $y = \alpha\mu h_1/\varepsilon = 4N^{-1}\ln N$. Then $B(y) = \frac{y}{1-e^{-y}}$ is bounded and it follows that
$|\rho D^+\hat{e}(x_i)| \leq CN^{-1}(\ln N)^2$ as required. Similarly, $|\rho D^+\hat{e}(x_i)| \leq CN^{-1}(\ln N)^2$, for $x_i \in (1-\sigma_2, 1]$.

When $\sigma_1 = 1/4$ and $\sigma_2 = 1/4$, the mesh is uniform. Using the above procedure one can prove $|\rho D^+\hat{e}(x_i)| \leq CN^{-1}(\ln N)^2$.

Case 2: $\alpha\mu^2 \leq \gamma\varepsilon$.

Similar to Case 1, one can easily obtain the required result. This completes the proof. □

Theorem 2. *Let u be the solution of (1)-(2) and U the numerical solution of (12). Then for each* i, $0 \leq i \leq N-1$ *we have*

$$\| \rho(D^+U(x_i) - u') \|_{\overline{\Omega}_i} \;\leq\; CN^{-1}(\ln N)^2, \quad \textit{if}, \quad \alpha\mu^2 \geq \gamma\varepsilon$$

and

$$\| \rho(D^+U(x_i) - u') \|_{\overline{\Omega}_i} \;\leq\; CN^{-1}(\ln N)^2, \quad \textit{if}, \quad \alpha\mu^2 \leq \gamma\varepsilon,$$

where C is independent of ε, μ *and* N.

Proof. From triangular inequality we have

$$|\rho(D^+U(x_i) - u'(x))| \leq |\rho D^+(U-u)(x_i)| + |\rho(D^+u(x_i) - u'(x))|.$$

From Lemma 9 we get $|\rho(D^+u(x_i) - u'(x))| \leq CN^{-1}\ln N$. To bound $|\rho D^+(U-u)(x_i)|$, it can be written as

$$\begin{aligned}|\rho D^+(U-u)(x_i)| &\leq |\rho D^+(V-v)(x_i)| + |\rho D^+(W-w)(x_i)| \\ &\leq CN^{-1}(\ln N)^2,\end{aligned}$$

by Lemmas 10 - 11. Hence the proof. □

Remark. Let $\bar{U}$ denote the piecewise linear interpolant over $\overline{\Omega}$ of the discrete solution U on $\overline{\Omega}^N$. Since $\rho\bar{U}$ is a linear function in the open interval $\Omega_i = (x_i, x_{i+1})$ for each $i, 0 \leq i \leq N-1$, we have $\rho\bar{U}'(x) = \rho D^+U(x_i)$ for all $x \in \Omega_i$. It then follows, from Theorem 2, that $\rho\bar{U}'$ is an $(\varepsilon,\mu)-$uniform approximation to $\rho u'(x)$ for each $x \in (x_i, x_{i+1})$. We now show that this approximation can be extended in a natural way to the entire domain $\overline{\Omega}$. We define the piecewise constant function $\bar{D}^+U$ on $[0,1)$ by $\rho\bar{D}^+U(x) = \rho D^+U(x_i)$, for $x \in [x_i, x_{i+1})$, $i = 0, ..., N-1$ and at the point $x = 1$ by $\rho\bar{D}^+U(1) = \rho D^+U(x_{N-1})$. Then, from the above theorem, $\rho\bar{D}^+U$ is an $(\varepsilon,\mu)-$uniform global approximation to $\rho u'$ in the sense that

$$\| \rho\bar{D}^+U - \rho u' \|_{\overline{\Omega}} \leq CN^{-1}(\ln N)^2.$$

3. Numerical results

In this section, we present an example to illustrate the result obtained in this paper.

Example.[7]. Consider the singularly perturbed boundary value problem

$$\varepsilon u''(x) + \mu(1+x)u'(x) - u(x) = (1+x)^2,\ x \in (0,1)$$
$$u(0) = 0, \quad u(1) = 0.$$

Let U^N be a numerical approximation for the exact solution u on the mesh Ω^N and N is the number of mesh points. For all integers $N, 2N$ satisfying $N \in R_N = [32, 64, 128, 256, 512, 1024]$, we compute the maximum pointwise two-mesh differences for the cases $\alpha\mu^2 \geq \gamma\varepsilon$ and $\alpha\mu^2 \leq \gamma\varepsilon$ respectively as

$$D^N_{\varepsilon,\mu} = \begin{cases} \max|(\varepsilon/\mu)(D^+U^N - \bar{D}^+U^{2N})(x_i)|, & \text{for} \quad 1 \leq i \leq N/4-1 \\ \max|(D^+U^N - \bar{D}^+U^{2N})(x_i)|, & \text{for} \quad N/4 \leq i \leq 3N/4 \\ \max|\mu(D^+U^N - \bar{D}^+U^{2N})(x_i)|, & \text{for} \quad 3N/4+1 \leq i \leq N, \end{cases}$$

and

$$D^N_{\varepsilon,\mu} = \begin{cases} \max|\sqrt{\varepsilon}(D^+U^N - \bar{D}^+U^{2N})(x_i)|, & \text{for} \quad 1 \leq i \leq N/4-1 \\ \max|(D^+U^N - \bar{D}^+U^{2N})(x_i)|, & \text{for} \quad N/4 \leq i \leq 3N/4 \\ \max|\sqrt{\varepsilon}(D^+U^N - \bar{D}^+U^{2N})(x_i)|, & \text{for} \quad 3N/4+1 \leq i \leq N. \end{cases}$$

We also compute the $\varepsilon-$uniform two-mesh differences as $D^N_\mu = \max_\varepsilon D^N_{\varepsilon,\mu}$ and the (ε,μ)-uniform two-mesh differences as $D^N = \max_\mu D^N_\mu$. From these values the local orders of convergence $p^N_{\varepsilon,\mu}$, the local order of $\varepsilon-$uniform convergence p^N_μ and the local order of $(\varepsilon,\mu)-$uniform convergence p^N are calculated using

$$p^N_{\varepsilon,\mu} = \log_2 \frac{D^N_{\varepsilon,\mu}}{D^{2N}_{\varepsilon,\mu}}, \quad p^N_\mu = \log_2 \frac{D^N_\mu}{D^{2N}_\mu} \quad \text{and} \quad p^N = \log_2 \frac{D^N}{D^{2N}}.$$

Table 1 contains the values of D^N_μ, p^N_μ and D^N, p^N for various values of ε, μ in case of $\mu^2 \geq \varepsilon$. Table 2 contains the value of D^N_μ, p^N_μ and D^N, p^N for various values of ε, μ in case of $\mu^2 \leq \varepsilon$. The numerical results in Table 1 and Table 2 are in agreement with the theoretical parameter-uniform error bound as given in Theorem 2.

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TABLE 1. Values of D_μ^N, p_μ^N and D^N, p^N for the first derivative of the solution u on $[0, x_{N/4})$, $[x_{N/4}, x_{3N/4}]$ and $(x_{3N/4}, 1]$ respectively.

N	$x_i \in [0, x_{N/2})$		$x_i \in [x_{N/4}, x_{3N/4}]$		$x_i \in (x_{3N/4}, 1]$	
	D_μ^N	p_μ^N	D_μ^N	p_μ^N	D_μ^N	p_μ^N
$\mu = 2^{-1}$, $\varepsilon \in \{2^{-25}, 2^{-2}\}$						
128	2.9531e-2	9.3173e-1	2.1332e-2	9.4454e-1	1.1628e-2	9.9443e-1
256	1.5481e-2	9.6560e-1	1.1084e-2	9.7198e-1	5.8365e-3	9.9716e-1
512	7.9273e-3	9.8276e-1	5.6507e-3	9.8585e-1	2.9240e-3	9.9862e-1
1024	4.0113e-3	-	2.8532e-3	-	1.4634e-3	-
$\mu = 2^{-10}$, $\varepsilon \in \{2^{-45}, 2^{-21}\}$						
128	7.6611e-2	2.4403e-1	1.6956e-1	4.3032e-1	1.0398e-1	6.4273e-1
256	6.4689e-2	4.7384e-1	1.2583e-1	7.1217e-1	6.6599e-2	7.2608e-1
512	4.6579e-2	6.3140e-1	7.6807e-2	9.4782e-1	4.0262e-2	7.8279e-1
1024	3.0069e-2	-	3.9818e-2	-	2.3402e-2	-
$\mu = 2^{-25}$, $\varepsilon \in \{2^{-51}, 2^{-75}\}$						
128	7.6443e-2	2.4403e-1	7.8278e-3	9.9158e-1	1.0386e-1	6.4265e-1
256	6.4547e-2	4.7380e-1	3.9368e-3	9.6596e-1	6.6526e-2	7.2604e-1
512	4.6478e-2	6.3144e-1	2.0154e-3	8.6637e-1	4.0219e-2	7.8273e-1
1024	3.0069e-2	-	3.9818e-2	-	2.3402e-2	-
	D^N	p^N	D^N	p^N	D^N	p^N
128	8.2160e-2	2.4377e-1	1.6956e-1	4.3032e-1	1.0840e-1	8.7802e-1
256	6.9387e-2	4.7401e-1	1.2583e-1	7.1217e-1	6.6599e-2	7.2608e-1
512	4.9956e-2	6.3167e-1	7.6807e-2	9.4782e-1	4.0262e-2	7.8279e-1
1024	3.2243e-2	-	3.9818e-2	-	2.3402e-2	-

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TABLE 2. Values of D_μ^N, p_μ^N and D^N, p^N for the first derivative of the solution u on $[0, x_{N/2})$, $[x_{N/4}, x_{3N/4}]$ and $(x_{3N/4}, 1]$ respectively.

N	$x_i \in [0, x_{N/2})$		$x_i \in [x_{N/4}, x_{3N/4}]$		$x_i \in (x_{3N/4}, 1]$	
	D_μ^N	p_μ^N	D_μ^N	p_μ^N	D_μ^N	p_μ^N
$\mu = 2^{-5}, \quad \varepsilon \in \{2^{-9}, 2^0\}$						
128	3.0650e-1	6.3996e-1	5.0600e-2	9.2457e-1	2.8816e-1	9.3951e-1
256	1.9669e-1	8.1681e-1	2.6658e-2	9.6250e-1	1.5025e-1	9.6950e-1
512	1.1166e-1	9.0750e-1	1.3680e-2	9.8129e-1	7.6730e-2	9.8474e-1
1024	5.9527e-2	-	6.9293e-3	-	3.8773e-2	-
$\mu = 2^{-10}, \quad \varepsilon \in \{2^{-19}, 2^{-1}\}$						
128	6.7777e-1	9.3951e-1	5.7701e-2	9.2215e-1	7.9460e-1	6.8981e-1
256	5.0759e-1	9.6950e-1	3.0450e-2	9.6129e-1	4.9260e-1	7.5766e-1
512	3.3678e-1	9.8474e-1	1.5639e-2	9.8074e-1	2.9135e-1	8.0230e-1
1024	2.0658e-1	-	7.9246e-3	-	1.6707e-1	-
$\mu = 2^{-25}, \quad \varepsilon \in \{2^{-49}, 2^{-1}\}$						
128	9.9469e-1	5.3115e-1	5.8021e-2	9.2215e-1	9.1608e-1	6.9689e-1
256	6.8833e-1	6.6471e-1	3.0619e-2	9.6137e-1	5.6513e-1	7.6215e-1
512	4.3421e-1	7.4879e-1	1.5725e-2	9.8068e-1	3.3321e-1	8.0498e-1
1024	2.5840e-1	-	7.9685e-3	-	1.9072e-1	-
	D^N	p^N	D^N	p^N	D^N	p^N
128	9.9469e-1	5.3115e-1	5.8021e-2	9.2215e-1	9.1608e-1	6.9689e-1
256	6.8833e-1	6.6471e-1	3.0619e-2	9.6137e-1	5.6513e-1	7.6215e-1
512	4.3421e-1	7.4879e-1	1.5725e-2	9.8068e-1	3.3321e-1	8.0498e-1
1024	2.5840e-1	-	7.9685e-3	-	1.9072e-1	-

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