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SINGULAR PERIODIC SOLUTIONS OF A CLASS OF ELASTODYNAMICS EQUATIONS

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ABSTRACT. A second order nonlinear ordinary differential equation is obtained by solving the initial-boundary value problem of a class of elastodynamics equations, which models the radially symmetric motion of a incompressible hyper-elastic solid sphere under a suddenly applied surface tensile load. Some new conclusions are presented. All existence conditions of nonzero solutions of the ordinary differential equation, which describes cavity formation and motion in the interior of the sphere, are presented. It is proved that the differential equation has singular periodic solutions only when the surface tensile load exceeds a critical value, in this case, a cavity would form in the interior of the sphere and the motion of the cavity with time would present a class of singular periodic oscillations, otherwise, the sphere remains a solid one. To better understand the results obtained in this paper, the modified Varga material is considered simultaneously as an example, and numerical simulations are given.

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1. Introduction

In this paper, we are concerned with the radially symmetric motion of a solid sphere, with the undeformed radius B, composed of an incompressible hyper-elastic material. It is assumed that the sphere is subjected to a prescribed uniform radial tensile load $p_0 > 0$ on its surface $R = B$ at time $t = 0$. In spherical coordinates, the point (R, Θ, Φ) in the undeformed configuration moves to the point (r, θ, ϕ) at time $t > 0$. Under the assumption of radially symmetric deformation, the deformed configuration is given by

$$r = r(R,t) > 0, 0 < R < B, r(0+,t) \geq 0; \Theta = \theta, \Phi = \phi, \tag{1}$$

where $r(R,t)$ is the radially deformed function to be determined.

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The mathematical model that describes the radial motion of the incompressible solid sphere under a prescribed uniform radial tensile load at its outer surface is as follows,

$$\frac{\partial \sigma_1(R,t)}{\partial R}\left(\frac{\partial r(R,t)}{\partial R}\right)^{-1} + \frac{2}{r(R,t)}(\sigma_1(R,t) - \sigma_2(R,t)) = \rho \frac{\partial^2 r(R,t)}{\partial t^2}, \quad (2)$$

$$\frac{\partial r(R,t)}{\partial R} = \frac{R^2}{r^2(R,t)}. \quad (3)$$

$$r(R,0) = R, \frac{\partial r(R,0)}{\partial t} = 0. \quad (4)$$

$$r(0+,t)\sigma_1(0+,t) = 0, \quad t \geq 0, \quad (5)$$

$$\sigma_1(B,t) = p_0 \left(\frac{B}{r(B,t)}\right)^2, \quad t \geq 0. \quad (6)$$

In this mathematical model, Eq.(2) is the equilibrium differential equation, in the absence of body force, that describes the radially symmetric motion of the solid sphere.

In Eq.(2), $\sigma_1(R,t)$ and $\sigma_2(R,t)$ are the radial and the circumference Cauchy stresses and respectively given by

$$\sigma_1(R,t) = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p(R,t), \quad (7a)$$

$$\sigma_2(R,t) = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p(R,t), \quad (7b)$$

in which $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain energy function associated with an incompressible hyper-elastic material, and $\lambda_1, \lambda_2, \lambda_3$ are the radial and the circumference stretches and respectively given by

$$\lambda_1 = \frac{\partial r(R,t)}{\partial R}, \quad \lambda_2 = \lambda_3 = \frac{r(R,t)}{R}. \quad (8)$$

and $p(R,t)$ is the hydrostatic pressure, ρ is a constant mass density of the material.

Obviously, Eq.(2) is a so-called nonlinear evolution equation.

Eq.(3) is obtained from the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ and Eq.(8).

Eq.(4) is the initial conditions, i.e., the sphere is in an undeformed state and at rest at time $t = 0$.

Eq.(5) denotes that if no cavity forms in the interior of the solid sphere, we have $r(0+,t) = 0$, if it is found that a cavity with radius $r(0+,t) = c > 0$ forms in the sphere, then the condition for traction-free cavity surface $\sigma_1(0+,t) = 0$ must hold.

Eq.(6) is the boundary condition, namely, a prescribed load p_0 is suddenly applied and maintained at the surface of the sphere.

Interestingly, if the right hand of Eq.(2), i.e., the acceleration term, is set to zero and Eq.(4) is not considered, then the above mathematical model describes the static deformation problems of a solid sphere under a prescribed uniform radial tensile load that has been extensively examined by many authors.

The first investigation was contributed by Ball [1] in 1982, who founded a mathematical model for the static deformation problems and formulated the cavity formation and growth as a bifurcation problem. In recent years, many significant works on the static bifurcation problem have been carried out, which may be found in [2]- [7]. In particular, the qualitative properties of the static problem for the modified Varga material was studied in [8] by using Singularity Theory and Bifurcation Theory.

On the other hand, while the static problems in hyper-elastic materials are well understood, the analogous dynamic problems are relatively unexplored due to the strong nonlinearity of the governing equations.

The first investigation of the radial oscillation was undertaken by Knowles [9, 10]. He respectively considered a cylindrical tube and a spherical shell composed of isotropic incompressible hyper-elastic materials, and reduced the equations of motion to second order ordinary differential equations. Other investigations on this aspect may be found in [11]-[15]. In particular, the mathematical model that describes cavitation in nonlinear elastodynamics for homogeneous isotropic neo-Hookean materials was first proposed by Chou-Wang and Horgan [16] in 1989, and it is pointed out that a cavity would form in the interior of the sphere and the motion of the cavity with time would present a nonlinearly periodic oscillation as the surface tensile load exceeds a critical value in their work. The results of [16] are well generalized and some new results are obtained in this work.

2. Solutions of the mathematical model

Integrating Eq.(3) with respect to R directly, we have

$$r = r(R,t) = \left[R^3 + c^3(t)\right]^{1/3}, \quad t \geq 0, \tag{9}$$

where $c(t) \geq 0$ is a integral constant to be determined. According to the relationship between r and R, for convenience, Eq.(9) is rewritten as

$$R = \left[r^3 - c^3(t)\right]^{1/3}, \tag{10}$$

so that the principal stretches (8) can be rewritten as

$$\lambda_1 = \left(1 - \frac{c^3}{r^3}\right)^{2/3}, \quad \lambda_2 = \lambda_3 = \left(1 - \frac{c^3}{r^3}\right)^{-1/3}. \tag{11}$$

Note. In this case, the principal Cauchy stresses (7a, b), the principal stretches (8) and the hydrostatic pressure $p(R,t)$ are all functions of variable r.

Let

$$\eta = \eta(r,c) = \left(1 - c^3/r^3\right)^{-1/3}, \tag{12}$$

we have $\lambda_1 = \eta^{-2}, \lambda_2 = \lambda_3 = \eta$. Further, the strain energy function $W = W(\lambda_1, \lambda_2, \lambda_3)$ can be rewritten as $W = W(\eta^{-2}, \eta, \eta)$.

From Eq.(9), the initial conditions (4), which the cavity radius $c(t)$ must satisfy, become

$$c(0) = 0, \quad \dot{c}(0) = 0. \tag{13}$$

Note. In this paper, dots over all letters denote derivative with respect to t.

Using Eq.(9), we also have

$$\frac{\partial^2 r(R,t)}{\partial t^2} = 2cr^{-5}(r^3 - c^3)(\dot{c})^2 + c^2 r^{-2}\ddot{c}. \tag{14}$$

Substituting Eqs.(7a, b) and (14) into Eq.(2) yields

$$\frac{\partial}{\partial r}\left[\eta^{-2}W_1(\eta^{-2},\eta,\eta) - p(r,t)\right] + \frac{2}{r}\left[\eta^{-2}W_1(\eta^{-2},\eta,\eta) - \eta W_2(\eta^{-2},\eta,\eta)\right]$$
$$= \rho\left[2cr^{-5}(r^3 - c^3)(\dot{c})^2 + c^2r^{-2}\ddot{c}\right]. \tag{15}$$

Integrating Eq.(15) with respect to r we obtain

$$p(r,t) = \eta^{-2}W_1(\eta^{-2},\eta,\eta) + 2\int_c^r \left[\eta^{-2}W_1(\eta^{-2},\eta,\eta) - \eta W_2(\eta^{-2},\eta,\eta)\right]\frac{d\xi}{\xi}$$
$$-p(c,t) - \rho\left[\left(\frac{c^4}{2r^4} - \frac{2c}{r} + \frac{3}{2}\right)(\dot{c})^2 + c\left(1 - \frac{c}{r}\right)\ddot{c}\right], \tag{16}$$

where in the integration, $\eta = \eta(\xi, c)$, and $W_i(\eta^{-2},\eta,\eta)$ denotes the partial derivative with respect to the $i-th$ variable.

For a prescribed incompressible hyper-elastic material (i.e. the corresponding strain energy function is determined), under the surface tensile load p_0, we see from Eq.(7a, b) that $p(c,t) = -\sigma_1(c,t)$. Thus, if $c(t) \equiv 0$, namely, no cavity forms in the interior of the sphere, then $r(R,t) = R$. From Eq.(16) and the boundary condition (5), we have $p(0,t) = -p_0$. On the other hand, if $c(t) \geq 0$, i.e., a cavity forms, then from Eq.(5) we have $p(c,t) = 0, t \geq 0$.

Remark. According to the normalization condition of the strain energy function, we have $W_1(1,1,1) = W_2(1,1,1) = 0$.

Multiplying both sides of Eq.(16) by $r(0+,t)$, setting $R = B$ and using Eq.(7a, b), from the boundary conditions (5) and (6) we obtain

$$\rho c\left[c\left(1 - \frac{c}{S}\right)\ddot{c} + \left(\frac{c^4}{2S^4} - \frac{2c}{S} + \frac{3}{2}\right)(\dot{c})^2\right] - cp_0\left(\frac{B}{S}\right)^2$$
$$-2c\int_c^S \left[\eta^{-2}W_1(\eta^{-2},\eta,\eta) - \eta W_2(\eta^{-2},\eta,\eta)\right]\frac{d\xi}{\xi} = 0, \tag{17}$$

where

$$S = r(B,t) = (B^3 + c(t)^3)^{1/3}. \tag{18}$$

Obviously, for any given $p_0 > 0$, $c(t) \equiv 0$ is a solution of Eq.(17), and thus

$$r(R,t) = R, \quad p(r,t) = -p_0, \quad t \geq 0 \tag{19}$$

are homogeneous solutions of the Eqs.(2) and (3). If it is found that there is a value of $c(t) > 0$ that satisfies Eq.(17) and the initial conditions (13), then

Eqs.(9) and (16) are the nontrivial solutions of the Eqs.(2) and (3) with the initial and boundary conditions.

Interestingly, the quantity $c(t)$ denotes the value of the cavity radius at time t, where $c(t)=0$ implies that the sphere remains solid in the current configuration. If it is found that $c(t)>0$, then it implies that there is a cavity with radius $r(0+,t)=c(t)>0$ centered at the sphere in the current configuration at time t.

Moreover, the second order nonlinear ordinary differential equation (17) provides a exact relationship between the prescribed load p_0 and the cavity radius $c(t)$. We call Eq.(17) the *motion equation of cavity.*

In order to obtain the existence conditions of cavity formation in the interior of the sphere, i.e., $c(t)>0$, and the motion rule of the formed cavity, we will study the qualitative properties of solutions of Eq.(17) with the initial conditions (13).

3. Qualitative analyses of Eq.(17)

For convenience, we define the dimensionless cavity radius and its velocity by

$$x(t)=c(t)/B, \quad \dot{x}(t)=\dot{c}(t)/B, \tag{20}$$

and so the initial conditions (13) become

$$x(0)=0, \quad \dot{x}(0)=0. \tag{21}$$

Let

$$h(x)=-2\int_c^S\left[\eta^{-2}W_1(\eta^{-2},\eta,\eta)-\eta W_2(\eta^{-2},\eta,\eta)\right]\frac{d\xi}{\xi} \tag{22}$$

and

$$\hat{W}(\eta)=W(\eta^{-2},\eta,\eta), \tag{23}$$

we have

$$\hat{W}_1(\eta)=\frac{d\hat{W}(\eta)}{d\eta}=2(-\eta^{-3}W_1(\eta^{-2},\eta,\eta)+W_2(\eta^{-2},\eta,\eta)). \tag{24}$$

From the relationship between η and ξ, Eq.(22) then becomes

$$h(x)=\int_{(1+x^3)^{1/3}}^{\infty}\frac{\hat{W}_1(\eta)}{\eta^3-1}d\eta. \tag{25}$$

According to the above notation, the motion equation of cavity (17) can be rewritten as

$$\rho B^2x^2\left(1-\frac{x}{(1+x^3)^{1/3}}\right)\ddot{x}+$$
$$\rho B^2x\left(\frac{x^4}{2(1+x^3)^{4/3}}-\frac{2x}{(1+x^3)^{1/3}}+\frac{3}{2}\right)\dot{x}^2+xh(x)-p_0x(1+x^3)^{-2/3}=0, \tag{26}$$

Obviously, $x(t)\equiv 0$ is a trivial solution of Eq.(26). Furthermore, it is not difficult to show that

$$\frac{d}{dx}\left(x^3-\frac{x^4}{(1+x^3)^{1/3}}\right)=2x^2\left(\frac{x^4}{2(1+x^3)^{4/3}}-\frac{2x}{(1+x^3)^{1/3}}+\frac{3}{2}\right),$$

so multiplying both sides of Eq.(26) by $x\dot{x}$ we obtain

$$\frac{d}{dx}\left(\rho B^2 x^3\left(1-\frac{x}{(1+x^3)^{1/3}}\right)\dot{x}^2+\right.$$
$$\left. 2\int_0^x \xi^2\left(h(\xi)-p_0(1+\xi^3)^{-2/3}\right)d\xi\right)=0. \tag{27}$$

On using the initial conditions (21), this yields

$$\rho B^2 x^3\left(1-\frac{x}{(1+x^3)^{1/3}}\right)\dot{x}^2+2\int_0^x \xi^2 h(\xi)d\xi-2p_0\left((1+x^3)^{1/3}-1\right)=0. \tag{28}$$

However, from Eqs.(26) and (28), we have the following expressions by setting $t\to 0+$, i.e.,

$$\dot{x}(0+)=\pm\left(\frac{2(p_0-h(0))}{3\rho B^2}\right)^{1/2},\ddot{x}(0+)=\frac{p_0-h(0)}{3\rho B^2}, \tag{29}$$

that is so say, the first derivative of $x(t)$ has a discontinuity at the initial moment $t=0$ (see (21)), and thus we can conclude that Eq.(26) is a **singular** second order nonlinear differential equation with the initial condition $x(0)=0$.

Next we will examine the dynamic behavior of Eq.(26). However, it is helpful to consider first the static bifurcation of a solid sphere composed of an incompressible hyper-elastic material under a prescribed uniform radial tensile load, which will be referred to later when dynamic behavior is analyzed.

3.1. Static bifurcation

Interestingly, if the right hand of Eq.(2), i.e., the acceleration term is set to zero and Eq.(4) is not considered, then the mathematical model describes the correspondingly static deformation problems. Using the similarly solving process in Section 2 and the notations in the front part of this section, we obtain an explicit function of the dimensionless cavity radius x and the suddenly applied radial tensile load p_0, i.e.,

$$xh(x)-p_0x(1+x^3)^{-2/3}=0, \tag{30}$$

where $h(x)$ is given by Eq.(25). Eq.(30) can also be obtained by setting $\ddot{x}\equiv 0$ and $\dot{x}\equiv 0$ in Eq.(26), in which x is independent of time.

Obviously, one can see that, for any $p_0>0$, $x=0$ is a trivial solution of Eq.(30). However, it remains to be determined whether or not there exists a nonzero value of x satisfying the following equation, which is obtained by Eq.(30), i.e.,

$$p_0=(1+x^3)^{2/3}h(x)=(1+x^3)^{2/3}\int_{(1+x^3)^{1/3}}^{\infty}\frac{\hat{W}_1(\eta)}{\eta^3-1}d\eta. \tag{31}$$

The critical load p_{cr}, which a cavity may be initiated at the center of the sphere, is found by formally setting $x \to 0+$ in Eq.(31), and so

$$p_{cr} = \int_1^\infty \frac{\hat{W}_1(\eta)}{\eta^3 - 1} d\eta. \tag{32}$$

The integral of Eq.(32) is improper, i.e., whether or not p_{cr} is finite, and thus cavitation may or may not take place, in other words, depends strictly on the concrete form of the strain energy function.

To insure p_{cr} is finite, some conditions must be imposed on the strain energy function W, as follows,

(i) $d^2\hat{W}(\eta)/d\eta^2$ must be finite as $\eta \to 1$;

(ii) The highest power of $W(\eta^{-2}, \eta, \eta)$ with respect to η cannot exceed 3 for large values of η.

Remark. Eq.(32) was first given by Ball [1]. For many material models such as the neo-Hookean material, the Gent-Thomas material, the Valanis-Landel material, the modified Varga material, and so on, p_{cr} is finite. However, for the Mooney-Rivlin material, p_{cr} is not finite. These results may be found in [2], [3],[5], [8].

Throughout this paper, we assume that the strain energy function $W(\eta^{-2}, \eta, \eta)$ satisfies the conditions **(i)** and **(ii)**.

From Eq.(30) one can see that there exists a unique bifurcation point $(0, p_{cr})$ on the trivial solution $x = 0$ if p_{cr} is finite. We now study the local properties of Eq.(30) at the bifurcation point $(0, p_{cr})$ by analyzing the curve $p_0 = p_0(x)$, given by Eq.(31), for small values of x.

The Taylor expansion of Eq.(31) at $x = 0$ is as follows,

$$p_0 = p_{cr} + kx^3 + o\left(x^3\right), \tag{33}$$

where

$$k = \frac{2}{3}\left(p_{cr} - \frac{1}{6}\frac{d^2\hat{W}(1)}{d\eta^2}\right). \tag{34}$$

Interestingly, we see from Eq.(33) that the nontrivial solution bifurcates locally to the left (supercritical) if $k > 0$ and bifurcates locally to the right (subcritical) if $k < 0$. See also [3], [8]. However, it is not difficult to show that $p_0 \to \infty$ as $x \to \infty$, this means that there also exists a secondary turning bifurcation point on the nontrivial solution.

Correspondingly, a cavity would form in the interior of the sphere as the surface tensile load p_0 exceeds the critical load p_{cr}.

Remark. For different material models, the local properties of Eq.(30) at the bifurcation point $(0, p_{cr})$ are quite different. Detail results may be found in [3], [6], [7], [8].

To better understand the above results, in this paper we consider the modified Varga materials an example. The corresponding strain energy function is given by

$$W(\lambda_1, \lambda_2, \lambda_3) = \mu[\lambda_1 + \lambda_2 + \lambda_3 + \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} - 6 +$$

$$\alpha(\lambda_1 - 1)^2 + \beta(\lambda_1 - 1)^3],$$

where μ, α, β are material parameters. The detail conclusions on static bifurcation of the modified Varga materials may be found in [8]. Example curves, p_0/μ vs x, are shown in Figs.1 and 2 for different material parameters.

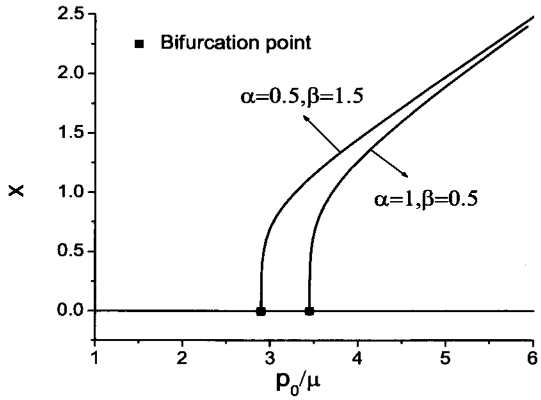

Fig.1. Example curves, p_0/μ vs x, bifurcate locally to the right for the modified Varga materials.

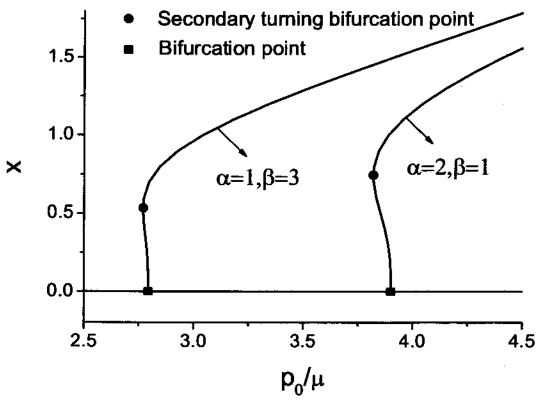

Fig.2. Example curves, p_0/μ vs x, bifurcate locally to the left for the modified Varga materials.

3.2. Dynamic behaviors

For the convenience of studying the dynamic behaviors of Eq.(28), i.e., Eq.(26), let

$$V(x,p_0)=2\int_0^x \xi^2 h(\xi)d\xi-2p_0\left((1+x^3)^{1/3}-1\right). \tag{35}$$

We see from Eq.(28) that if there exists $x\in(0,+\infty)$ such that the inequality $V(x,p_0)<0$ holds, the existence conditions and the range of nonzero solutions of Eq.(28) are then determined, since the first term without $\dot{x}^2$ in Eq.(28) is positive as $x>0$. Furthermore, we can obtain the nonzero solutions of Eq.(26) with the initial conditions (21).

From Eqs.(30), (31) and (35), we see that the value of x which satisfies the inequality $V(x,p_0)<0$, depends not only on the prescribed surface tensile load $p_0>0$, but also on the strain energy function of the hyper-elastic material.

From the following equation

$$\begin{aligned} V_x(x,p_0)&=x^2\left(h(x)-p_0(1+x^3)^{-2/3}\right)\\ &=x^2\left(\int_{(1+x^3)^{1/3}}^{\infty}\frac{\hat{W}_1(\eta)}{\eta^3-1}d\eta-p_0(1+x^3)^{-2/3}\right)=0, \end{aligned} \tag{36}$$

we know that, in fact, the nonzero critical points of $V_x(x,p_0)$ are nonzero solutions of Eq.(31).

For strain energy functions corresponding to the hyper-elastic materials, the strongly elliptic condition, i.e., $d^2W(\eta)/d\eta^2>0$, must be satisfied (cf. Ball [1] pp.563, (3.7)). When $p_0=0$, we have $\lim\limits_{x\to0+}V(x,0)=\lim\limits_{x\to0+}V_x(x,0)=0$ and $\lim\limits_{x\to\infty}V(x,0)=\lim\limits_{x\to\infty}V_x(x,0)=\infty$, moreover, $V(x,0)$ is a strictly increasing function of x.

Thus, for the given $p_0>0$, from $\lim\limits_{x\to0+}V(x,p_0)=0$, we see that $V(x,p_0)<0$ is equivalent to $\min\limits_{x\in(0,+\infty)}V(x,p_0)<0$. It is easy to show that $\lim\limits_{x\to\infty}V(x,p_0)=\infty$, and thus $V(x,p_0)$ has no maximum.

According to the continuous dependence of $V(x,p_0)$ with respect to x, for the given $p_0>0$, we first determine the nonzero critical point of $V(x,p_0)$ by solving the equation $V_x(x,p_0)=0$, and then discuss whether the critical point is an extreme point or an inflexion point.

To discuss the variation of $V(x,p_0)$ with respect to p_0, we examine the relationship between x and p_0.

From the above analyses on Eq.(30), we know that $V(x,p_0)$ has no extreme value as $x\in(0,+\infty)$ for sufficiently small values of p_0, that is to say, for any $x\in(0,+\infty)$, we have $V(x,p_0)>0$, and so Eq.(28) (i.e., Eq.(26)) has only a zero solution. From the expression of $V(x,p_0)$, it is not difficult to see that the value of $V(x,p_0)$ decreases gradually along with the increasing values of p_0. But as the value of p_0 attains a certain value, the property of $V(x,p_0)$ may have a change. Assume that p_0 increases and attains a certain value such that $V(x,p_0)$ has a nonzero critical point. We will consider the following two possible cases:

(a) The nontrivial solution of Eq.(30) bifurcates locally to the right, see the example curves shown in Fig.1.

In this case, only when $p_0 > p_{cr}$, $V(x, p_0)$ has a nonzero critical point and it is an extreme point, written as $(\bar{x}, p_0)$. However, $(\bar{x}, p_0)$ must be the minimum point of $V(x, p_0)$. Since $\lim_{x\to 0+} V(x, p_0) = 0$, we have $V(\bar{x}, p_0) < 0$, and Eq.(28) has a nonzero solution, furthermore, this solution is also a solution of Eq.(26) because it satisfies the initial conditions (21). The critical tensile load, which corresponds to the appearance of nonzero minimum of $V(x, p_0)$ (in other words, the limit state that a cavity forms in the sphere), is given by Eq.(32).

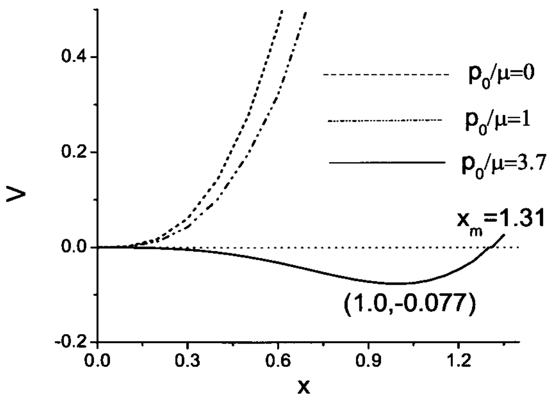

Fig.3. Relation curves, V vs x, for different values of p_0 and for $\alpha = 1, \beta = 0.5$.

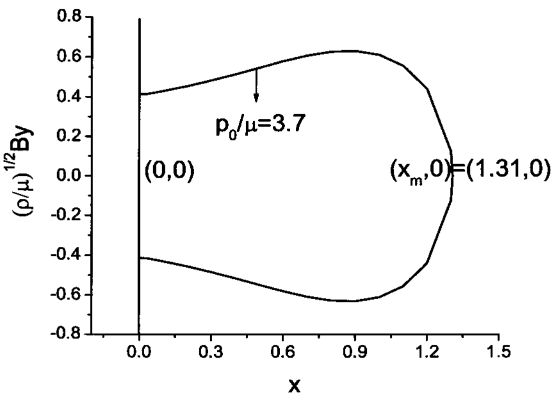

Fig.4. Phase diagram of Eq.(26) satisfying the initial conditions (21) for the modifired Varga material ($\alpha = 1, \beta = 0.5, y = \dot{x}$).

As p_0 increases gradually, the minimum of $V(x, p_0)$ becomes small and small, that is to say, the range of the nonzero solution of Eq.(28) extends more and more.

Further, as the prescribed p_0 exceeds p_{cr}, assume that $V(x, p_0)$ takes the minimum at $\bar{x}$ (nonzero), from $V(\bar{x}, p_0) < 0$ and $\lim_{x\to\infty} V(x, p_0) = +\infty$, we can conclude that there exists a nonzero value of $x \in (\bar{x}, +\infty)$, written as x_m, so that $V(x_m, p_0) = 0$. Note that x_m is the maximum of the nonzero solution of Eq.(26) for the corresponding value of p_0.

For the modified Varga materials, example curves, $V(x, p_0)$ vs x, are shown in Fig.3 as p_0 takes different values. Phase diagram of Eq.(26) satisfying the initial conditions (21) are shown in Fig.4.

(b) The nontrivial solution of Eq.(30) bifurcates locally to the left, see the example curves shown in Fig.2.

In this case, there exists a critical value of p_0 corresponding to the secondary turning bifurcation point, written as p_n, such that $V(x, p_0)$ has a nonzero critical point and it is an inflexion point as $p_0 = p_n$. However, as the value of p_0 exceeds p_n, the inflexion point splits into a local maximum $\bar{x}_1$ and a local minimum $\bar{x}_2$ of $V(x, p_0)$, in which $\bar{x}_1 > \bar{x}_2$ (note that the two local extreme points all vary with p_0).

We can conclude that the values of $V(x, p_0)$ at the two local extreme points must be greater than zero, because its value at the inflexion point is positive. If p_0 increases continuously, the local maximum and the local minimum all decrease gradually. As p_0 attains a certain value (written as p_s), the local minimum of $V(x, p_0)$ is zero.

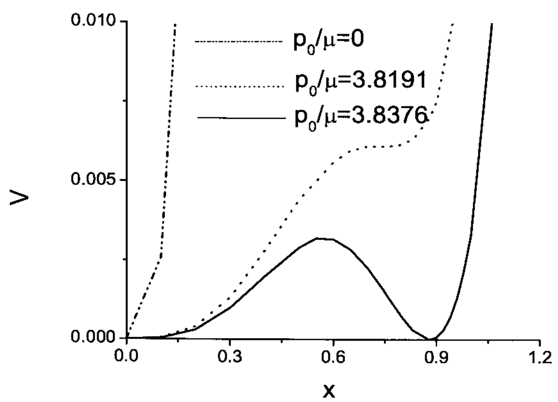

Fig.5. Relation curves, V vs x, for different values of p_0 and for $\alpha = 2, \beta = 1$.

If p_0 increases more, the local minimum of $V(x, p_0)$ is negative and turns into the global minimum at $\bar{x}_2$, moreover, the local maximum is still positive, and it closes to the origin gradually. Although Eq.(28) has nonzero solution at the moment, we say that this solution is not a solution of Eq.(26), because it does not satisfy the initial conditions (21). As p_0 increases more and more, the local minimum (i.e., the global minimum at the moment) of $V(x,p_0)$ decreases continuously, moreover, the local maximum closes to the origin unceasingly and reaches to it ultimately, the corresponding tensile load is p_{cr} (the same as Eq.(32)), such that Eq.(28) has a nonzero solution. Since this solution satisfies the initial conditions (21) as $p_0 > p_{cr}$, it is also a nonzero solution of Eq.(26) in this case.

For the modified Varga materials, example curves, $V(x,p_0)$ vs x, are shown in Figs.5 and 6 as p_0 takes different values. Phase diagram of Eq.(26) satisfying the initial conditions (21) are shown in Fig.7.

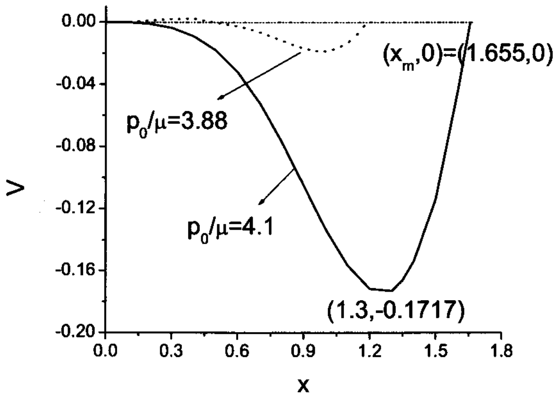

Fig.6. Relation curves, V vs x, for different values of p_0 and for $\alpha = 2, \beta = 1$.

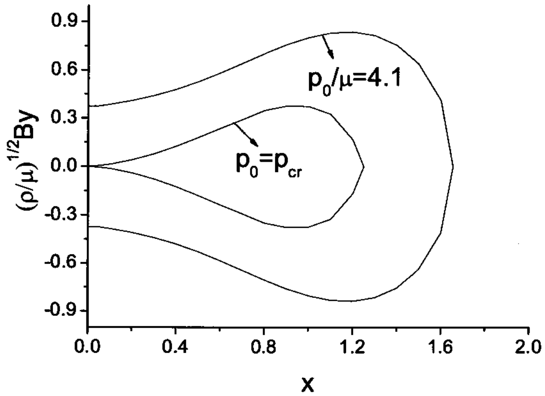

Fig.7. Phase diagrams of Eq.(26) satisfying the initial conditions (21) for the modifired Varga material ($\alpha = 2, \beta = 1, y = \dot{x}$).

In sum, from the properties of Eq.(28) and the above analyses, we know that Eq.(26) has nonzero solutions $x(t)$ (namely, dimensionless cavity radius) only when $p_0 > p_{cr}$. As shown in Figs.4 and 7, we can see that the solution $x(t)$ begins increasing with respect to time from the initial value $x(0) = 0$ at the initial time $t = 0$, and that the increasing velocity of cavity radius $\dot{x}(t)$ reaches directly to $\left(\frac{2(p_0-h(0))}{3\rho B^2}\right)^{1/2}$ given by Eq.(29) from $\dot{x}(0) = 0$; as the solution $x(t)$ increases and reaches to the maximum x_m at time $t = T/2$, where T can be obtained by Eq.(28) implicitly, the increasing velocity of cavity radius decreases to zero; thereafter, $x(t)$ decreases gradually, but the decreasing velocity of cavity radius begins increasing, and then decreasing, as $x(t)$ decreases to zero, $\dot{x}(t)$ is $x(T^-) = -\left(\frac{2(p_0-h(0))}{3\rho B^2}\right)^{1/2}$ also given by (29). As time increases continuously, the nonzero solution of Eq.(26) will repeat this cycle. In view of the first derivative of $x(t)$ has a discontinuity at the initial moment $t = 0$, thus we can say that the nonzero solution of Eq.(26) with the initial conditions (21) must be a singular periodic solution of time, in other words, when the prescribed load p_0 exceeds p_{cr} given by Eq.(32), a cavity forms in the interior of the sphere, and that the motion of the formed cavity with respect to time is a class of singular nonlinear periodic oscillations. The oscillation center is $\bar{x}$ that satisfies Eq.(31) for the given $p_0 > p_{cr}$, T is called a period of nonlinear oscillation.

Interestingly, the phase diagrams of Eq.(26) are quite different as the nontrivial solution of Eq.(30) bifurcates locally to the right or to the left. If the nontrivial solution of Eq.(30) bifurcates locally to the right, Eq.(26) has only zero solution as $p_0 = p_{cr}$, in other word, no cavity forms in the interior of the sphere, and the sphere is in the critical state of cavity formation; while if the nontrivial solution of Eq.(30) bifurcates locally to the left, Eq.(26) has a nonsingular periodic solution as $p_0 = p_{cr}$, that is to say, a cavity has formed in the sphere and then presented a classical nonlinear periodic oscillation.

Remark. It is worth pointing out here that the conclusions obtained in this paper are similar to those for other incompressible hyper-elastic materials such as the neo-Hookean material, the Gent-Thomas material, the Valanis-Landel material and so on.

4. Conclusions

In this paper, a mathematical model that describes the radially symmetric motion problem of an incompressible hyper-elastic solid sphere is reduced to a second order nonlinear ordinary differential equation, which describes cavity formation and motion in the interior of the sphere. Firstly, the conditions of static bifurcation are presented. Secondly, it is proved that the differential equation has singular periodic solutions only when $p_0 > p_{cr}$, namely, a cavity forms in the interior of the sphere and the motion of the cavity with time presents a class of singular periodic oscillations only when $p_0 > p_{cr}$, as shown in Figs.4 and 7. However, as the surface tensile load is exactly equal to a certain critical value, it is also proved that the differential equation has only zero solution if the nontrivial solution of the static bifurcation equation bifurcate locally to the right and has periodic solutions if the nontrivial solution of the static bifurcation equation bifurcate locally to the left. To better understand the results obtained in this paper, the modified Varga material is considered, and numerical simulations are given.

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