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CALCULATING ZEROS OF THE GENERALIZED GENOCCHI POLYNOMIALS

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ABSTRACT. Kim [4] defined the generalized Genocchi numbers $G_{n,\chi}$. In this paper, we introduce the generalized Genocchi polynomials $G_{n,\chi}(x)$. One purpose of this paper is to investigate the zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$. We also display the shape of generalized Genocchi polynomials $G_{n,\chi}(x)$.

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1. Introduction

Many mathematicians have studied Genocchi polynomials and Genocchi numbers (see [1,2,3,4]). Genocchi polynomials and Genocchi numbers posses many interesting properties and arising in many areas of mathematics and physics. In this paper, we introduce the generalized Genocchi polynomials $G_{n,\chi}(x)$. In order to study the generalized Genocchi polynomials $G_{n,\chi}(x)$, we must understand the structure of the generalized Genocchi polynomials $G_{n,\chi}(x)$. Therefore, using computer, a realistic study for the generalized Genocchi polynomials $G_{n,\chi}(x)$ is very interesting.

It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$ in complex plane. The outline of this paper is as follows. We introduce the generalized Genocchi polynomials $G_{n,\chi}(x)$.

In Section 2, we describe the beautiful zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$ using a numerical investigation. Finally, we investigate the roots of the generalized Genocchi polynomials $G_{n,\chi}(x)$.

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First, we introduce the Genocchi numbers and Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function:

$$F(t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi), \text{ cf. [4]} \tag{1}$$

where we use the technique method notation by replacing G^n by $G_n (n \geq 0)$ symbolically. Here is the list of the first Genocchi's numbers.

$$\begin{aligned} &G_1 = 1, \quad G_2 = -1, \quad G_3 = 0, \quad G_4 = -1, \quad G_5 = 0, \quad G_6 = -3, \\ &G_7 = 0, \quad G_8 = 17, \quad G_9 = 0, \quad G_{10} = -155, \quad G_{11} = 0, \quad G_{12} = 2073, \\ &G_{14} = -38227 \quad G_{16} = 929569, \quad G_{18} = -28820619 \quad G_{20} = 1109652905, \cdots \end{aligned}$$

In general, it satisfies $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given $G_n = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}$, where B_n are Bernoulli numbers and E_n are Euler numbers which are defined by $\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$. For $x \in \mathbb{R}$ (= the field of real numbers), we consider the Genocchi polynomials $G_n(x)$ as follows:

$$F(x,t) = \frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{2}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$.

Let m be odd. Using the multiplication theorem, we obtain

$$G_n(x) = m^{n-1} \sum_{i=0}^{m-1} (-1)^i G_n \left(\frac{i+x}{m} \right).$$

Here is the list of the first Genocchi's polynomials.

$$\begin{aligned} &G_1(x) = 1, \quad G_2(x) = 2x - 1, \quad G_3(x) = 3x^2 - 3x, \quad G_4(x) = 4x^3 + 6x^2 - 1 \\ &G_5(x) = 5x^4 - 10x^3 + 5x, \quad G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3, \cdots \end{aligned}$$

Next, we introduce the generalized Genocchi polynomials $G_{n,\chi}(x)$. Let χ be Dirichlet character with conductor $f \in \mathbb{N}$ (f= odd). In [4], Kim, Jang, and Pak defined the generalized Genocchi numbers $G_{n,\chi}$ with character χ as follows ([4]):

$$\sum_{n=0}^{\infty} G_{n,\chi} \frac{t^n}{n!} = \frac{\sum_{a=0}^{f-1} (-1)^a \chi(a) e^{at} 2t}{e^{ft}+1}. \tag{3}$$

If $\chi = \chi^0$ in (3), then we have the $F(t)$. We introduce the generalized Genocchi polynomials $G_{n,\chi}(x)$ with character χ as follows:

$$\sum_{n=0}^{\infty} G_{n,\chi}(x)\frac{t^n}{n!} = e^{xt}\frac{\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{at}2t}{e^{ft}+1}. \tag{4}$$

If $\chi = \chi^0$ in (4), then we have the $F(x,t)$. Then we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\chi}(x)\frac{t^n}{n!} &= e^{xt}\frac{\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{at}2t}{e^{ft}+1} \\
&= \frac{1}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\left(\frac{2ft}{e^{ft}+1}e^{\frac{(a+x)ft}{f}}\right) \\
&= \frac{1}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{n=0}^{\infty}\left(G_n\left(\frac{a+x}{f}\right)\frac{f^n t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{a+x}{f}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

For $n \geq 0$, we have

$$G_{n,\chi}(x) = f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{a+x}{f}\right). \tag{5}$$

When $x = 0$, we write $G_{n,\chi} = G_{n,\chi}(0)$, which are called the generalized Genocchi numbers. By definition of the generalized Genocchi polynomials $G_{n,\chi}(x)$ with character χ, we obtain

$$G_{n,\chi} = f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{a}{f}\right).$$

By above definition, we obtain

$$\begin{aligned}
\sum_{l=0}^{\infty} G_{l,\chi}(x)\frac{t^l}{l!} &= \frac{\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{at}2t}{e^{ft}+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,\chi}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m\frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} G_{n,\chi}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}G_{n,\chi}x^{l-n}\right)\frac{t^l}{l!}.
\end{aligned}$$

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

Theorem 1. *For any positive integer n, we have*

$$G_{n,\chi}(x) = \sum_{k=0}^{n} \binom{n}{k} G_{k,\chi} x^{n-k}.$$

Let χ be Dirichlet character with conductor $f = 3$. We obtain the first value of the generalized Genocchi numbers $G_{n\chi}$:

$$G_{1,\chi} = 1, \quad G_{2,\chi} = 3, \quad G_{3,\chi} = -6, \; G_{4,\chi} = -39, \quad G_{5,\chi} = 110, \; G_{6,\chi} = 1089,$$
$$G_{7,\chi} = -4214, \quad G_{8,\chi} = -55743, \quad G_{9,\chi} = 276678, \quad G_{10,\chi} = 4576065,$$
$$G_{11,\chi} = -27753022, \; G_{12,\chi} = -550835487, \quad G_{13,\chi} = 3948004606,$$
$$G_{14,\chi} = 91419220641, \; G_{15,\chi} = -756031185030, \; G_{16,\chi} = -20007447302271,$$
$$G_{17,\chi} = 187521633674294, \quad G_{18,\chi} = 5582849109900417,$$
$$G_{19,\chi} = -58481734930175438, \quad G_{20,\chi} = -1934560218174688095, \cdots$$

With $f = 5$ we have

$$G_{1,\chi} = 2, \; G_{2,\chi} = 10, \; G_{3,\chi} = -18, \; G_{4,\chi} = -310, \; G_{5,\chi} = 810, \quad G_{6,\chi} = 23430,$$
$$G_{7,\chi} = -84882, \quad G_{8,\chi} = -3320270, \quad G_{9,\chi} = 15454098, \quad G_{10,\chi} = 756835550,$$
$$G_{11,\chi} = -4305202506, \quad G_{12,\chi} = -253051752630, \; G_{13,\chi} = 1701175877466,$$
$$G_{14,\chi} = 116659545802870, \quad G_{15,\chi} = -904914918671490,$$
$$G_{16,\chi} = -70920486447871390, \quad G_{17,\chi} = 623471780847860514,$$
$$G_{18,\chi} = 54970968246387909390, \quad G_{19,\chi} = -540111169018279765434,$$
$$G_{20,\chi} = -52912373781201449500550, \quad \cdots$$

Since

$$\sum_{n=0}^{\infty} G_n\left(\frac{f-a-x}{f}\right)\frac{(-t)^n}{n!} = F\left(\frac{f-a-x}{f}, -t\right) = \frac{-2t}{e^{-t}+1} e^{(\frac{f-a-x}{f})(-t)}$$
$$= -F\left(\frac{a+x}{f}, t\right) = -\sum_{n=0}^{\infty} G_n\left(\frac{a+x}{f}\right)\frac{t^n}{n!}$$

Hence we have the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$G_n\left(\frac{a+x}{f}\right) = (-1)^{n+1} G_n\left(\frac{f-a-x}{f}\right).$$

By using (5), we have,

$$\begin{aligned} G_{n,\chi}(f-2a-x) &= f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{a+f-2a-x}{f}\right) \\ &= f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{f-a-x}{f}\right) \end{aligned} \tag{6}$$

By (6) and Theorem 1, we obtain

$$\begin{aligned} G_{n,\chi}(x) &= f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{a+x}{f}\right) \\ &= (-1)^{n+1}f^{n-1}\sum_{a=0}^{f-1}(-1)^a\chi(a)G_n\left(\frac{f-a-x}{f}\right) \end{aligned}$$

Hence we have the following theorem.

Theorem 3. For $n \geq 0$, we have

$$G_{n,\chi}(x) = (-1)^{n+1}G_{n,\chi}(f-2a-x).$$

We also obtain the following corollary.

Corollary 4. *If* $G_{n,\chi}(x) = 0$, *then* $G_{n,\chi}(f-2a-x) = 0$.

2. Zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$

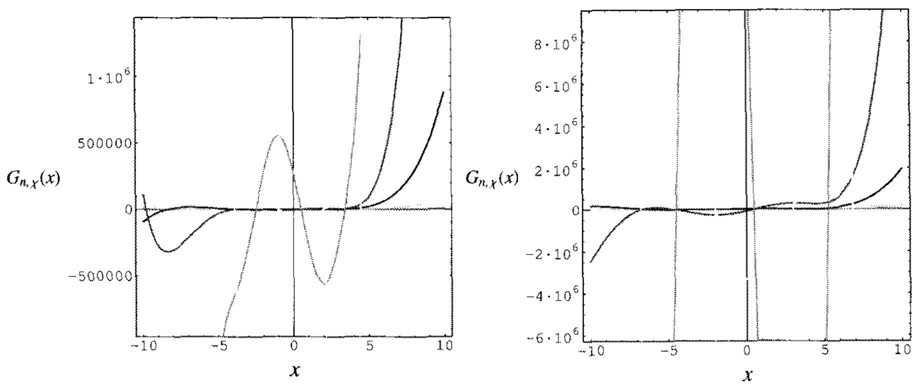

FIGURE 1. Conductor $f = 3, 5$

In this section, we display the shapes of the generalized Genocchi polynomials $G_{n,\chi}(x)$ and we investigate the zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$. For $n = 1, \cdots, 10$, we can draw a plot of the generalized Genocchi polynomials $G_{n,\chi}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $G_{n,\chi}(x)$, $-10 \leq x \leq 10$. (Figures 1).

We investigate the beautiful zeros of the $G_{n,\chi}(x)$ by using a computer. We plot the zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$ for $n = 40, f = 3, 5, 7, 9$ and $x \in \mathbb{C}$. (Figure 2).

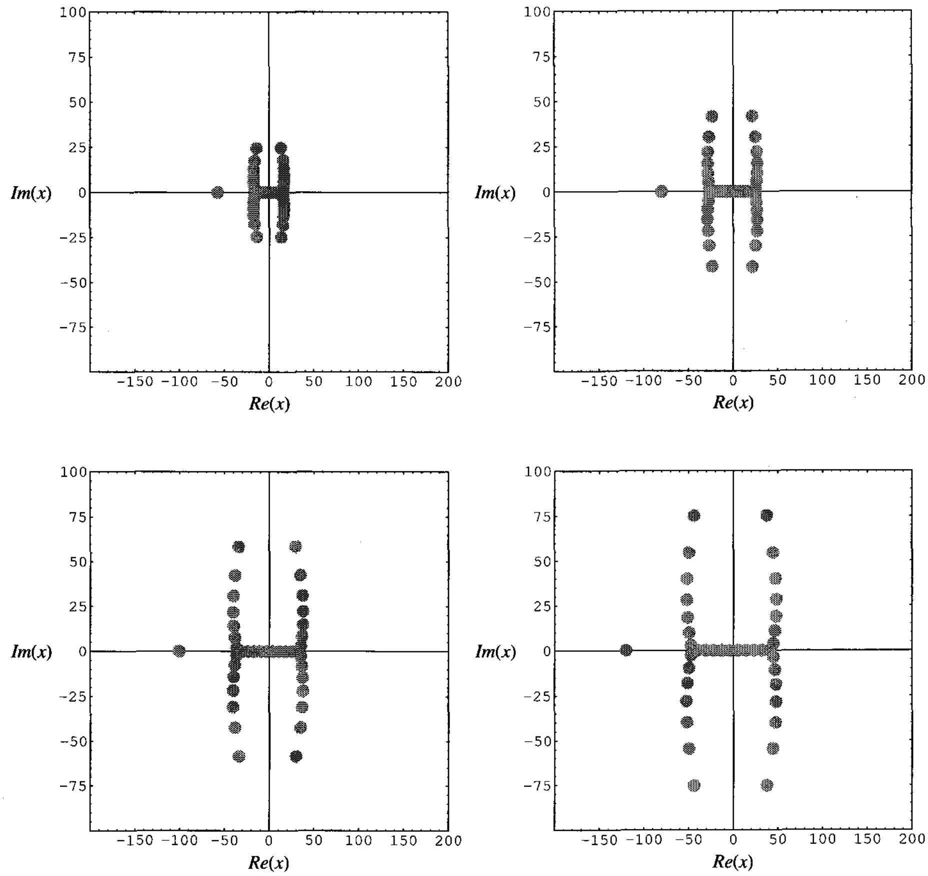

FIGURE 2. Conductor $f = 3, 5, 7, 9$

We plot the zeros of the generalized Genocchi polynomials $G_{n,\chi}(x)$ with conductor $f = 3$ for $n = 10, 20, 30, 40$ and $x \in \mathbb{C}$. (Figure 3).

Table 1. Numbers of real and complex zeros of $G_{n,\chi}(x)$

degree n	$f=3$		$f=5$	
	real zeros	complex zeros	real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	3	0	3	0
5	4	0	4	0
6	5	0	5	0
7	4	2	4	2
8	3	4	3	4
9	4	4	4	4
10	5	4	5	4
11	6	4	6	4

Table 2. Approximate solutions of $G_n(x,\chi)=0, f=3, x\in\mathbb{R}$

degree n	x
2	-1.50000000
3	$-3.56155281, 0.56155281$
4	$-5.2861239, -1.02078010, 1.80690401$
5	$-6.8522587, -2.3501657, 0.50679203, 2.69563238$
6	$-8.3446779, -3.3660605, -1.00227267, 2.0175678, 3.1954433$
7	$-9.8058479, -4.0009315, -2.5186445, 0.50075600$
8	$-11.2551632, -1.00025214, 1.9992757$
9	$-12.7003785, -2.4991238, 0.50008403, 3.4259981$
10	$-14.1442442, -3.942989, -1.00002801, 1.9999792, 4.464600$
11	$-15.5876528, -5.069919, -2.5000125, 0.50000934, 3.5074228, 5.085502$

Stacks of zeros of $G_{n,\chi}(x)$ for $1\le n\le 40$ from a 3-D structure are presented. (Figure 4). Our numerical results for approximate solutions of real zeros of $G_{n,\chi}(x)$ are displayed. (Tables 1, 2, 3).

We observe a remarkably regular structure of the complex roots of the generalized Genocchi polynomials. We hope to verify a remarkably regular structure of the complex roots of the generalized Genocchi polynomials $G_{n,\chi}(x)$. (Table 1). Next, we calculated an approximate solution satisfying $G_{n,\chi}(x), x\in\mathbb{R}$. The results are given in Table 2 and Table 3.

3. Directions for further research

Finally, we shall consider the more general problems. Prove that $G_{n,\chi}(x)=0$ has $n-1$ distinct solutions. Find the numbers of complex zeros $C_{G_{n,\chi}(x)}$ of $G_{n,\chi}(x), Im(x)\neq 0$.

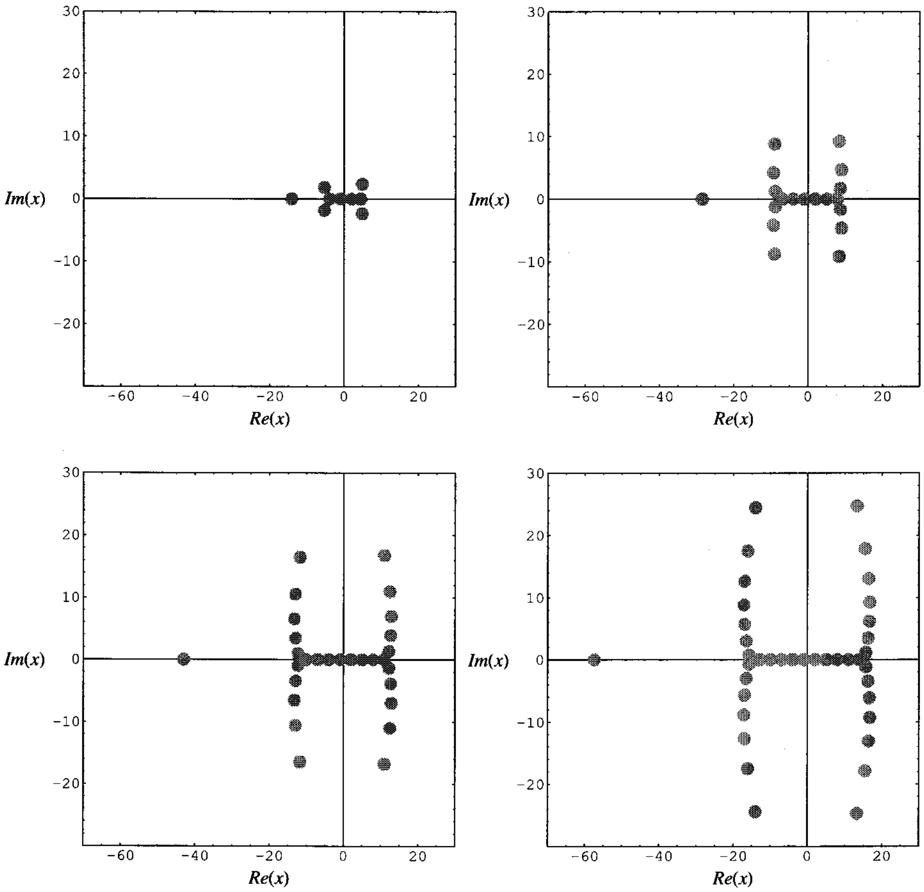

FIGURE 3. Zeros of $G_{n,\chi}(x)$ for $n = 10, 20, 30, 40, f = 3$

Since $n - 1$ is the degree of the polynomial $G_{n,\chi}(x)$, the number of real zeros $R_{G_{n,\chi}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{G_{n,\chi}(x)} = n - 1 - C_{G_{n,\chi}(x)}$, where $C_{G_{n,\chi}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{G_{n,\chi}(x)}$ and $C_{G_{n,\chi}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane.

We prove that $G_n(x), x \in \mathbb{C}$, has $Re(x) = \frac{1}{2}$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The question is: what happens with the reflection symmetry, when one considers the generalized Genocchi polynomials $G_{n,\chi}(x)$? (Figures 2, 3). The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the generalized Genocchi polynomials

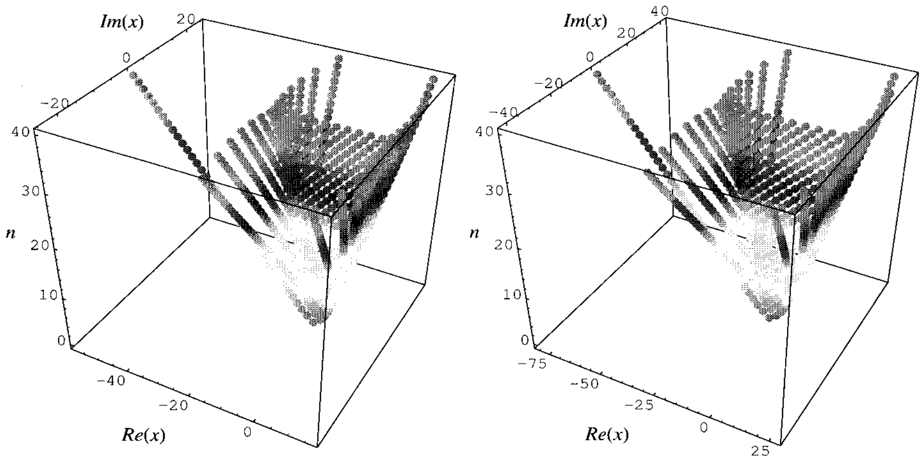

FIGURE 4. Stacks of zeros of $G_{n,\chi}(x), 1 \le n \le 40, f = 3, 5$

$G_{n,\chi}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [5,6,7].

Table 3. Approximate solutions of $G_n(x, \chi) = 0, f = 5, x \in \mathbb{R}$

degree n	x
2	-2.50000000
3	$-5.5413813, 0.54138127$
4	$-8.0197391, -1.9535803, 2.4733194$
5	$-10.1983296, -4.1327431, 0.5018939, 3.8291788$
6	$-12.227536, -5.798466, -2.0003922, 3.1345942, 4.391800$
7	$-14.199741, -6.721783, -4.638162, 0.5000773$
8	$-16.162163, -1.9999772, 2.9963465$
9	$-18.130871, -4.489063, 0.5000031, 5.268329$
10	$-20.107497, -6.794130, -1.9999994, 3.0000825, 6.821727$
11	$-22.089840, -8.520678, -4.500647, 0.5000001, 5.534728, 7.689807$

REFERENCES

1. D. Dumont, *Interpretations combinatoires des nombres de Genocchi* , Duke Math. J. **41** (1974), 305-318.
2. R. Ehrenborg, E. Steingrimsson, *Yet another triangle for the Genocchi numbers*, Europ. J. Combin. **21** (2000), 593-600.

3. G. Kreweras, *An Additive Generation for the Genocchi Numbers and Two of its Enumerative Meanings*, Bull. Inst. Combin. Appl. **20** (1997), 99-103.
4. T. Kim, L.-C. Jang, H. K. Pak, *A note on q-Euler and Genocchi numbers* , Proc. Japan Acad. **77 A** (2001), 139-141.
5. C.S.Ryoo, *A numerical computation on the structure of the roots of q-extension of Genocchi polynomials*, Applied Mathematics Letters **21** (2008), 348-354.
6. C. S. Ryoo, T. Kim, R. P. Agarwal, *A numerical investigation of the roots of q-polynomials*, Inter. J. Comput. Math. **83** (2006), 223-234.
7. C. S. Ryoo, H. song, R. P. Argawal , *On the real roots of the Changhee-Barnes' q-Bernoulli polynomials*, Advan. Stud. Contemp. Math. **9** (2004), 153-163.

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