단면적이 변하는 곡선보의 진동해석

Free Vibration Analysis of Curved Beams with Varying Cross-Section

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요 지

미분구적법을 이용하여 전단변형을 고려하지 않은 단면적이 변하는 곡선 보의 면내 자유진동을 해석하였다. 다양한 경계조건 및 굽힘 각에 따른 진동수를 계산하였고, 그 결과를 다른 수치해석들과 비교하였다. 미분구적법은 비교적 적은 요소를 사용하고도 정확한 해석결과를 보여주었고, 수정된 결과를 추가적으로 제시하였다.

핵심용어 : 곡선보, 미분구적법, 진동수, 수치해석

Abstract

The differential quadrature method(DQM) is applied to the free in-plane vibration analysis of circular curved beams with varying cross-section neglecting transverse shearing deformation. Natural frequencies are calculated for the beams with various opening angles and end conditions. Results obtained by the DQM are compared with available results by other methods in the literature. It is found that the DQM gives good accuracy even with a small number of grid points. In addition, the corrected results are given for the beams not previously presented for this problem.

Keywords : *curved beam, DQM, fundamental frequency, numerical method*

1. 서 론

Owing to their importance in many fields of technology and engineering, the vibration behavior of elastic curved beams has been the subject of a large number of investigations. Despite of a number of advantages, a curved member behaves in an extremely complex manner as compared to a straight member, and practicing engineers have often been discouraged by the complexity because of the initial curvature. However, the mathematical difficulties associated with curved members have been largely overcome with the application of digital computers and the development of numerical methods.

The early investigators into the in-plane vibration of rings were Hoppe(1871) and Love(1944). Love (1944) improved on Hoppe's theory by allowing for stretching of the ring. Lamb(1888) investigated the statics of a curved bar with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog(1928) used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with clamped ends, and his work was extended by Volterra and Morell(1961) for the vibration of arches having center lines in the form of cycloids, catenaries, or parabolas. Archer(1960) carried out for a mathematical study of the in-plane inextensional vibrations of an incomplete circular ring of small cross section with the basic equations of motion as given in Love(1944)

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and gave a prescribed time-dependent displacement at the other end for the case of clamped ends. Nelson(1962) applied the Rayleigh-Ritz method in conjunction with Lagrangian multipliers to the case of a circular ring segment having simply supported ends. Auciello and De Rosa(1994) reviewed the free vibrations of circular arches and briefly illustrated a number of other approaches. Recently, Kim and Park(2006) have proposed a new efficient 2-noded hybrid-mixed element for curved beam vibrations having the uniform and non-uniform cross sections, and Kang and Kim(2007) have analyzed the out-of-plane vibration of curved beams considering shear deformation using DQM.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti(1971). This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. This method is used in the present work to analyze the free in-plane inextensional vibrations for curved beams including varying cross-section. If the cross-section varies in a continuous way, then the analytical solution can become very complex, because a sixth order differential equation with non-constant coefficients must be solved. Therefore, approximate approaches seem unavoidable. Natural frequencies are calculated for the beams having a range of nondimensional parameters representing variations in continuously varying cross-section with various opening angles and end conditions. Results are compared with numerical solutions by the Rayleigh-Ritz, the cells discretization method(C. D. M.), or the SAP 90 finite element solutions for cases in which they are available.

2. Governing Differential Equations

The curved beam considered is shown in Fig. 1. A point on the centroidal axis is defined by the angle θ, measured from the left support. The tangential and

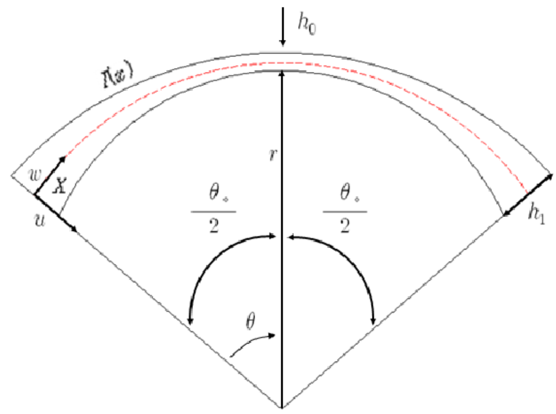

Fig. 1 Coordinate system for curved beam

radial displacements of the arch axis are w and u, respectively. Here, r is the radius of the centroidal axis. These displacements are considered to be positive in the directions indicated.

A mathematical study of the in-plane inextensional vibrations of a curved beam neglecting transverse shearing deformation is carried out starting with the basic equations of motion as given by Love(1944). Following Love(1944), the analysis is simplified by restricting attention to problems where there is no extension of the center line. This condition requires that w and u be related by

$$u=\frac{\partial w}{\partial\theta} \tag{1}$$

The bending moment related to the change in curvature can be written as

$$M=\frac{EI(x)}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2}+u\right)=\frac{EI(x)}{r^2}\left(\frac{\partial^3 w}{\partial\theta^3}+\frac{\partial w}{\partial\theta}\right) \tag{2}$$

where E is the Young's modulus of elasticity for the material of the beam, and $I(x)$ is the area moment of inertia of the varying cross section. If rotatory inertia is neglected, the moment equation takes the form

$$\frac{\partial M}{\partial\theta}+N^* r=0 \tag{3}$$

where N^* is the internal shear force.

Equation (3) using the equation (2) becomes

$$\frac{E}{r^2}\frac{\partial}{\partial\theta}\left(I(x)\frac{\partial^3 w}{\partial\theta^3}+I(x)\frac{\partial w}{\partial\theta}\right)+N^* r=0 \tag{4}$$

or

$$N^*=-\frac{E}{r^3}\left(\frac{\partial I(x)}{\partial\theta}\frac{\partial^3 w}{\partial\theta^3}+I(x)\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial I(x)}{\partial\theta}\frac{\partial w}{\partial\theta}+I(x)\frac{\partial^2 w}{\partial\theta^2}\right) \tag{5}$$

The equation of motion in the radial direction takes the form

$$\frac{\partial N^*}{\partial\theta}+T=mr\frac{\partial^2 u}{\partial t^2} \tag{6}$$

where T is the normal force, and m is the mass per unit length. Using equation (5), equation (6) can be written as

$$-\frac{E}{r^3}\frac{\partial}{\partial\theta}\left(\frac{\partial I(x)}{\partial\theta}\frac{\partial^3 w}{\partial\theta^3}+I(x)\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial I(x)}{\partial\theta}\frac{\partial w}{\partial\theta}+I(x)\frac{\partial^2 w}{\partial\theta^2}\right)+T=mr\frac{\partial^2}{\partial t}\left(\frac{\partial w}{\partial\theta}\right) \tag{7}$$

or

$$T=\frac{E}{r^3}\left(2\frac{\partial I(x)}{\partial\theta}\frac{\partial^4 w}{\partial\theta^4}+2\frac{\partial I(x)}{\partial\theta}\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial^2 I(x)}{\partial\theta^2}\frac{\partial^3 w}{\partial\theta^3}+I(x)\frac{\partial^5 w}{\partial\theta^5}+\frac{\partial^2 I(x)}{\partial\theta^2}\frac{\partial w}{\partial\theta}+I(x)\frac{\partial^3 w}{\partial\theta^3}\right)+mr\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial\theta^2}\right) \tag{8}$$

In the tangential direction, the equation of motion take the form

$$\frac{\partial T}{\partial\theta}-N^*=mr\frac{\partial^2 w}{\partial t^2} \tag{9}$$

The substitution of equations (8) and (5) into equation (9) leads to a single sixth order partial differential equation for w in the form

$$\begin{aligned}&\frac{E}{r^3}\Big(2\frac{\partial I(x)}{\partial\theta}\frac{\partial^5 w}{\partial\theta^5}+2\frac{\partial^2 I(x)}{\partial\theta^2}\frac{\partial^4 w}{\partial\theta^4}+2\partial\frac{I(x)}{\partial\theta}\frac{\partial^3 w}{\partial\theta^3}\\&+2\frac{\partial^2 I(x)}{\partial\theta^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial^2 I(x)}{\partial\theta^2}\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^3 I(x)}{\partial\theta^3}\frac{\partial^3 w}{\partial\theta^3}\\&+I(x)\frac{\partial^6 w}{\partial\theta^6}+\frac{\partial I(x)}{\partial\theta}\frac{\partial^5 w}{\partial\theta^5}+\frac{\partial^2 I(x)}{\partial\theta^2}\frac{\partial^2 w}{\partial\theta^2}\\&+\frac{\partial^3 I(x)}{\partial\theta^3}\frac{\partial w}{\partial\theta}+I(x)\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial I(x)}{\partial\theta}\frac{\partial^3 w}{\partial\theta^3}\Big)\\&+mr\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{E}{r^3}\left(\frac{\partial I(x)}{\partial\theta}\frac{\partial^3 w}{\partial\theta^3}+I(x)\frac{\partial^4 w}{\partial\theta^4}\right.\\&\left.+\frac{\partial I(x)}{\partial\theta}\frac{\partial w}{\partial\theta}+I(x)\frac{\partial^2 w}{\partial\theta^2}\right)=mr\frac{\partial^2 w}{\partial t^2}\end{aligned} \tag{10}$$

or

$$\begin{aligned}&\frac{E}{r^4}\Big[I(x)\frac{\partial^6 w}{\partial\theta^6}+\frac{\partial^5 w}{\partial\theta^5}\left(3\frac{\partial I(x)}{\partial\theta}\right)+\frac{\partial^4 w}{\partial\theta^4}\left(3\frac{\partial^2 I(x)}{\partial\theta^2}+2I(x)\right)\\&+\frac{\partial^3 w}{\partial\theta^3}\left(\frac{\partial^3 I(x)}{\partial\theta^3}+4\frac{\partial I(x)}{\partial\theta}\right)+\frac{\partial^2 w}{\partial\theta^2}\left(3\frac{\partial^2 I(x)}{\partial\theta^2}+I(x)\right)\\&+\frac{\partial w}{\partial\theta}\left(\frac{\partial^3 I(x)}{\partial\theta^3}+\frac{\partial I(x)}{\partial\theta}\right)\Big]=m\frac{\partial^2}{\partial t^2}\left(w-\frac{\partial^2 w}{\partial\theta^2}\right)\end{aligned} \tag{11}$$

If the cross-sectional area of the curved beam is a constant, the differential equation of this uniform beam can be written as(Love, 1944)

$$\frac{EI}{r^4}\left(\frac{\partial^6 w}{\partial\theta^6}+2\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}\right)=m\frac{\partial^2}{\partial t^2}\left(w-\frac{\partial^2 w}{\partial\theta^2}\right) \tag{12}$$

Equation (11) can be rewritten using $w(\theta,t)=W(\theta)T(t)$

$$\begin{aligned}&\frac{W^{(IV)}}{\theta_0^6}(f(X))+\frac{W^{(V)}}{\theta_0^5}\left(3\frac{f'(X)}{\theta_0}\right)+\frac{W^{(VI)}}{\theta_0^4}\\&\left(3\frac{f''(X)}{\theta_0^2}+2f(X)\right)+\frac{W'''}{\theta_0^3}\left(\frac{f'''(X)}{\theta_0^3}+4\frac{f'(X)}{\theta_0}\right)\\&+\frac{W''}{\theta_0^2}\left(3\frac{f''(X)}{\theta_0^2}+f(X)\right)+\frac{W'}{\theta_0}\left(\frac{f'''(X)}{\theta_0^3}+\frac{f'(X)}{\theta_0}\right)\\&=\frac{mr^4}{EI_0}\omega^2\left(-W+\frac{W''}{\theta_0^2}\right)\end{aligned} \tag{13}$$

where $I(x)$ is $I_0 f(X)$, and each prime denotes one differentiation with respect to the dimensionless distance coordinate X, defined as equations (14) and (15), respectively.

Here $f(X)$ and I_0 are the function of the cross-section variation law and the area moment of inertia of the varying cross section associated with the height of the cross-section h_0 at the crown, respectively. In the following the simple case in which the cross-section varies linearly is examined, because it seems the only law which has been studied by Auciello and De Rosa(1994).

Consider then the beam structure with a rectangular cross-section shown in Fig. 1, in which the height of the cross-section varies linearly from h_1 at the supports to h_0 at the crown, according to the law

$$I(x) = I_0 f(X) \text{ and } f(X) = [1+|2\eta(X-0.5)|]^3$$
$$0 \leq X \leq 1 \quad (14)$$

where h_1 is $(1+\eta)h_0$, and η is the ratio of the heights. The dimensionless distance coordinate defines as

$$X = \frac{\theta}{\theta_0} \quad (15)$$

The usual boundary conditions have been examined such as clamped-clamped, simply-simply supported, and simply supported-clamped ends.

If the curved beam is clamped at $\theta=0$ and $\theta=\theta_0$, then the boundary conditions take the form

$$u=0,\ w=0,\ \psi=0 \text{ at } \theta=0 \text{ or } \theta=\theta_0 \quad (16)$$

or

$$w(0) = w'(0) = w''(0) = w(\theta_0) = w'(\theta_0) = w''(\theta_0) = 0 \quad (17)$$

where ψ is th rotation of the cross- section.

If the curved beam is simply supported at $\theta=0$ and $\theta=\theta_0$, then the boundary conditions can be expressed in the following form

$$w=0 \quad (18)$$

$$u = -\frac{\partial w}{\partial \theta} = 0 \quad (19)$$

$$M = \frac{EI}{r^2}\left(\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta}\right) = 0 \quad (20)$$

or

$$w(0) = w'(0) = w'''(0) = w(\theta_0) = w'(\theta_0) = w'''(\theta_0) = 0 \quad (21)$$

3. Differential Quadrature Method(DQM)

In many cases, moderately accurate solutions which can be calculated rapidly are desired at a few points in the respective physical domains. These solutions have traditionally been obtained by the standard finite difference and finite element methods have to be computed based on a large number of points. Consequently, computational efforts are often considerable for these standard methods. In order to overcome the aforementioned complexities, an efficient procedure called differential quadrature method (DQM) was introduced by Bellman and Casti(1971). By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al.(1989). Kukreti et al.(1992) calculated the fundamental frequencies of tapered plates, and Farsa et al.(1993) applied the method to calculate the fundamental frequencies of general anisotropic and laminated plates. Recently, Kang and Han(1998) applied the method to the static analysis of circular curved beams using classical and shear deformable beam theories, and Kang and Kim(2007) have analyzed the out-of-plane vibration of curved beams considering shear deformation using DQM.

From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij} f(x_j) \text{ for } i, j = 1,2,...,N \quad (22)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x)=x^{k-1} \text{ for } k=1,2,...,N \tag{23}$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij}x_j^{k-1}=(k-1)(k-2)\cdots(k-n)x_i^{k-n-1}$$
$$\text{for } i,k=1,2,...,N \tag{24}$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse as described by Hamming(1973).

4. Application

Applying the DQM to equation (13) gives

$$\frac{1}{\theta_0^6}\sum_{j=1}^{N}F_{ij}W_j(f(X_i))+\frac{1}{\theta_0^5}\sum_{j=1}^{N}E_{ij}W_j(\frac{3}{\theta_0}\sum_{j=1}^{N}A_{ij}f(X_j))$$
$$+\frac{1}{\theta_0^4}\sum_{j=1}^{N}D_{ij}W_j(\frac{3}{\theta_0^2}\sum_{j=1}^{N}B_{ij}f(X_j)+2f(X_i)$$
$$+\frac{1}{\theta_0^3}\sum_{j=1}^{N}C_{ij}W_j(\frac{1}{\theta_0^3}\sum_{j=1}^{N}C_{ij}f(X_j)+\frac{4}{\theta_0}\sum_{j=1}^{N}A_{ij}f(X_j))$$
$$+\frac{1}{\theta_0^2}\sum_{j=1}^{N}B_{ij}W_j(\frac{3}{\theta_0^2}\sum_{j=1}^{N}B_{ij}f(X_j)+f(X_i))$$
$$+\frac{1}{\theta_0}\sum_{j=1}^{N}A_{ij}W_j(\frac{1}{\theta_0^3}\sum_{j=1}^{N}C_{ij}f(X_j)+\frac{1}{\theta_0}\sum_{j=1}^{N}A_{ij}f(X_i))$$
$$=\lambda^2(-W_i+\frac{1}{\theta_0^2}\sum_{j=1}^{N}B_{ij}W_j) \tag{25}$$

where $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}$, and F_{ij} are the weighting coefficients for the first-, the second-, the third-, the fourth-, the fifth-, and the sixth-order derivatives, respectively, along the dimensionless axis. and λ^2 is the non-dimensional frequency, $\frac{mr^4\omega^2}{EI_0}$.

The boundary conditions for clamped ends, given by equation (17), can be expressed in differential quadrature form as follows:

$$W_1=0 \text{ at } X=0 \tag{26}$$
$$W_N=0 \text{ at } X=1 \tag{27}$$
$$\sum_{j=1}^{N}A_{2j}W_j=0 \text{ at } X=0+\delta \tag{28}$$
$$\sum_{j=1}^{N}A_{(N-1)j}W_j=0 \text{ at } X=1-\delta \tag{29}$$
$$\sum_{j=1}^{N}B_{3j}W_j=0 \text{ at } X=0+2\delta \tag{30}$$
$$\sum_{j=1}^{N}B_{(N-2)j}W_j=0 \text{ at } X=1-2\delta \tag{31}$$

Similarly, the boundary conditions for simply supported ends given by equation (21), can be expressed in differential quadrature form as follows:

$$W_1=0 \text{ at } X=0 \tag{32}$$
$$W_N=0 \text{ at } X=1 \tag{33}$$
$$\sum_{j=1}^{N}A_{2j}W_j=0 \text{ at } X=0+\delta \tag{34}$$
$$\sum_{j=1}^{N}A_{(N-1)j}W_j=0 \text{ at } X=1-\delta \tag{35}$$
$$\sum_{j=1}^{N}C_{3j}W_j=0 \text{ at } X=0+2\delta \tag{36}$$
$$\sum_{j=1}^{N}C_{(N-2)j}W_j=0 \text{ at } X=1-2\delta \tag{37}$$

Here, δ denotes a very small distance measured along the dimensionless axis from the boundary ends. Mixed boundary conditions can be easily accommodated by combining these equations; simply change the weighting coefficients. This set of equations together with the appropriate boundary conditions can be solved for the in-plane free vibration of non-uniform curved beams.

5. Numerical Results and Comparisons

Based on the above derivations, Fundamental frequency parameters, $\lambda=(\omega^2mr^4/EI_0)^{1/2}$, for non-uniform curved beams are evaluated for the rectangular cross sections under the various boundary conditions, and the numerical results by the DQM are compared with other numerical solutions by Auciello and De Rosa(1994). In the following the simple case in which the cross-section varies linearly is examined, because it seems the only law which has been studied in the literature.

Tables 1 and 2 present the results of convergence studies relative to the number of grid point N and the parameter δ, respectively. Table 1 shows that the accuracy of the numerical solution increases with increasing N and passes through a maximum. The optimal value for N is found to be 11 to 13 using $\delta=1\times10^{-6}$ comparing with Rayleigh-Ritz's solutions. Table 2 shows the sensitivity of the numerical solution to the choice of δ using 13 grid points. From Table 2, the solution accuracy decreases due to numerical instabilities if δ becomes too big comparing with Rayleigh-Ritz's solutions. The optimal value for δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. Therefore, all results are calculated using 13 grid points and $\delta=1\times10^{-6}$ along the dimensionless axis.

The values of $\eta(=\frac{h_1}{h_0}-1)$ are taken to be from 0.1 to 0.5 for the comparisons. The results are summarized in Tables 3~5. As it can be seen from Tables 3~5, the numerical results by the DQM show excellent agreement with the solutions by Auciello and De Rosa(1994) for the case of

Table 3 Fundamental frequency parameters, $\lambda=\sqrt{\omega^2(mr^4/EI_0)}$, for in-plane vibration of non-uniform curved beams with clamped-clamped ends

η	θ_0^*(Auciello and De Rosa, 1994)	Rayleigh-Ritz	C.D.M.	SAP90	DQM
0.1	10(5)	2034.0	2138.3	-	2038.92
	20(10)	506.34	532.59		507.809
	30(15)	223.70	235.26		224.306
	40(20)	124.73	131.20		125.092
	50(25)	-	-		79.1806
	60(30)				54.2619
0.2	10(5)	2096.8	2263.8	2070.9	2079.23
	20(10)	515.48	563.93	565.80	517.909
	30(15)	227.64	249.15	250.01	228.622
	40(20)	126.94	139.0	139.50	127.615
	50(25)	-	-	-	80.7977
	60(30)				55.3875
0.3	10(5)	2125.9	2387.4	2392.8	2140.79
	20(10)	529.43	594.79	569.19	533.251
	30(15)	233.84	262.84	263.49	235.588
	40(20)	130.41	146.68	147.06	131.424
	50(25)	-	-	-	83.2348
	60(30)				57.0601
0.4	10(5)	2198.8	2509.29	-	2220.90
	20(10)	547.74	625.23		553.314
	30(15)	241.94	276.35		244.486
	40(20)	134.94	154.25		136.389
	50(25)	-	-		86.4331
	60(30)				59.2347
0.5	10(5)	-	-	-	2318.38
	20(10)				577.567
	30(15)				255.210
	40(20)				142.376
	50(25)				90.1803
	60(30)				61.8451

Table 1 Fundamental frequency parameters, $\lambda=\sqrt{\omega^2(mr^4/EI_0)}$, for in-plane vibration of non-uniform curved beams with clamped-clamped ends including a range of N; $\eta=0.1$ and $\delta=1\times10^{-6}$

θ_0(degrees)	Rayleigh-Ritz	C.D.M.	N(DQM)				
			9	11	13	15	17
10	2034.0	2138.3	1919.95	2045.85	2038.92	2044.21	2056.32
20	506.54	532.59	478.232	509.548	507.809	509.159	513.331
30	223.70	235.26	211.258	225.061	224.306	224.465	222.264
40	124.73	131.20	117.831	125.507	125.092	125.409	124.701

Table 2 Fundamental frequency parameters, $\lambda=\sqrt{\omega^2(mr^4/EI_0)}$, for in-plane vibration of non-uniform curved beams with clamped-clamped ends including a range of δ; $\eta=0.1$ and $N=13$

θ_0(degrees)	Rayleigh-Ritz	C.D.M.	δ(DQM)				
			1×10^{-4}	1×10^{-5}	1×10^{-6}	1×10^{-7}	1×10^{-8}
20	506.54	532.59	1857.04	2038.48	2038.92	2039.35	2039.99
30	223.70	235.26	462.087	507.728	507.809	507.900	508.469
40	124.73	131.20	204.113	224.279	224.306	224.320	224.596
40	124.73	131.20	113.808	125.061	125.092	125.199	125.223

Table 4 Fundamental frequency parameters, $\lambda=\sqrt{\omega^2(mr^4/EI_0)}$, for in-plane vibration of non-uniform curved beams with simply-simply supported ends

η	θ_0^*(Auciello and De Rosa, 1994)	Rayleigh-Ritz	C.D.M.	SAP90	DQM
0.1	10(5)	1299.0	1354.4	-	1309.24
	20(10)	322.86	3366.70		325.408
	30(15)	142.15	148.25		143.301
	40(20)	78.890	82.31		79.5917
	50(25)	-	-		50.0862
	60(30)				34.0912
0.2	10(5)	1315.1	1416.1	1418.8	1338.99
	20(10)	326.88	352.08	352.79	332.913
	30(15)	143.90	155.05	155.39	144.760
	40(20)	79.875	86.105	86.325	81.4338
	50(25)	-	-	-	51.2697
	60(30)				34.9014
0.3	10(5)	1340.7	1476.2	1478.2	1381.09
	20(10)	333.20	367.05	367.72	343.308
	30(15)	146.70	161.66	161.99	151.211
	40(20)	81.434	89.799	90.006	84.0040
	50(25)	-	-	-	52.9110
	60(30)				36.0373
0.4	10(5)	1374.2	1534.9	-	1431.41
	20(10)	341.58	381.66		356.104
	30(15)	150.39	168.12		156.856
	40(20)	83.346	93.40		87.1511
	50(25)	-	-		54.8801
	60(30)				37.3944
0.5	10(5)	-	-	-	1490.20
	20(10)				370.639
	30(15)				163.299
	40(20)				90.7254
	50(25)				57.1671
	60(30)				38.9364

Table 5 Fundamental frequency parameters, $\lambda=\sqrt{\omega^2(mr^4/EI_0)}$, for in-plane vibration of non-uniform curved beams with simply supported-clamped ends

η	θ_0(degrees)	Rayleigh-Ritz	C.D.M.	SAP90	DQM
0.1	10	1712.0	1716.6	-	1684.46
	20	426.10	427.21		419.210
	30	187.96	188.46		184.948
	40	104.45	104.91		102.978
	50	66.011	66.257		65.0414
	60	45.111	45.280		44.4577
0.2	10	1800.3	1806.7	1810.2	1760.00
	20	447.69	449.70	449.97	438.113
	30	197.88	198.42	198.76	192.927
	40	110.10	110.48	110.73	107.674
	50	69.582	69.804	69.990	68.0390
	60	47.482	47.724	47.871	46.5319
0.3	10	1887.0	1895.1	1897.7	1854.96
	20	470.02	471.75	472.48	461.763
	30	207.36	208.19	208.39	203.797
	40	115.47	115.95	116.14	113.532
	50	72.970	73.283	73.431	71.7761
	60	49.944	50.121	50.243	49.0978
0.4	10	1976.2	1981.9	-	1966.39
	20	492.42	493.42		489.594
	30	217.14	217.78		216.121
	40	121.03	121.33		120.406
	50	76.476	76.701		76.1608
	60	52.351	52.478		52.0976
0.5	10	-	-	-	2092.93
	20				521.111
	30				230.030
	40				128.171
	50				81.0492
	60				55.4735

clamped-clamped, simply-simply supported, and simply supported-clamped ends. Tables 3~5 also show that the numerical results by the DQM are good agreement with those by the SAP90 FEM. However, the SAP90 FEM was quite expensive because 90 finite elements were employed, as described by Auciello and De Rosa(1994). From Tables 3~5, the frequency parameters by the DQM are generally lower than both those by the cells discretization method(C. D. M.) and the finite element SAP90, but, generally higher than those by the Ritz method. In general, as the values of $\eta(=\frac{h_1}{h_0}-1)$ ratios of beam cross sections become larger, the frequencies become higher and more significant for the vibration. There need some corrections for the opening angles, θ_0^*, in Tables 3 and 4. The opening angles presented by Auciello and De Rosa(1994) should be the half of the opening angles presented by the DQM. But, the angles by Auciello and De Rosa(1994) have been expressed as the same of the opening angles in their results. Therefore, the angles, θ_0^*, presented by Auciello and De Rosa(1994) should be the same as the angles in Tables 3 and 4. However, the opening angles, θ_0, in Tables 5 and 6 are correct ones. In Table 6, the frequency parameters by the DQM for the uniform curved beams also show excellent agreement with those by other numerical

Table 6 Fundamental frequency parameters, $\lambda=\sqrt{\omega^2(mr^4/EI_0)}$, for in-plane vibration of uniform curved beams with clamped-clamped ends; η= 0.0

θ_0(degrees)	Glalerkin	Rayleigh-Ritz	C.D.M.	SAP IV	DQM
10	2073.9	2021.9	-	2021.9	2021.65
20	516.56	503.5	500.56	503.5	503.479
30	228.18	222.36	-	222.36	222.349
40	127.26	123.97	123.24	123.97	123.856

Table 7 The first three frequency parameters, λ_i, for in-plane vibration of the curved beams with clamped-clamped ends; η=0.1 and 0.0

θ_0 (degrees)	η	Method	λ_i		
			λ_1	λ_2	λ_3
40	0.1	Rayleigh-Ritz	124.73	239.52	-
		C. D. M.	131.20	237.17	-
		DQM	125.09	234.18	463,58
180	0.0	Archer(1960)	4.3841	9.6514	17.930
		DQM	4.3840	9.7029	18.771

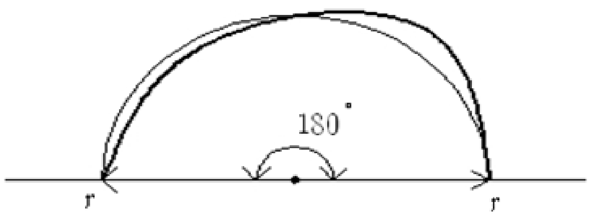

Fig. 2 Sketch of first mode of the uniform curved beam vibration(η=0.0)

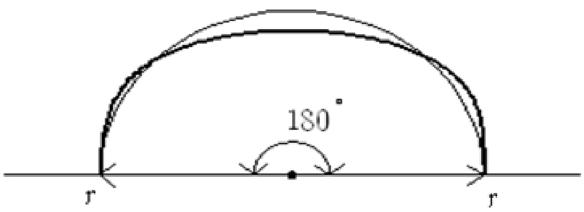

Fig. 3 Sketch of second mode of the uniform curved beam vibration(η=0.0)

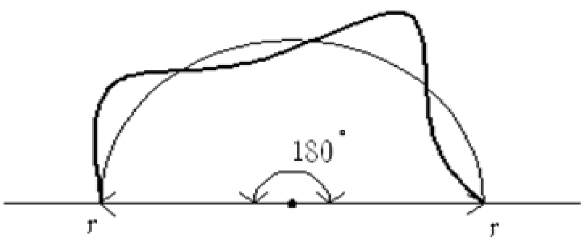

Fig. 4 Sketch of third mode of the uniform curved beam vibration(η=0.0)

solutions. In Table 7, the first three frequency parameters, λ_i, obtained by the DQM are also compared with the parameters by other methods for the cases of η=0.1 and 0.0 with clamped-clamped ends. Figs. 2~4 also show the sketch of the modes of the uniform curved beam vibration(η=0.0) with clamped-clamped ends.

6. Conclusions

The differential quadrature method(DQM) is presented to determine the fundamental frequencies of the circular curved beams with varying cross-section neglecting transverse shearing deformation. The frequency parameters are calculated for the beams with various opening angles and end conditions. The results are compared with existing numerical solutions by other methods(Rayleigh-Ritz, C.D.M., or FEM) for cases in which they are available. It has been shown that compare to the finite element method, the DQM requires less grid points to obtain the frequencies of the beams. The DQM using only thirteen discrete points along the non-dimensional X-axe gives the good results and requires the small computation times for the evaluation of the vibration characteristics reported in totality by Kukreti et al.(1992). In addition, the corrected results are given for the beams not previously presented for this problem.

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Nomenclature

The following symbols are used in this paper:

A_{ij} weighting coefficients for the first derivatives

B_{ij} weighting coefficients for the second derivatives

C_{ij} weighting coefficients for the third derivatives

D_{ij} weighting coefficients for the fourth derivatives

E_{ij} weighting coefficients for the fifth derivatives

F_{ij} weighting coefficients for the sixth derivatives

E Young's modulus of elasticity

$f(X)$ function of cross-section variation law

$f(x)$ general function

$f(x_j)$ function value at point x_j

h_0, h_1 height of cross-section varies linearly

$I(x)$ area moment of inertia of varying cross section

I_0 area moment of inertia of varying cross section associated with h_0 at the crown

L differential operator

M bending moment

m mass per unit length

N number of discrete points

N^* internal shear force

r radius of centroidal axis

T normal force

u radial displacement

W_{ij} weighting coefficients

w tangential displacement

X dimensionless position coordinate

x_j discrete point in domain

δ small dimensionless distance measured from boundary ends of member

$\eta(=\frac{h_1}{h_0}-1)$ cross-section ratio

θ angle from left support to generic point

θ_0 opening angle of member

θ_0^* correction of opening angle

λ^2 fundamental frequency parameter, $\frac{mr^4\omega^2}{EI_0}$

ψ rotation of cross- section area

ω circular frequency(rad/s)

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