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L-pre-separation axioms in $(2, L)$-topologies based on complete residuated lattice-valued logic

Fathei M. Zeyada, M. Azab Abd-Allah and A.K. Mousa*

***Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt**

Abstract

In the present paper we introduce and study L-pre-T_0-, L-pre-T_1-, L-pre-T_2 (L-pre-Hausdorff)-, L-pre-T_3 (L-pre-regularity)-, L-pre-T_4 (L-pre-normality)-, L-pre-strong-T_3-, L-pre-strong-T_4-, L-pre-R_0-, L-pre-R_1-separation axioms in $(2, L)$-topologies where L is a complete residuated lattice. Sometimes we need more conditions on L such as the completely distributive law or that the "$\wedge$" is distributive over arbitrary joins or the double negation law as we illustrate through this paper. As applications of our work the corresponding results (see [1, 2]) are generalized and new consequences are obtained.

Key Words : $(2, L)$-topology; Complete residuated lattice; Pre-open set; Pre-separation axioms.

1. Introduction

In [4, 5, 9, 13, 17] the concept of (M, L)-fuzzy topology was appeared as a function $\tau : M^X \longrightarrow L$ where X is an ordinary set; and M and L are some types of lattices. We prefer to use the symbol (M, L)-topology instead of (M, L)-fuzzy topology.

The concept of $(2, L)$-topology (L-fuzzifying topology) appeared in [4] by Höhle under the name "L-fuzzy topology" (cf. Definition 4.6, Proposition 4.11 in [4]). In the case of $L = I$ where I is the closed unit interval $[0, 1]$ the terminology "L-fuzzifying topology" traces back to Ying (cf. Definition 2.1 in [14]).

In $(2, I)$-topology (fuzzifying topology) separation axioms were introduced and studied in [2, 8, 12, 14]. Recently the concepts of L-T_0-, L-T_1-, L-T_2 (L-Hausdorff)-, L-T_3 (L-regularity)-, L-T_4 (L-normality)-, L-R_0-, L-R_1-separation axioms in $(2, L)$-topology (L-fuzzifying topology) where L is a complete residuated lattice were introduced and studied in [19].

In the present paper we introduce and study the concept of $(2, L)$-pre-open sets and the concepts of L-pre-T_0-, L-pre-T_1-, L-pre-T_2 (L-pre-Hausdorff)-, L-pre-T_3 (L-pre-regularity)-, L-pre-T_4 (L-pre-normality)-, L-pre-strong-T_3-, L-pre-strong-T_4-, L-pre-R_0-, L-pre-R_1-separation axioms in $(2, L)$-topologies where L is a complete residuated lattice. Sometimes we need more conditions on L such as the completely distributive law or that the "$\wedge$" is distributive over arbitrary joins or the double negation law as we illustrate through this paper.

Since the complete residuated lattice $I = [0, 1]$ as defined in [14, 15] satisfies all the conditions in our work, then our results are generalizations of the corresponding results in [1, 2]. However we illustrate this in conclusion.

In [20], it was proved that the concept of complete residuated lattice and the concept of strictly two-sided commutative quantale are equivalent.

The contents are arranged as follows: In Section 2, we recall basic definitions and results in complete residuated lattice and in $(2, L)$-topology. In Section 3, we introduce and study $(2, L)$-pre-open sets, $(2, L)$-pre-neighborhoods in $(2, L)$-topologies. In Section 4, we introduce and study $(2, L)$-pre-derived, $(2, L)$-pre-closure, operators in $(2, L)$-topologies. Section 5 is devoted to introduce and study the basic properties of L-pre-T_0-, L-pre-T_1-, L-pre-T_2 (L-pre-Hausdorff)-, L-pre-T_3 (L-pre-regular)-, L-pre-T_4 (L-pre-normal)-, L-pre-strong-T_3-, L-pre-strong-T_4-, L-pre-R_0-, L-pre-R_1-separation axioms in $(2, L)$-topologies and study the relation of it with L-separation axioms. In Section 6, relations among these separation axioms are considered. Finally in Section 7 a conclusion is given.

2. Preliminaries

First, we introduce the definition of complete residuated lattice.

Definition 2.1 [10, 16, 20]. A structure $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is called a complete residuated lattice iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

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(2) $(L, *, \top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L, and

(b) For every $a \in L, a * \top = a$,

(3) $\longrightarrow$ is couple with $*$ as: $a * b \leq c$ if and only if $a \leq b \longrightarrow c \ \forall a, b, c \in L$.

Definition 2.2 [5, 11]. A structure $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is called a strictly two-sided commutative quantale iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

(2) $(L, *, \top)$ is a commutative monoid,

(3) (a) $*$ is distributive over arbitrary joins, i.e., $a * \bigvee_{j \in J} b_j = \bigvee_{j \in J} (a * b_j) \ \forall a \in L, \ \forall \{b_j | j \in J\} \subseteq L$,

(b) $\longrightarrow$ is a binary operation on L defined by: $a \longrightarrow b = \bigvee_{\lambda * a \leq b} \lambda \ \forall a, b \in L$.

Theorem 2.3 [20]. A structure $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is called a complete residuated lattice iff it is a strictly two-sided commutative quantale.

Corollary 2.4 [20]. A structure $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is a complete MV-algebra iff $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is a complete residuated lattice satisfies the additional property

(MV) $(a \longrightarrow b) \longrightarrow b = a \vee b \ \forall a, b \in L$.

We recall now the laws of completely distributive and double negation for L.

Definition 2.5 [3]. L is called completely distributive if the following law is satisfied: $\forall \{A_j | j \in J\} \subseteq 2^L$, where 2^L is the power subset of L we have
$\bigwedge_{j \in J} \bigvee A_j = \bigvee_{f \in \prod_{j \in J} A_j} (\bigwedge_{j \in J} f(j))$.

Definition 2.6 [6]. The double negation law in L is given as follows: $(a \longrightarrow \bot) \longrightarrow \bot = a \ \forall \ a \in L$.

Lemma 2.7 [21]. For every $a \in L$ we have $a \leq (a \longrightarrow \bot) \longrightarrow \bot$.

Now we recall the definitions of L-equality and L-inclusion.

Definition 2.8 [6]. Let $f, g \in L^X$. Then the L-equality between f and g is denoted by $[[f, g]]$ and defined as follows: $[[f, g]] = \bigwedge_{x \in X} \big((f(x) \longrightarrow g(x)) \wedge (g(x) \longrightarrow f(x))\big)$.

Definition 2.9 [19]. Let $f, g \in L^X$. Then the L-inclusion of f in g is denoted by $[[f, g[[$ and defined as follows: $[[f, g[[= \bigwedge_{x \in X} \big(f(x) \longrightarrow g(x)\big)$.

In the following we recall the concept of (M, L)-topology and illustrate that the L-fuzzifying topology is in fact the $(2, L)$-topology.

Definition 2.10 (Höhle [4], Höhle and Šostak [6], Kubiak [9], Šostak [13], [17]). An (M, L)-fuzzy topology (we prefer to say an (M, L)-topology) is a mapping $\tau : M^X \longrightarrow L$ such that

(1) $\tau(1_X) = \tau(1_\emptyset) = \top$,

(2) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B) \ \forall A, B \in M^X$,

(3) $\tau(\bigvee_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau(A_j) \ \forall \{A_j | j \in J\} \subseteq M^X$.

The pair (M^X, τ) is called an (M, L)-fuzzy topological space (we prefer to say an (M, L)-topological space).

When $M = \{0, 1\}$, Definition 2.10 will reduce to that of (2, L)-topology (L-fuzzifying topology).

Some basic concepts and results in $(2, L)$-topology (L-fuzzifying topology) which are useful in the present paper are given as follows:

Definition 2.11 [19]. Let (X, τ) be a $(2, L)$-topological space. The family of all $(2, L)$-closed sets will be denoted by $F_\tau \in L^{(2^X)}$, and defined as follows: $F_\tau(A) = \tau(X - A)$ where $X - A$ is the complement of A.

Definition 2.12 [5]. Let $x \in X$. The $(2, L)$-neighborhood system of x is denoted by $\varphi_x \in L^{(2^X)}$ and defined as follows: $\varphi_x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$.

Remark 2.13. Höhle proved in Proposition 3.13 [5] that if L satisfies the completely distributive law, then $\tau(A) = \bigwedge_{x \in A} \varphi_x(A)$.

Proposition 2.14 [19]. Let (X, τ) be a $(2, L)$-topological space and let $A, B \in 2^X$. Then $\forall x \in X$,

(1) $\varphi_x(X) = \top, \varphi_x(\emptyset) = \bot$,

(2) $A \subseteq B \Longrightarrow \varphi_x(A) \leq \varphi_x(B)$,

(3) $\varphi_x(A \cap B) \leq \varphi_x(A) \wedge \varphi_x(B)$,

(4) If $\wedge$ is distributive over arbitrary joins, then $\varphi_x(A \cap B) \geq \varphi_x(A) \wedge \varphi_x(B)$,

(5) $\varphi_x(A) \leq \bigvee_{y \in X - B} (\varphi_y(A) \vee \varphi_x(B)) \ \forall B \in 2^X$.

Definition 2.15 [19]. The $(2, L)$-closure operator is denoted by $Cl_\tau \in (L^X)^{2^X}$, and defined as follows: $Cl_\tau(A)(x) = \varphi_x(X - A) \longrightarrow \bot$.

Proposition 2.16 [19]. Let (X, τ) be a $(2, L)$-topological space, then:

(1) If L satisfies the double negation law, then $\varphi_x(A) = Cl_\tau(X - A)(x) \longrightarrow \bot \ \forall A \in 2^X, \ \forall x \in X$,

(2) $Cl_\tau(\emptyset) = 1_\emptyset$ where $1_\emptyset \in L^X$ and defined as follows: $1_\emptyset(x) = \bot \ \forall x \in X$,

(3) $A \leq Cl_\tau(A) \ \forall A \in 2^X$,

(4) If $A, B \in 2^X, A \subseteq B$, then $Cl_\tau(A) \leq Cl_\tau(B)$,

(5) If $\wedge$ is distributive over arbitrary joins, then $Cl_\tau(A\cup B) \leq Cl_\tau(A) \vee Cl_\tau(B)\ \ \forall A, B \in 2^X$.

Lemma 2.17 [19]. For every $a, b \in L$ we have $a \leq b \longrightarrow a$.

Lemma 2.18 [21]. For every $a, b \in L$ we have $\bigwedge_{j\in J}(a_j \longrightarrow b) = (\bigvee_{j\in J} a_j) \longrightarrow b$.

Definition 2.19 [21]. Let $A, B \in 2^X$. The binary crisp predicat $D : 2^X \times 2^X \longrightarrow \{\bot, \top\}$, called crisp jointness, is given as follows:

$$D(A,B) = \begin{cases} \top, & \text{if } A\cap B \neq \emptyset \\ \bot, & \text{if } A\cap B = \emptyset. \end{cases}$$

For the definitions of L-T_0-, L-T_1-, L-T_2 (L-Hausdorff)-, L-T_3 (L-regularity)-, L-T_4 (L-normality)-, L-R_0-, L-R_1-separation axioms in $(2,L)$-topology and the notations of $K(x,y), H(x,y), M(x,y), V(x,A), W(A,B)$, where $x, y \in X, A, B \in 2^X$ we refer to [19].

3. Fundamental concepts (a): $(2,L)$-pre-open sets, $(2,L)$-pre-neighborhoods in $(2,L)$-topology

Definition 3.1. Let (X,τ) be a $(2,L)$-topological space. And let $f \in L^X$. Then the extended $(2,L)$-closure operator is denoted by $\widetilde{Cl_\tau} \in (L^X)^{L^X}$ and defined as follows:

$\widetilde{Cl_\tau}(f)(y) = \bigvee_{\alpha\in L}\big(\alpha \wedge Cl_\tau(f_\alpha)(y)\big)$, where $f_\alpha = \{x \in X |\ f(x) \geq \alpha\}$.

Definition 3.2. Let (X,τ) be a $(2,L)$-topological space. Then the extended (2, L)-interior operator is denoted by $\widetilde{Int_\tau} \in (L^X)^{L^X}$ and defined as follows:

$\widetilde{Int_\tau}(f)(x) = \widetilde{Cl_\tau}(f \longrightarrow \bot)(x) \longrightarrow \bot\ \ \forall f \in L^X$.

Definition 3.3. Let (X,τ) be a $(2,L)$-topological space. The family of all $(2,L)$-pre-open sets, is denoted by $\tau_P \in L^{(2^X)}$, and defined as $\tau_P(A) = [[A, \widetilde{Int_\tau}(Cl_\tau(A))[[$.

Definition 3.4. Let (X,τ) be a $(2,L)$-topological space. The family of all $(2,L)$-pre-closed sets will be denoted by $F_P \in L^{(2^X)}$, and defined as follows:

$F_P(A) = \tau_P(X-A)$ where $X-A$ is the complement of A.

Lemma 3.5. For any $A \in 2^X, f, g \in L^X$ we have

(1) If $f \leq g$, then $f \longrightarrow \bot \geq g \longrightarrow \bot$,

(2) If $f \leq g$, then $[[\widetilde{Cl_\tau}(f), \widetilde{Cl_\tau}(g)[[= \top$,

(3) If $f \leq g$, then $[[\widetilde{Int_\tau}(f), \widetilde{Int_\tau}(g)[[= \top$,

(4) $[[Int_\tau(A), \widetilde{Int_\tau}(Cl_\tau(A))[[= \top$.

Proof. (1) Since $f \leq g$, then we have

$$f \longrightarrow \bot = \bigvee_{\lambda * f\leq\bot}\lambda \geq \bigvee_{\lambda * g\leq\bot}\lambda = g \longrightarrow \bot.$$

(2) Since $f \leq g$, then $f_\alpha \subseteq g_\alpha$ and from Proposition 2.16 (4) we have

$$\begin{aligned}\widetilde{Cl_\tau}(f)(y) &= \bigvee_{\alpha\in L}\big(\alpha\wedge Cl_\tau(f_\alpha)(y)\big)\\ &\leq \bigvee_{\alpha\in L}\big(\alpha\wedge Cl_\tau(g_\alpha)(y)\big) = \widetilde{Cl_\tau}(g)(y).\end{aligned}$$

(3) From Proposition 2.16 (4) and (1), (2) above we have

$\widetilde{Int_\tau}(f)(x) = \widetilde{Cl_\tau}(f \longrightarrow \bot)(x) \longrightarrow \bot \leq \widetilde{Cl_\tau}(g \longrightarrow \bot)(x) \longrightarrow \bot = \widetilde{Int_\tau}(g)(x)$.

(4) From Proposition 2.16 (3) and (3) above we have $\widetilde{Int_\tau}(Cl_\tau(A))(x) \geq \widetilde{Int_\tau}(A)(x) = Int_\tau(A)(x)$. □

Theorem 3.6. Let (X,τ) be a $(2,L)$-topological space.Then

(1) $\tau_P(X) = \top, \tau_P(\emptyset) = \top$,

(2) $F_P(X) = \top, F_P(\emptyset) = \top$,

(3) For every $\{A_j | j \in J\} \subseteq 2^X$, $\tau_P(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J}\tau_P(A_j)$,

(4) For every $\{A_j | j \in J\} \subseteq 2^X$, $F_P(\bigcap_{j\in J} A_j) \geq \bigwedge_{j\in J} F_P(A_j)$.

Proof. The proof of (1) and (2) are straightforward.

(3) From Proposition 2.16 (4) and Lemma 3.5 (4) we have $[[\widetilde{Int_\tau}(Cl_\tau(A_j)), \widetilde{Int_\tau}(Cl_\tau(\bigcup_{j\in J} A_j))[[= \top$. Therefore, from Lemma 2.18 we have

$$\begin{aligned}\tau_P(\bigcup_{j\in J}A_j) &= [[\bigcup_{j\in J}A_j, \widetilde{Int_\tau}(Cl_\tau(\bigcup_{j\in J}A_j))[[\\ &= \bigwedge_{x\in X}\Big((\bigvee_{j\in J}(A_j)(x)) \longrightarrow \widetilde{Int_\tau}(Cl_\tau(\bigcup_{j\in J}A_j))(x)\Big)\\ &= \bigwedge_{j\in J}\bigwedge_{x\in X}\Big(A_j(x) \longrightarrow \widetilde{Int_\tau}(Cl_\tau(\bigcup_{j\in J}A_j))(x)\Big)\\ &\geq \bigwedge_{j\in J}\bigwedge_{x\in X}\Big(A_j(x) \longrightarrow \widetilde{Int_\tau}(Cl_\tau(A_j))(x)\Big)\\ &= \bigwedge_{j\in J}[[A_j, \widetilde{Int_\tau}(Cl_\tau(A_j))[[= \bigwedge_{j\in J}\tau_P(A_j).\end{aligned}$$

(4) Follows from (3) above. □

Definition 3.7. Let $x \in X$. The $(2,L)$-pre-neighborhood system of x is denoted by $\varphi_x^P \in L^{(2^X)}$ and defined as follows: $\varphi_x^P(A) = \bigvee_{x\in B\subseteq A}\tau_P(B)$.

Proposition 3.8. Let (X, τ) be a $(2, L)$-topological space and let $A, B \in 2^X$. Then $\forall x \in X$, we have

(1) (a) $\varphi_x^P(X) = \top$, (b) $\varphi_x^P(\emptyset) = \bot$,

(2) If $A \subseteq B$, then $\varphi_x^P(A) \leq \varphi_x^P(B)$,

(3) $\varphi_x^P(A \cap B) \leq \varphi_x^P(A) \wedge \varphi_x^P(B)$,

(4) $\varphi_x^P(A) \leq \bigvee_{y \in X-B} (\varphi_y^P(A) \vee \varphi_x^P(B))$.

Proof. (1) (a) $\varphi_x^P(X) = \bigvee_{x \in B \subseteq X} \tau_P(B) = \top$ because $\tau_P(X) = \top$.

(1) (b) $\varphi_x^P(\emptyset) = \bigvee_{x \in H \subseteq \emptyset} \tau_P(H) = \bot$.

(2) $\varphi_x^P(A) = \bigvee_{x \in H \subseteq A} \tau_P(H) \leq \bigvee_{x \in H \subseteq B} \tau_P(H) = \varphi_x^P(B)$.

(3) Follows from (2) above.

(4) Let x be a fixed point in X and let A, B, G be subsets of X such that $x \in G \subseteq A$. Now,

(a) for every $B \in 2^X$ such that $G \cap (X - B) \neq \emptyset$, $\bigvee_{y \in X-B} \varphi_y^P(A) \geq \tau_P(G)$ $\big($Indeed, since $G \cap (X - B) \neq \emptyset$, then there exists $y_\circ \in G \cap (X - B)$. Now, $\varphi_{y_\circ}^P(A) = \bigvee_{y_\circ \in H \subseteq A} \tau_P(H) \geq \tau_P(G)$. Hence $\bigvee_{y \in X-B} \varphi_y^P(A) \geq \varphi_{y_\circ}^P(A) \geq \tau_P(G).\big)$ and

(b) for every $B \in 2^X$ such that $G \cap (X - B) = \emptyset$, $\varphi_x^P(B) = \bigvee_{x \in M \subseteq B} \tau_P(M) \geq \tau_P(G)$.

Hence, $\bigvee_{y \in X-B} (\varphi_y^P(A) \vee \varphi_x^P(B)) \geq \bigvee_{x \in G \subseteq A} \tau_P(G) = \varphi_x^P(A)$. □

Theorem 3.9. Let (X, τ) be a $(2, L)$-topological space. Then we have

(1) $\big[\big[\tau, \tau_P\big[\big[= \top$, and

(2) $\big[\big[F_\tau, F_P\big[\big[= \top$.

Proof. (1) One can verify that $\tau(A) \leq \big[\big[A, Int_\tau(A)\big[\big[$. Then applying Lemma 3.5 (4) we have $\tau(A) \leq \big[\big[A, Int_\tau(A)\big[\big[\leq \big[\big[A, \widetilde{Int}_\tau(Cl_\tau(A))\big[\big[= \tau_P(A)$.

(2) The proof is obtained from (1) above . □

Theorem 3.10. Let (X, τ) be a $(2, L)$-topological space. Then we have

(1) $\tau_P(A) \leq \bigwedge_{x \in A} \bigvee_{x \in B \subseteq A} \tau_P(B)$, and

(2) If L satisfies the completely distributive law, then $\tau_P(A) \geq \bigwedge_{x \in A} \bigvee_{x \in B \subseteq A} \tau_P(B)$.

Proof. (1) Obvious.

(2) Let $\eta_x = \{B | x \in B \subseteq A\}$. Then for every $f \in \prod_{x \in A} \eta_x$ we have $\bigcup_{x \in A} f(x) = A$ and so from Theorem 3.6 (3) we have $\tau_P(A) = \tau_P(\bigcup_{x \in A} f(x)) \geq \bigwedge_{x \in A} \tau_P(f(x))$.

Hence,
$\tau_P(A) \geq \bigvee_{f \in \prod_{x \in A} \eta_x} \bigwedge_{x \in A} \tau_P(f(x)) = \bigwedge_{x \in A} \bigvee_{x \in B \subseteq A} \tau_P(B)$. □

Corollary 3.11. Let (X, τ) be a $(2, L)$-topological space. If L satisfies the completely distributive law, then $\tau_P(A) = \bigwedge_{x \in A} \varphi_x^P(A)$.

4. Fundamental concepts (b): $(2, L)$-pre-derived, $(2, L)$-pre-closure operators in $(2, L)$-topology

Definition 4.1. Let (X, τ) be a $(2, L)$-topological space. The $(2, L)$-pre-derived operator is denoted by $d_P \in (L^X)^{2^X}$, and defined as follows:

$d_P(A)(x) = \varphi_x^P((X - A) \cup \{x\}) \longrightarrow \bot$.

Theorem 4.2. Let (X, τ) be a $(2, L)$-topological space. Then we have

(1) $F_P(A) \leq \big[\big[d_P(A), A\big[\big[$, and

(2) If L satisfies the double negation law and the completely distributive law, then $F_P(A) = \big[\big[d_P(A), A\big[\big[$.

Proof. (1) From Lemma 2.7 we have

$$
\begin{aligned}
&\big[\big[d_P(A), A\big[\big[\\
&= \bigwedge_{x \in X} (d_P(A)(x) \longrightarrow A(x)) \\
&= \Big(\bigwedge_{x \in X-A} (d_P(A)(x) \longrightarrow \bot)\Big) \wedge \Big(\bigwedge_{x \in A} (d_P(A)(x) \longrightarrow \top)\Big) \\
&= \Big(\bigwedge_{x \in X-A} (d_P(A)(x) \longrightarrow \bot)\Big) \wedge \top \\
&= \bigwedge_{x \in X-A} ((\varphi_x^P((X - A) \cup \{x\}) \longrightarrow \bot) \longrightarrow \bot) \\
&\geq \bigwedge_{x \in X-A} \varphi_x^P((X - A) \cup \{x\}) = \bigwedge_{x \in X-A} \varphi_x^P((X - A)) \\
&= \bigwedge_{x \in X-A} \bigvee_{x \in B \subseteq X-A} \tau_P(B) \geq \tau_P(X - A) = F_P(A).
\end{aligned}
$$

(2) The inequalities in proof of (1) above become equalities from the double negation law and from Theorem 3.10 (since L satisfies the completely distributive law) respectively so that the result hold. □

Definition 4.3. Let (X, τ) be a $(2, L)$-topological space. The (2,L)-pre-closure operator is denoted by $Cl_P \in (L^X)^{2^X}$, and defined as follows:

$Cl_P(A)(x) = \varphi_x^P(X - A) \longrightarrow \bot$.

Proposition 4.4. Let (X, τ) be a $(2, L)$-topological space. Then for any x, A, B, we have

(1) If L satisfies the double negation law, then $\varphi_x^P(A) = Cl_P(X - A)(x) \longrightarrow \bot$,

(2) $Cl_P(\emptyset) = 1_\emptyset$, where $1_\emptyset \in L^X$ is defined as follow: $1_\emptyset(x) = \bot, \quad \forall x \in X$,

(3) $\big[\big[A, Cl_P(A)\big[\big[= \top$,

(4) If $A \subseteq B$, then $\big[\big[Cl_P(A), Cl_P(B)\big[\big[= \top$,

(5) $Cl_P(A)(x) = \bigwedge_{B\in 2^X} \big(\varphi^P_x(B) \longrightarrow D(A,B)\big)$,

(6) $\big[\big[Cl_P(A), A \cup d_P(A)\big]\big] = \top$,

(7) $F_P(A) \leq \big[\big[A, Cl_P(A)\big]\big]$,

(8) If L satisfies the double negation law and the completely distributive law, then $F_P(A) = \big[\big[A, Cl_P(A)\big]\big]$,

(9) $Cl_P(A)(x) = \bigwedge_{x\notin B\supseteq A} (F_P(B) \longrightarrow \bot)$.

Proof. (1) Since $Cl_P(X-A)(x) = \varphi^P_x(A) \longrightarrow \bot$, then $\varphi^P_x(A) = (\varphi^P_x(A) \longrightarrow \bot) \longrightarrow \bot = Cl_P(X-A)(x) \longrightarrow \bot$.

(2) $Cl_P(\emptyset)(x) = \varphi^P_x(X) \longrightarrow \bot = \top \longrightarrow \bot = \bot \; \forall x \in X$. Then $Cl_P(\emptyset) = 1_\emptyset$.

(3) If $x \in A$, then $Cl_P(A)(x) = \varphi^P_x(X-A) \longrightarrow \bot = \bot \longrightarrow \bot = \top = A(x)$.
If $x \notin A$, then $A(x) = \bot \leq Cl_P(A)(x)$.

(4) If $A \subseteq B$. Then from Proposition 3.8 (2) we have $Cl_P(A)(x) = \varphi^P_x(X-A) \longrightarrow \bot \leq \varphi^P_x(X-B) \longrightarrow \bot = Cl_P(B)(x)$.

(5) From the double negation law and Lemma 2.18 we have

$$\begin{aligned}
&\bigwedge_{B\in 2^X} \big(\varphi^P_x(B) \longrightarrow D(A,B)\big) \\
&= \bigwedge_{B\in 2^X,\, A\cap B=\emptyset} (\varphi^P_x(B) \longrightarrow D(A,B)) \wedge \\
&\qquad \bigwedge_{B\in 2^X,\, A\cap B\neq\emptyset} (\varphi^P_x(B) \longrightarrow D(A,B)) \\
&= \bigwedge_{A\cap B=\emptyset} (\varphi^P_x(B) \longrightarrow \bot) \wedge \bigwedge_{A\cap B\neq\emptyset} (\varphi^P_x(B) \longrightarrow \top) \\
&= \bigwedge_{A\cap B=\emptyset} (\varphi^P_x(B) \longrightarrow \bot) \wedge \top \\
&= \bigwedge_{A\cap B=\emptyset} \big(\varphi^P_x(B) \longrightarrow \bot\big) \\
&= \bigwedge_{B\subseteq X-A} \big(\varphi^P_x(B) \longrightarrow \bot\big) \\
&= \bigvee_{B\subseteq X-A} \varphi^P_x(B) \longrightarrow \bot \\
&= \varphi^P_x(X-A) \longrightarrow \bot = Cl_P(A)(x).
\end{aligned}$$

(6) If $x \in A$, then from (3) above we have $(A \cup d_P(A))(x) = A(x) = \top = Cl_P(A)(x)$. Now suppose $x \notin A$. Then we have

$$\begin{aligned}
(A \cup d_P(A))(x) &= A(x) \vee d_P(A)(x) \\
&= d_P(A)(x) \\
&= \varphi^P_x((X-A)\cup\{x\}) \longrightarrow \bot \\
&= \varphi^P_x((X-A)) \longrightarrow \bot \\
&= Cl_P(A)(x).
\end{aligned}$$

(7) From (3) above and Lemma 2.7 we have

$$\begin{aligned}
\big[\big[A, Cl_P(A)\big]\big] &= \big[\big[A, Cl_P(A)\big[\big[\wedge \big[\big[Cl_P(A), A\big[\big[\\
&= \top \wedge \big[\big[Cl_P(A), A\big[\big[\\
&= \bigwedge_{x\in X} (Cl_P(A)(x) \longrightarrow A(x)) \\
&= \Big(\bigwedge_{x\in X-A} (Cl_P(A)(x) \longrightarrow \bot)\Big) \wedge \\
&\qquad \Big(\bigwedge_{x\in A} (Cl_P(A)(x) \longrightarrow \top)\Big) \\
&= \Big(\bigwedge_{x\in X-A} (Cl_P(A)(x) \longrightarrow \bot)\Big) \wedge \top \\
&= \bigwedge_{x\in X-A} ((\varphi^P_x(X-A) \longrightarrow \bot) \longrightarrow \bot) \\
&\geq \bigwedge_{x\in X-A} \varphi^P_x(X-A) \\
&= \bigwedge_{x\in X-A} \; \bigvee_{x\in B\subseteq X-A} \tau_P(B) \\
&\geq \tau_P(X-A) = F_P(A).
\end{aligned}$$

(8) The inequalities in proof of (7) above become equalities from the double negation law and from Corollary 3.11 (since L satisfies the completely distributive law) respectively so that the result hold.

(9) From Lemma 2.18 we have

$$\begin{aligned}
Cl_P(A)(x) &= \varphi^P_x(X-A) \longrightarrow \bot \\
&= \bigvee_{x\in X-B\subseteq X-A} \tau_P(X-B) \longrightarrow \bot \\
&= \bigvee_{x\notin B\supseteq A} F_P(B) \longrightarrow \bot \\
&= \bigwedge_{x\notin B\supseteq A} (F_P(B) \longrightarrow \bot). \quad \square
\end{aligned}$$

From Proposition 4.4 (3), (7) and (8) we have the following result.

Corollary 4.5. Let (X,τ) be a $(2,L)$-topological space. Then we have

(1) $F_P(A) \leq \big[\big[Cl_P(A), A\big[\big[$, and

(2) If L satisfies the double negation law and the completely distributive law, then $F_P(A) = \big[\big[Cl_P(A), A\big[\big[$.

5. L-pre-separation axioms in $(2, L)$-topologies

First, in the framework of $(2,2)$-topology (general topology) [7] the concepts of separation axioms were considered as crisp concepts ($\{\bot, \top\}$-concepts), for example in the framework of $(2,2)$-topology the space (X, τ) is T_1 (resp. is not T_1) if and only if $T_1(X,\tau) = \top$ (resp. $T_1(X,\tau) = \bot$).

Second, in the framework of $(2, I)$-topology (fuzzifying topology) [1, 8, 12, 14] the concepts of separation axioms were considered as fuzzy concepts ($[0,1]$-concepts), for example in the framework of $(2, I)$-topology any topological space (X, τ) is T_1 with a degree in $[0,1]$.

Third, in the framework of $(2, L)$-topology (L-fuzzifying topology) [19] the authors were considered the concepts of separation axioms as L-concepts.

In the following, in the framework of $(2, L)$-topology (L-fuzzifying topology) we consider the concepts of pre-separation axioms as L-concepts, for example in the framework of $(2, L)$-topology any topological space (X, τ) is pre-T_1 with a degree in L.

Remark 5.1. For simplicity we put the following notations:

(1) $K^P(x,y) = \big(\bigvee_{y\notin A} \varphi_x^P(A)\big) \vee \big(\bigvee_{x\notin A} \varphi_y^P(A)\big)$,

(2) $H^P(x,y) = \big(\bigvee_{y\notin A} \varphi_x^P(A)\big) \wedge \big(\bigvee_{x\notin B} \varphi_y^P(B)\big)$,

(3) $M^P(x,y) = \bigvee_{A\cap B=\emptyset} \big(\varphi_x^P(A) \wedge \varphi_y^P(B)\big)$,

(4) $V^P(x,C) = \bigvee_{A\cap B=\emptyset,\ C\subseteq B} \big(\varphi_x^P(A) \wedge \tau_P(B)\big)$,

(5) $W^P(A,B) = \bigvee_{A\subseteq C,\ B\subseteq D,\ C\cap D=\emptyset} \big(\tau_P(C) \wedge \tau_P(D)\big)$,

where $x, y \in X, A, B, C, D \in 2^X$ and τ_P is the family of all $(2, L)$-pre-open sets in a $(2, L)$-topology τ on X.

Definition 5.2. Let Ω be the class of all $(2, L)$-topological spaces. The unary L-predicates L-pre-$T_i \in L^\Omega$ are denoted by $T_i^P, i = 0,1,2,3,4$, L-pre-strong-$T_i \in L^\Omega$ are denoted by $T_i^{PS}, i = 3,4$ and L-pre-$R_i \in L^\Omega$ are denoted by $R_i^P, i = 0,1$ and defined as follows:

(1) $T_0^P(X,\tau) = \bigwedge_{x\neq y} K^P(x,y)$,

(2) $T_1^P(X,\tau) = \bigwedge_{x\neq y} H^P(x,y)$,

(3) $T_2^P(X,\tau) = \bigwedge_{x\neq y} M^P(x,y)$,

(4) $T_3^P(X,\tau) = \bigwedge_{x\notin C} \big(F_\tau(C) \longrightarrow V^P(x,C)\big)$,

(5) $T_4^P(X,\tau) = \bigwedge_{A\cap B=\emptyset} \big((F_\tau(A) \wedge F_\tau(B)) \longrightarrow W^P(A,B)\big)$,

(6) $T_3^{PS}(X,\tau) = \bigwedge_{x\notin C} \big(F_P(C) \longrightarrow V^P(x,C)\big)$,

(7) $T_4^{PS}(X,\tau) = \bigwedge_{A\cap B=\emptyset} \big((F_P(A) \wedge F_P(B)) \longrightarrow W^P(A,B)\big)$,

(8) $R_0^P(X,\tau) = \bigwedge_{x\neq y} \big(K^P(x,y) \longrightarrow H^P(x,y)\big)$,

(9) $R_1^P(X,\tau) = \bigwedge_{x\neq y} \big(K^P(x,y) \longrightarrow M^P(x,y)\big)$.

Lemma 5.3. Let $(X,\tau) \in \Omega$. Then for any $x, y \in X$, we have

(1) $K(x,y) \leq K^P(x,y)$,
(2) $H(x,y) \leq H^P(x,y)$,
(3) $M(x,y) \leq M^P(x,y)$,
(4) $V(x,C) \leq V^P(x,C)$,
(5) $W(A,B) \leq W^P(A,B)$.

Proof. From Theorem 3.9 one can deduce that $\varphi_x(A) \leq \varphi_x^P(A)\ \ \forall A \in 2^X$, hence the proof is immediate. □

Theorem 5.4. Let $(X,\tau) \in \Omega$. Then we have
(1) $T_i(X,\tau) \leq T_i^P(X,\tau), i = 1,2,3,4$, and
(2) $T_i^{PS}(X,\tau) \leq T_i^P(X,\tau), i = 3,4$.

Proof. (1) It is obtained from Lemma 5.3.
(2) It follows from Theorem 3.9 (2). □

Lemma 5.5. Let $(X,\tau) \in \Omega$. Then for any $x, y \in X$, we have
(1) $M^P(x,y) \leq H^P(x,y)$, and
(2) $H^P(x,y) \leq K^P(x,y)$.

Proof. (1) Let $S_1 = \{(A,B) \in 2^X \times 2^X |\ A \cap B = \emptyset\}$ and let $S_2 = \{(A,B) \in 2^X \times 2^X |\ y \notin A, x \notin B\}$. Then $S_1 \subseteq S_2$. Hence,

$$\begin{aligned} H^P(x,y) &= \big((\bigvee_{y\notin A} \varphi_x^P(A)) \wedge (\bigvee_{x\notin B} \varphi_y^P(B))\big) \\ &\geq \bigvee_{y\notin A} \bigvee_{x\notin B} (\varphi_x^P(A) \wedge \varphi_y^P(B)) \\ &= \bigvee_{y\notin A, x\notin B} (\varphi_x^P(A) \wedge \varphi_y^P(B)) \\ &\geq \bigvee_{A\cap B=\emptyset} (\varphi_x^P(A) \wedge \varphi_y^P(B)) = M^P(x,y). \end{aligned}$$

(2)
$$\begin{aligned} H^P(x,y) &= \big((\bigvee_{y\notin A} \varphi_x^P(A)) \wedge (\bigvee_{x\notin B} \varphi_y^P(B))\big) \\ &\leq (\bigvee_{y\notin A} \varphi_x^P(A)) \\ &\leq \big((\bigvee_{y\notin A} \varphi_x^P(A)) \vee (\bigvee_{x\notin A} \varphi_y^P(A))\big) = K^P(x,y). \end{aligned}$$ □

From Lemma 5.5 (1), (2) we have the following result.

Corollary 5.6. Let $(X,\tau) \in \Omega$. Then for any $x, y \in X$, we have $M^P(x,y) \leq K^P(x,y)$.

Theorem 5.7. For any $(X,\tau)\in\Omega$, we have $R_1^P(X,\tau)\le R_0^P(X,\tau)$.

Proof. From Lemma 5.5 (1) we have

$$\begin{aligned} R_1^P(X,\tau) &= \bigwedge_{x\neq y}(K^P(x,y)\longrightarrow M^P(x,y))\\ &\le \bigwedge_{x\neq y}(K^P(x,y)\longrightarrow H^P(x,y))\\ &= R_0^P(X,\tau). \ \square \end{aligned}$$

Theorem 5.8. For any $(X,\tau)\in\Omega$, we have
(1) $T_1^P(X,\tau)\le R_0^P(X,\tau)$, and
(2) $T_1^P(X,\tau)\le T_0^P(X,\tau)$.

Proof. (1) From Lemma 2.17 we have $H^P(x,y)\le K^P(x,y)\longrightarrow H^P(x,y)$ so that $T_1^P(X,\tau)\le R_0^P(X,\tau)$.
(2) The proof follows from Lemma 5.5 (2). $\square$

From Theorem 5.8 (1), (2) we have the following result.

Corollary 5.9. Let $(X,\tau)\in\Omega$. Then we have $T_1^P(X,\tau)\le R_0^P(X,\tau)\wedge T_0^P(X,\tau)$.

Theorem 5.10. Let $(X,\tau)\in\Omega$. If $T_0^P(X,\tau)=\top$, then we have $T_1^P(X,\tau)=R_0^P(X,\tau)\wedge T_0^P(X,\tau)$.

Proof. Since $T_0^P(X,\tau)=\top$, then for every $x,y\in X$ such that $x\neq y$ we have $K^P(x,y)=\top$. Now, since $\top\longrightarrow\alpha=\alpha\ \forall\alpha\in L$ (Indeed $\top\longrightarrow\alpha=\bigvee_{\lambda*\top\le\alpha}\lambda=\bigvee_{\lambda\le\alpha}\lambda=\alpha$) we have

$$\begin{aligned} &R_0^P(X,\tau)\wedge T_0^P(X,\tau)\\ &=(\bigwedge_{x\neq y}(K^P(x,y)\longrightarrow H^P(x,y)))\wedge\top\\ &=\bigwedge_{x\neq y}H^P(x,y)=T_1^P(X,\tau). \ \square \end{aligned}$$

Theorem 5.11. For any $(X,\tau)\in\Omega$, we have
(1) $T_2^P(X,\tau)\le R_1^P(X,\tau)$, and
(2) $T_2^P(X,\tau)\le T_1^P(X,\tau)$.

Proof. The proof is similar to the proof of Theorem 5.8. $\square$

From Theorem 5.11 (1), (2) we have the following result.

Corollary 5.12. Let $(X,\tau)\in\Omega$. Then we have $T_2^P(X,\tau)\le R_1^P(X,\tau)\wedge T_1^P(X,\tau)$.

Theorem 5.13. Let $(X,\tau)\in\Omega$. If $T_0^P(X,\tau)=\top$, then we have $T_2^P(X,\tau)=R_1^P(X,\tau)\wedge T_0^P(X,\tau)$.

Proof. The proof is similar to the proof of Theorem 5.10. $\square$

Theorem 5.14. Let $(X,\tau)\in\Omega$. Then we have

(1) $T_2^P(X,\tau)\le T_0^P(X,\tau)$,

(2) $T_2^P(X,\tau)\le R_0^P(X,\tau)$,

(3) $R_1^P(X,\tau)\vee T_1^P(X,\tau)\le R_0^P(X,\tau)$.

Proof. (1) The proof follows from Theorem 5.8 (2) and Theorem 5.11 (2).

(2) The proof follows from Theorem 5.7 and Theorem 5.11 (1).

(3) The proof follows from Theorem 5.7 and Theorem 5.8 (1). $\square$

From Theorem 5.14 (1), (2) we have the following result.

Corollary 5.15. For any $(X,\tau)\in\Omega$, we have $T_2^P(X,\tau)\le R_0^P(X,\tau)\wedge T_0^P(X,\tau)$.

Theorem 5.16. If L satisfies the completely distributive law, then for any $(X,\tau)\in\Omega$, we have $T_1^P(X,\tau)=\bigwedge_{x\in X}F_P(\{x\})$.

Proof. Let $x_1,x_2\in X$ such that $x_1\neq x_2$. Then from Corollary 3.11 we have

$$\begin{aligned} \bigwedge_{x\in X}F_P(\{x\}) &= \bigwedge_{x\in X}\tau_P(X-\{x\})\\ &= \bigwedge_{x\in X}(\bigwedge_{y\in X-\{x\}}\varphi_y^P(X-\{x\}))\\ &\le \bigwedge_{y\in X-\{x_2\}}\varphi_y^P(X-\{x_2\})\\ &\le \varphi_{x_1}^P(X-\{x_2\})=\bigvee_{x_2\notin A}\varphi_{x_1}^P(A). \end{aligned}$$

Similarly, we have $\bigwedge_{x\in X}F_P(\{x\})\le\bigvee_{x_1\notin B}\varphi_{x_2}^P(B)$. Hence,

$$\begin{aligned} \bigwedge_{x\in X}F_P(\{x\}) &\le \bigwedge_{x_1\neq x_2}\ \bigvee_{x_1\notin B,\ x_2\notin A}(\varphi_{x_1}^P(A)\wedge\varphi_{x_2}^P(B))\\ &= T_1^P(X,\tau). \end{aligned}$$

On the other hand,

$$\begin{aligned}
&T_1^P(X,\tau)\\
&= \bigwedge_{x_1,\, x_2\in X,\, x_1\neq x_2} \big(\big(\bigvee_{x_2\notin A} \varphi^P_{x_1}(A)\big)\wedge\big(\bigvee_{x_1\notin B} \varphi^P_{x_2}(A)\big)\big)\\
&= \bigwedge_{x_1\neq x_2} \big(\varphi^P_{x_1}(X-\{x_2\})\wedge \varphi^P_{x_2}(X-\{x_1\})\big)\\
&\leq \bigwedge_{x_1\neq x_2} \varphi^P_{x_1}(X-\{x_2\})\\
&= \bigwedge_{x_2\in X}\ \bigwedge_{x_1\in X-\{x_2\}} \varphi^P_{x_1}(X-\{x_2\})\\
&= \bigwedge_{x_2\in X} \tau_P(X-\{x_2\})\\
&= \bigwedge_{x\in X} \tau_P(X-\{x\}) = \bigwedge_{x\in X} F_P(\{x\}).\ \square
\end{aligned}$$

Theorem 5.17. Let $(X,\tau)\in\Omega$. If L satisfies the double negation law, then we have $T_0^P(X,\tau) = \bigwedge_{x\neq y} ((Cl_P(\{y\})(x)\longrightarrow\bot)\vee(Cl_P(\{x\})(y)\longrightarrow\bot))$.

Proof. From Proposition 4.4 (1) we have

$$\begin{aligned}
T_0^P(X,\tau) &= \bigwedge_{x\neq y}\big(\big(\bigvee_{y\notin A}\varphi^P_x(A)\big)\vee\big(\bigvee_{x\notin A}\varphi^P_y(A)\big)\big)\\
&= \bigwedge_{x\neq y}\Big(\varphi^P_x(X-\{y\})\vee\varphi^P_y(X-\{x\})\Big)\\
&= \bigwedge_{x\neq y}((Cl_P(\{y\})(x)\longrightarrow\bot)\vee\\
&\qquad (Cl_P(\{x\})(y)\longrightarrow\bot)).\ \square
\end{aligned}$$

Definition 5.18. Let $(X,\tau)\in\Omega$ and let $x\in X$. Then $\beta^P_x\in L^{2^X}$ is called a local base for τ_P at $x\in X$ iff the following conditions are satisfied:

(1) $[[\beta^P_x, \varphi^P_x[[= \top$,

(2) $\varphi^P_x(A)\leq \bigvee_{x\in B\subseteq A}\beta^P_x(B)$,

(3) If $A\subseteq B$, then $\beta^P_x(A)\leq\beta^P_x(B)\ \ \forall x\in X$.

Lemma 5.19. Let $(X,\tau)\in\Omega$ and let β^P_x be a local base for τ_P at $x\in X$. Then for any $A\in 2^X$, we have $\varphi^P_x(A)=\bigvee_{x\in B\subseteq A}\beta^P_x(B)$.

Proof. From condition (1) in Definition 5.18 and Proposition 3.8 (2) we have $\varphi^P_x(A)\geq\varphi^P_x(B)\geq\beta^P_x(B)$ for every $B\in 2^X$, such that $x\in B\subseteq A$. So, $\varphi^P_x(A)\geq \bigvee_{x\in B\subseteq A}\beta^P_x(B)$.
On the other hand from condition (2) in Definition 5.18 we have $\varphi^P_x(A)\leq\bigvee_{x\in B\subseteq A}\beta^P_x(B)$. Hence, $\varphi^P_x(A) = \bigvee_{x\in B\subseteq A}\beta^P_x(B)$. $\square$

Theorem 5.20. Let $(X,\tau)\in\Omega$ and let $x\in X$, and let β^P_x be a local base for τ_P at $x\in X$. If L satisfies the completely distributive law, then we have

$$T_1^P(X,\tau)\leq\bigwedge_{x\neq y}\big(\bigvee_{y\in X-A}\beta^P_x(A)\big).$$

Proof. Let $B\in 2^X$ such that $x\in B\subseteq X-\{y\}$. Then $y\notin B$ so that $\bigvee_{y\notin A}\beta^P_x(A)\geq\beta^P_x(B)$ so that $\bigvee_{y\notin A}\beta^P_x(A)\geq \bigvee_{x\in B\subseteq X-\{y\}}\beta^P_x(B)=\varphi^P_x(X-\{y\})$ so that from Corollary 3.11 we have

$$\begin{aligned}
\bigwedge_{x\neq y}\big(\bigvee_{y\notin A}\beta^P_x(A)\big) &\geq \bigwedge_{x\neq y}\varphi^P_x(X-\{y\})\\
&= \bigwedge_{y\in X}\ \bigwedge_{x\in X-\{y\}}\varphi^P_x(X-\{y\})\\
&= \bigwedge_{y\in X}\tau_P(X-\{y\})\\
&= \bigwedge_{y\in X}F_P(\{y\}) = T_1^P(X,\tau).\ \square
\end{aligned}$$

Theorem 5.21. Let $(X,\tau)\in\Omega$, and let β^P_x be a local base for τ_P at x. Then we have $T_1^P(X,\tau)\geq \bigwedge_{x\neq y}\big(\bigvee_{y\in X-A}\beta^P_x(A)\big)$.

Proof. From condition (1) in Definition 5.18 we have

$$\begin{aligned}
&\big(\bigvee_{y\in X-A}\beta^P_x(A)\big)\wedge\big(\bigvee_{x\in X-B}\beta^P_x(B)\big)\\
&\leq\big(\bigvee_{y\in X-A}\varphi^P_x(A)\big)\wedge\big(\bigvee_{x\in X-B}\varphi^P_x(B)\big) = H^P(x,y).
\end{aligned}$$

Then $\bigwedge_{x\neq y}\big(\bigvee_{y\in X-A}\beta^P_x(A)\big)\leq\bigwedge_{x\neq y}H^P(x,y) = T_1^P(X,\tau)$. $\square$

From Theorems 5.20 and 5.21 one can have the following theorem.

Theorem 5.22. Let $(X,\tau)\in\Omega$ and let $x\in X$, and let β^P_x be a local base for τ_P at $x\in X$. If L satisfies the completely distributive law, then we have

$$T_1^P(X,\tau)=\bigwedge_{x\neq y}\big(\bigvee_{y\in X-A}\beta^P_x(A)\big).$$

Theorem 5.23. Let $(X,\tau)\in\Omega$, let $x\in X$, and let β^P_x be a local base for τ_P at x. If the meet is distributive over arbitrary joins, then we have

$$T_2^P(X,\tau)=\bigwedge_{x\neq y}\bigvee_{A\in 2^X}\Big(\beta^P_x(A)\wedge\varphi^P_y(X-A)\Big).$$

Proof. From Lemma 5.19 we have

$$\begin{aligned}
&\bigwedge_{x\neq y}\bigvee_{A\in 2^X}\Big(\beta_x^P(A)\wedge\varphi_y^P(X-A)\Big)\\
&=\bigwedge_{x\neq y}\bigvee_{A\in 2^X}\Big(\beta_x^P(A)\wedge(\bigvee_{y\in B\subseteq X-A}\beta_y^P(B))\Big)\\
&=\bigwedge_{x\neq y}\bigvee_{A\in 2^X}\bigvee_{y\in B\subseteq X-A}\Big(\beta_x^P(A)\wedge\beta_y^P(B)\Big)\\
&=\bigwedge_{x\neq y}\bigvee_{A\cap B=\emptyset}\bigvee_{x\in C\subseteq A}\bigvee_{y\in D\subseteq B}\Big(\beta_x^P(C)\wedge\beta_y^P(D)\Big)\\
&=\bigwedge_{x\neq y}\bigvee_{A\cap B=\emptyset}\Big(\varphi_x^P(A)\wedge\varphi_y^P(B)\Big)=T_2^P(X,\tau).\ \square
\end{aligned}$$

Theorem 5.24. Let $(X,\tau)\in\Omega$, let $x\in X$, and let β_x^P be a local base for τ_P at x. If the meet is distributive over arbitrary joins, and L satisfies the double negation law, then we have $T_2^P(X,\tau)=\bigwedge_{x\neq y}\bigvee_{A\in 2^X}\Big(\beta_x^P(A)\wedge\big(Cl_P(A)(y)\longrightarrow\bot)\Big)$.

Proof. It follows from Theorem 5.23 and Proposition 4.4 (1). $\square$

Definition 5.25. The unary L-predicates $PT_3^{(i)}, PST_3^{(i)}\in L^{\Omega}$, where $i=1,2$ are defined as follows:

(1) $PT_3^{(1)}(X,\tau)=\bigwedge_{x\notin D}\Big(F_\tau(D)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}(Cl_P(A)(y)\longrightarrow\bot))\big)\Big)$,

(2) $PT_3^{(2)}(X,\tau)=\bigwedge_{x\in A}\Big(\tau(A)\longrightarrow\bigvee_{B\in 2^X}\big(\varphi_x^P(B)\wedge[[Cl_P(B),A[[\,)\Big)$,

(3) $PST_3^{(1)}(X,\tau)=\bigwedge_{x\notin D}\Big(F_P(D)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}(Cl_P(A)(y)\longrightarrow\bot))\big)\Big)$,

(4) $PST_3^{(2)}(X,\tau)=\bigwedge_{x\in A}\Big(\tau_P(A)\longrightarrow\bigvee_{B\in 2^X}\big(\varphi_x^P(B)\wedge[[Cl_P(B),A[[\,)\Big)$.

Theorem 5.26. If L satisfies the completely distributive law and the double negation law, then for each $(X,\tau)\in\Omega$, we have $T_3^P(X,\tau)=PT_3^{(1)}(X,\tau)$.

Proof. Now, from proposition 4.4 (1) we have

$$\begin{aligned}
&PT_3^{(1)}(X,\tau)\\
&=\bigwedge_{x\notin D}\Big(F_\tau(D)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}(Cl_P(A)(y)\longrightarrow\bot))\big)\Big)\\
&=\bigwedge_{x\notin D}\Big(\tau(X-D)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}\varphi_x^P(X-A))\big)\Big)
\end{aligned}$$

and

$$T_3^P(X,\tau)=\bigwedge_{x\notin D}\Big(\tau(X-D)\longrightarrow\bigvee_{A\cap B=\emptyset,D\subseteq B}\big(\varphi_x^P(A)\wedge\tau_P(B)\big)\Big).$$

So, the result holds if we prove that $\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}\varphi_y^P(X-A))\big)=\bigvee_{A\cap B=\emptyset,\ D\subseteq B}\big(\varphi_x^P(A)\wedge\tau_P(B)\big)$. $(*)$

In the left side of $(*)$ if $A\cap D\neq\emptyset,\ \exists\, y\in D$ such that $y\notin X-A$ so that $\bigwedge_{y\in D}\varphi_y^P(X-A)=\bot$.

Second, $\bigvee_{A\cap B=\emptyset,\ A\in 2^X}\Big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}\varphi_x^P(X-A))\Big)=\bigvee_{A\cap B=\emptyset,\ A\in 2^X}\Big(\varphi_x^P(A)\wedge(\bigwedge_{y\in D}(\bigvee_{y\in B\subseteq X-A}\tau_P(B)))\Big)$. Now we prove that

$$\bigwedge_{y\in D}\bigvee_{y\in B\subseteq X-A}\tau_P(B)=\bigvee_{A\cap B=\emptyset,\ D\subseteq B}\tau_P(B).$$

Let $y\in D$. Assume $S=\{H\in 2^X|\ H\cap A=\emptyset, D\subseteq H\}$ and $\wp_y=\{M\in 2^X|\ y\in M\subseteq X-A\}$. Then $S\subseteq\wp_y$ so that $\bigvee_{B\in\wp_y}\tau_P(B)\geq\bigvee_{B\in S}\tau_P(B)$ so that

$$\bigwedge_{y\in D}\bigvee_{y\in B\subseteq X-A}\tau_P(B)\geq\bigvee_{A\cap B=\emptyset,\ D\subseteq B}\tau_P(B).$$

Let $\wp_y^P=\{\tau_P(M)|\ M\in\wp_y\}$. Then

$$\bigwedge_{y\in D}\bigvee\wp_y^P=\bigwedge_{y\in D}\bigvee_{B\in\wp_y}\tau_P(B)=\bigvee_{f\in\prod_{y\in D}\wp_y^P}\bigwedge_{y\in D}f(y).$$

Then for each $f\in\prod_{y\in D}\wp_y^P$, $\exists\, K$ s.t. $K=\cup\{f(y)|\ f(y)\in\wp_y,\ y\in D\}$ such that, $D\subseteq K\subseteq X-A$ and $\bigwedge_{y\in D}f(y)\leq\tau_P\big(\cup\{f(y)|\ f(y)\in\wp_y,\ y\in D\}\big)=\tau_P(K)$ so that

$$\bigwedge_{y\in D}f(y)\leq\tau_P(K)\leq\bigvee_{A\cap B=\emptyset,\ D\subseteq B}\tau_P(B)$$

so that

$$\bigwedge_{y\in D}\ \bigvee_{y\in B\subseteq X-A}\tau_P(B) = \bigvee_{f\in\prod_{y\in D}\wp_y^P}\ \bigwedge_{y\in D} f(y) \leq \bigvee_{A\cap B=\emptyset,\ D\subseteq B}\tau_P(B).\ \square$$

Theorem 5.27. If L satisfies the completely distributive law and the double negation law, then for any $(X,\tau)\in\Omega$, we have $T_3^P(X,\tau)=PT_3^{(2)}(X,\tau)$.

Proof. From Theorem 5.26 we have

$$\begin{aligned}
&T_3^P(X,\tau)\\
&=\bigwedge_{x\notin D}\Big(F_\tau(D)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge (\bigwedge_{y\in D}(Cl_P(A)(y)\longrightarrow\bot))\big)\Big)\\
&=\bigwedge_{x\notin X-B}\Big(F_\tau(X-B)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge (\bigwedge_{y\in X-B}(Cl_P(A)(y)\longrightarrow\bot))\big)\Big)\\
&=\bigwedge_{x\in B}\Big(\tau(B)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge (\bigwedge_{y\in X}(Cl_P(A)(y)\longrightarrow B(y)))\big)\Big)\\
&=\bigwedge_{x\in B}\Big(\tau(B)\longrightarrow\bigvee_{A\in 2^X}\big(\varphi_x^P(A)\wedge [[Cl_P(A),B[[\ \big)\Big)\\
&=PT_3^{(2)}(X,\tau).\ \square
\end{aligned}$$

Theorem 5.28. If L satisfies the completely distributive law and the double negation law, then for any $(X,\tau)\in\Omega$, we have $T_3^{PS}(X,\tau)=PST_3^{(i)}(X,\tau),\ i=1,2.$

Proof. The proof similar to the proof of Theorems 5.26 and 5.27 respectively. $\square$

6. Relations among L-separation axioms and L-pre-separation axioms in $(2,L)$-topologies

Lemma 6.1. For every $a,b\in L$ we have
$a\longrightarrow\bigwedge_{j\in J}b_j=\bigwedge_{j\in J}(a\longrightarrow b_j)$.

Proof.

$$\begin{aligned}
a\longrightarrow\bigwedge_{j\in J}b_j &= \bigvee_{\lambda * a\leq\bigwedge_{j\in J}b_j}\lambda=\bigvee_{\forall j\in J\ \lambda*a\leq b_j}\lambda\\
&=\bigvee_{\forall j\in J\ \lambda\leq a\longrightarrow b_j}\lambda=\bigvee_{\lambda\leq\bigwedge_{j\in J}(a\longrightarrow b_j)}\lambda\\
&=\bigwedge_{j\in J}(a\longrightarrow b_j).\ \square
\end{aligned}$$

Theorem 6.2. If L satisfies the completely distributive law, then for any $(X,\tau)\in\Omega$, we have $T_3^P(X,\tau)*T_1(X,\tau)\leq T_2^P(X,\tau)$.

Proof. From Lemma 6.1 we have

$$\begin{aligned}
&\bigwedge_{x\neq y}\Big(\tau(X-\{y\})\longrightarrow\bigvee_{A\cap B=\emptyset,\ y\in B}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\varphi_y^P(B)))\Big)\\
&\leq\bigwedge_{x\neq y}\Big(\bigwedge_{y\in X}\tau(X-\{y\})\longrightarrow\bigvee_{A\cap B=\emptyset,\ y\in B}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\varphi_y^P(B)))\Big)\\
&=\bigwedge_{x\neq y}\Big(T_1(X,\tau)\longrightarrow\bigvee_{A\cap B=\emptyset,\ y\in B}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\varphi_y^P(B)))\Big)\\
&=T_1(X,\tau)\longrightarrow\bigwedge_{x\neq y}(\bigvee_{A\cap B=\emptyset}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\varphi_y^P(B))))\\
&\leq T_1(X,\tau)\longrightarrow\bigwedge_{x\neq y}(\bigvee_{A\cap B=\emptyset}(\varphi_x^P(A)\wedge\varphi_y^P(B)))\\
&=T_1(X,\tau)\longrightarrow T_2^P(X,\tau).
\end{aligned}$$

Since,

$$\begin{aligned}
&T_3^P(X,\tau)\\
&=\bigwedge_{x\notin D}\Big(\tau(X-D)\longrightarrow\bigvee_{A\cap B=\emptyset,D\subseteq B}(\varphi_x^P(A)\wedge\tau_P(B))\Big)\\
&=\bigwedge_{x\notin D}\Big(\tau(X-D)\longrightarrow\bigvee_{A\cap B=\emptyset,D\subseteq B}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\tau_P(B)))\Big)\\
&\leq\bigwedge_{x\notin\{y\}}\Big(\tau(X-\{y\})\longrightarrow\bigvee_{A\cap B=\emptyset,\ y\in B}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\varphi_y^P(B)))\Big)\\
&=\bigwedge_{x\neq y}\Big(\tau(X-\{y\})\longrightarrow\bigvee_{A\cap B=\emptyset,\ y\in B}(\bigwedge_{y\in B}(\varphi_x^P(A)\wedge\varphi_y^P(B)))\Big),
\end{aligned}$$

then from above, $T_3^P(X,\tau)\leq T_1(X,\tau)\longrightarrow T_2^P(X,\tau)$
so that $T_3^P(X,\tau)*T_1(X,\tau)\leq T_2^P(X,\tau)$. $\square$

Theorem 6.3. If L satisfies the completely distributive law, then for any $(X,\tau)\in\Omega$, we have $T_3^{PS}(X,\tau)*T_1^P(X,\tau)\leq T_2^P(X,\tau)$.

Proof. The proof similar to the proof of Theorem 6.2. $\square$

From Theorems 6.2 and 6.3 we have the following result.

Corollary 6.4. If L satisfies the completely distributive law, then for any $(X,\tau)\in\Omega$, we have $\big(T_3^{PS}(X,\tau)*T_1^P(X,\tau)\big)\vee\big(T_3^P(X,\tau)*T_1(X,\tau)\big)\le T_2^P(X,\tau)$.

Lemma 6.5. For every $a,b\in L$ we have
$(\bigwedge_{j\in J} a_j)*b\le\bigwedge_{j\in J}(a_j*b)$.

Proof. For every $a,b\in L$ we obtain $a_j*b\le a_j*b\;\forall j\in J$ implies $(\bigwedge_{j\in J} a_j)*b\le a_j*b\;\;\forall j\in J$ hence $(\bigwedge_{j\in J} a_j)*b\le\bigwedge_{j\in J}(a_j*b)$. □

Theorem 6.6. If L satisfies the completely distributive law, then for any $(X,\tau)\in\Omega$, we have $T_4^P(X,\tau)*T_1(X,\tau)\le T_3^P(X,\tau)$.

Proof.

$$
\begin{aligned}
&T_4^P(X,\tau)\\
&=\bigwedge_{E\cap D=\emptyset}\Big((F_\tau(E)\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{E\subseteq A,\ D\subseteq B,\ A\cap B=\emptyset}(\tau_P(A)\wedge\tau_P(B))\Big)\\
&\le\bigwedge_{x\notin D}\Big((F_\tau(\{x\})\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{x\in A,\ D\subseteq B,\ A\cap B=\emptyset}(\tau_P(A)\wedge\tau_P(B))\Big)\\
&=\bigwedge_{x\notin D}\Big((F_\tau(\{x\})\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\bigvee_{x\in A}\tau_P(A)\wedge\tau_P(B))\Big)\\
&\le\bigwedge_{x\notin D}\Big((F_\tau(\{x\})\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\bigvee_{x\in K\subseteq A}\tau_P(K)\wedge\tau_P(B))\Big)\\
&=\bigwedge_{x\notin D}\Big((F_\tau(\{x\})\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\varphi_x^P(A)\wedge\tau_P(B))\Big)\\
&\le\bigwedge_{x\notin D}\Big((\bigwedge_{x\in X}F_\tau(\{x\})\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\varphi_x^P(A)\wedge\tau_P(B))\Big)\\
&=\bigwedge_{x\notin D}\Big((T_1(X,\tau)\wedge F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\varphi_x^P(A)\wedge\tau_P(B))\Big)\\
&\le\bigwedge_{x\notin D}\Big((T_1(X,\tau)*F_\tau(D))\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\varphi_x^P(A)\wedge\tau_P(B))\Big)\\
&\le T_1(X,\tau)\longrightarrow\bigwedge_{x\notin D}\Big(F_\tau(D)\longrightarrow\\
&\qquad\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\varphi_x^P(A)\wedge\tau_P(B))\Big)\\
&=T_1(X,\tau)\longrightarrow T_3^P(X,\tau),
\end{aligned}
$$

so that $T_4^P(X,\tau)*T_1(X,\tau)\le T_3^P(X,\tau)$ (Indeed, put $T_1(X,\tau)=\alpha$, $j=(x,D)$, $J=\{(x,D)|\ x\in X,\ D\in 2^X,\ x\notin D\}$, $B_{(x,D)}=F_\tau(D)$, $M_{(x,D)}=\bigvee_{D\subseteq B,\ A\cap B=\emptyset}(\varphi_x^P(A)\wedge\tau_P(B))$ and $A_j=\{\lambda|\ \lambda*\alpha\le(B_j\longrightarrow M_j)\}$. Then

$$
\begin{aligned}
\bigwedge_{j\in J}((\alpha*B_j)\longrightarrow M_j)&=\bigwedge_{j\in J}\ \bigvee_{\lambda*(\alpha*B_j)\le M_j}\lambda\\
&=\bigwedge_{j\in J}\ \bigvee_{(\lambda*\alpha)*B_j\le M_j}\lambda\\
&=\bigwedge_{j\in J}\ \bigvee_{\lambda*\alpha\le(B_j\longrightarrow M_j)}\lambda\\
&=\bigvee_{f\in\prod_{j\in J}A_j}\ \bigwedge_{j\in J}f(j).
\end{aligned}
$$

Now, $\forall\ f\in\prod_{j\in J}A_j$, there exists $K_f=\bigwedge_{j\in J}f(j)$ such that $K_f*\alpha=(\bigwedge_{j\in J}f(j))*\alpha\le\bigwedge_{j\in J}(f(j)*\alpha)\le\bigwedge_{j\in J}(B_j\longrightarrow M_j)$. Then

$$
\begin{aligned}
\bigwedge_{j\in J}((\alpha*B_j)\longrightarrow M_j)&\le\bigvee_{f\in\prod_{j\in J}A_j}K_f\\
&\le\bigvee_{\lambda*\alpha\le\bigwedge_{j\in J}(B_j\longrightarrow M_j)}\lambda\\
&=\alpha\longrightarrow\bigwedge_{j\in J}(B_j\longrightarrow M_j).).\quad\square
\end{aligned}
$$

Theorem 6.7. If L satisfies the completely distributive law, then for any $(X,\tau)\in\Omega$, we have $T_4^{PS}(X,\tau)*T_1^P(X,\tau)\le T_3^{PS}(X,\tau)$.

Proof. The proof similar to the proof of Theorem 6.6. □

7. Conclusion

(1) Let $L = [0,1]$ and let $* \in [0,1]^{([0,1]\times[0,1])}$ is defined as follows:
$\alpha * \beta = max(0, \alpha + \beta - 1)$, then the structure $(L, \vee, \wedge, *, \longrightarrow, 0, 1)$ is a completely distributive complete MV-algebra so that "Theorem 3.1 (1) [1], Theorem 3.2 (1) [1]" (resp. "Theorem 3.3 (1) (a), (2) (a) [1]", Theorem 4.1 [1], Theorem 4.2 [1], Theorem 5.1 (1) [1], Theorem 3.1 [2], Theorem 3.3 [2] Theorem 3.4 [2], Theorem 3.5 [2], Theorem 3.7 [2], Theorem 3.13 [2], Theorem 3.14 [2], Theorem 3.15 (1), (2) [2], Theorem 4.1 [2], Theorem 4.2 [2], Theorem 4.3 [2], Theorem 4.4 [2]) is obtained as a special case of Theorem 3.6 (resp. Theorem 3.9, Theorem 3.10, Proposition 3.8, Proposition 4.4, Theorem 5.4, Theorem 5.17, Theorem 5.16, Theorem 5.23, "Theorems 5.8, 5.10", Theorem 5.26, Theorem 5.27, Theorem 5.28, Theorem 6.2, Theorem 6.6, Theorem 6.3, Theorem 6.7) above.

(2) Let $L = [0,1]$ and let $* \in [0,1]^{([0,1]\times[0,1])}$ is defined as follows: $\alpha * \beta = \alpha\beta$. Then $(L, \vee, \wedge, *, \longrightarrow, 0, 1)$ is a completely distributive complete residuated lattice. Note that the double negation law is not satisfied since $(\alpha \longrightarrow 0) \longrightarrow 0 = 0 \longrightarrow 0 = 1 \neq \alpha$ if $\alpha \in (0,1)$. Hence, Propositions 3.8, 4.4 (2)–(7), (9), Theorems 3.6, 3.9, 3.10, 4.2 (1), 5.4, 5.7, 5.8, 5.10, 5.11, 5.13, 5.14, 5.16, 5.20, 5.21, 5.22, 5.23, 6.2, 6.3, 6.6, 6.7 are satisfied as corollaries from our results.

(3) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete MV-algebra, then Propositions 3.8, 4.4 (2)–(7), (9), Theorems 3.6, 3.9, 4.2 (1), 5.4, 5.7, 5.8, 5.10, 5.11, 5.13, 5.14, 5.17 are satisfied as corollaries from our results because from Corollary (1) [20], any complete MV-algebra, is a complete residuated lattice. Furthermore any complete $MV-$algebra satisfies the double negation law.

(4) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete MV-algebra, such that the " $\wedge$ " is distributive over arbitrary joins, then Propositions 3.8, 4.4 (2)–(7), (9), Theorems 3.6, 3.9, 4.2 (1), 5.4, 5.7, 5.8, 5.10, 5.11, 5.13, 5.14, 5.17, 5.21, 5.23, 5.24 are satisfied as corollaries from our results.

(5) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a completely distributive complete $MV-$algebra, then all results in Section 3, 4, 5 and 6 are satisfied as corollaries from our results.

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Fathei M. Zeyada
Research Area: Fuzzy topology, general topology
E-mail : zcomatcs@yahoo.com

M. Azab Abd-Allah
Research Area: Fuzzy topology, General topology
E-mail : mazab57@yahoo.com

A.K. Mousa
Department of Mathematics
Faculty of Science
Al-Azhar University
Assiut 71524
Egypt Research Area: Fuzzy topology, General topology
E-mail : $a_k_$musa@yahoo.com