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Note on Fuzzy Random Renewal Process and Renewal Rewards Process

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Abstract

Recently, Zhao et al. [Fuzzy Optimization and Decision Making (2007) 6, 279-295] characterized the interarrival times as fuzzy random variables and presented a fuzzy random elementary renewal theorem on the limit value of the expected renewal rate of the process in the fuzzy random renewal process. They also depicted both the interarrival times and rewards are depicted as fuzzy random variables and provided fuzzy random renewal reward theorem on the limit value of the long run expected reward per unit time in the fuzzy random renewal reward process. In this note, we simplify the proofs of two main results of the paper.

Key Words : Renewal process · Renewal reward process · Fuzzy random variable · Fuzzy variable · Stochastic renewal process

1. Introduction

The theory of fuzzy sets introduced by Zadeh [8] has been extensively studied and applied in statistics and probability areas in recent years. But there are only a few papers investigating the renewal process in fuzzy environments. Popova and Wu [6] considered a renewal rewards process with random inter-arrival times and fuzzy random rewards. Hwang [2] considered a renewal process having inter-arrival times which are fuzzy random variables and proved a theorem for the rate of a renewal process having inter-arrival times which are fuzzy random variables. Zhao and Liu [9] proved a "fuzzy elementary renewal theorem" which showed that the expected number of renewals per unit time is just the expected reciprocal of the inter-arrival time, and fuzzy renewal rewards theorem" which showed that the expected reward per unit time is simply the expected value of the ratio of the reward spent in one cycle to the length of the cycle under the assumption that both inter-arrival times and rewards are continuous fuzzy variables. Recently, Zhao et al.[11] presented the interarrival times are characterized as fuzzy random variables and a fuzzy random elementary renewal theorem on the limit value of the expected renewal rate of the process in the fuzzy random renewal process. In this note, we simplify the proofs of two main results of the paper.

2. Fuzzy variables

To begin with, we recall some concepts and results on fuzzy variables. Let ξ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta)$,Cr), where Θ is a universe, P(Θ) is the power set of Θ and Cr is a credibility measure defined on P(Θ).

Definition 2.1. [10] Let ξ be a fuzzy variable on the credibility space . Then its membership function is derived from the credibility measure by

$$\mu_\xi(x) = (2\text{Cr}\{\xi = x\}) \wedge 1, x \in \mathcal{R}.$$

Definition 2.2. [4] Let ξ be a fuzzy variable on the credibility space $(\Theta, \mathcal{P}(\Theta)$,Cr), and $\alpha \in (0, 1]$. Then

$$\xi'_\alpha = \inf\{x|\mu(x) \geq \alpha\} \quad \text{and} \quad \xi''_\alpha = \sup\{x|\mu(x) \geq \alpha\}$$

are called the α-pessimistic value and the α-optimistic value of ξ, respectively.

Definition 2.3. [4] Let ξ be a fuzzy variable on the credibility space $(\Theta, \mathcal{P}(\Theta)$,Cr). The expected value $E[\xi]$ is defined as

$$E[\xi] = \int_0^\infty \text{Cr}\{\xi \geq r\}dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite. Especially, if ξ is a nonnegative fuzzy variable (i.e., $\text{Cr}\{\xi < 0\} = 0$), then $E[\xi] = \int_0^\infty \text{Cr}\{\xi \geq r\}dr$.

Proposition 2.4. [4] Let ξ be a fuzzy variable with finite expected value $E[\xi]$, then we have

$$E[\xi] = \frac{1}{2}\int_0^1 [\xi'_\alpha + \xi''_\alpha]d\alpha$$

where ξ'_α and ξ''_α are the α-pessimistic value and the α-optimistic value of ξ , respectively.

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3. Fuzzy random variables

We assume that $(\Omega, \mathcal{A}, \mathrm{Pr})$ is a probability space. Let $\mathcal{F}$ be a collection of fuzzy variables defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$.

Definition 3.1. [5] A fuzzy random variable is a function $\xi : \Omega \to \mathcal{F}$ such that for any Borel set B of $\mathcal{R}$, $\mathrm{Cr}\{\xi(\omega) \in B\}$ is a measurable function of ω.

Definition 3.2. [1] Let ξ be a fuzzy random variable, and B a Borel set of $\mathcal{R}$. Then the chance of fuzzy random event $\xi \in B$ is a function from $(0, 1]$ to $[0, 1]$, defined as

$$\mathrm{Ch}\{\xi \in B\}(\alpha) = \sup_{\mathrm{Pr}\{A\} \geq \alpha} \inf_{\omega \in A} \mathrm{Cr}\{\xi(\omega) \in B\}.$$

Definition 3.3. A fuzzy random variable ξ is said to be positive if and only if $\mathrm{Cr}\{\xi(\omega) \leq 0\} = 0$, for any $\omega \in \Omega$.

Definition 3.4. [5] Let ξ be a fuzzy random variable defined on the probability space $(\Omega, \mathcal{A}, \mathrm{Pr})$. Then its expected value is defined by

$$\begin{aligned} E[\xi] &= \int_\Omega \left[\int_0^\infty \mathrm{Cr}\{\xi(\omega) \geq r\} dr \right. \\ &- \left. \int_{-\infty}^0 \mathrm{Cr}\{\xi(\omega) \leq r\} dr \right] \mathrm{P}(d\omega) \end{aligned}$$

provided that at least one of the two integrals is finite. Especially, if ξ is a positive fuzzy random variable, then $E[\xi] = \int_\Omega \int_0^\infty \mathrm{Cr}\{\xi(\omega) \geq r\} dr \mathrm{P}(d\omega)$.

Definition 3.5. [11] The fuzzy random variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

(a) $\xi_1(\omega), \xi_2(\omega), \cdots, \xi_n(\omega)$ are independent fuzzy variables for each ω;

(b) $E[\xi_1(\omega)], E[\xi_2(\omega)], \cdots, E[\xi_n(\omega)]$ are independent random variables.

Definition 3.6. [11] The fuzzy random variables ξ and η are identically distributed if

$$\mathrm{Ch}\{\xi \in B\}(\alpha) = \mathrm{Ch}\{\eta \in B\}(\alpha)$$

for any $\alpha \in (0, 1]$ and Borel set B of real numbers.

Proposition 3.7. [5] If $\xi_1, \xi_2, \cdots, \xi_n$ are iid fuzzy random variables defined on the probability space $(\Omega, \mathcal{A}, \mathrm{Pr})$, then, for each r , we have

(i) $\mathrm{Cr}\{\xi(\omega) \geq r\}, i = 1, 2, \cdots, n$ are iid random variables;

(ii) $\mu_{\xi_i(\omega)}(r), i = 1, 2, \cdots, n$ are iid random variables;

(ii) $E[\xi_i(\omega)], i = 1, 2, \cdots, n$ are iid random variables.

The assertion (i) still holds if the symbol "$\geq$" is replaced with "$\leq$","$<$", or "$>$".

4. Fuzzy random renewal process

In stochastic process theory, the interarrival time between two events is usually assumed to be a random variable, while it is assumed to be a fuzzy variable in fuzzy process theory [9]. However, randomness and fuzziness are often required to be considered simultaneously in the same process. In this section, we will discuss this issue and provide some useful theorems.

Let ξ_n denote the interarrival times between the $(n-1)$th and nth events, $n = 1, 2, \cdots$, respectively. Define $S_0 = 0$ and

$$S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \forall n \geq 1.$$

If the interarrival times are fuzzy random variables on the probability space $(\Omega, \mathcal{A}, \mathrm{Pr})$, then the process $\{S_n, n \geq 1\}$ is called a fuzzy random process.

Let $N(t)$ denote the total number of events that have occurred by time t. Then we have

$$N(t) = \max_{n \geq 0} \{n \mid 0 < S_n \leq t\}.$$

Note that $N(t)(\omega) = n$ implies that $S_n(\omega) \leq t < S_{n+1}(\omega), \forall \omega \in \Omega$. thus $N(t)(\omega)$ is a fuzzy variable with membership function

$$\mu_{N(t)(\omega)}(n) = 2(\mathrm{Cr}\{S_n(\omega) \leq t < S_{n+1}(\omega)\}) \wedge 1, n = 0, 1, 2, \cdots$$

By the measurability of the right side of (9), it follows that $N(t)$ is a fuzzy random variable. We call $N(t)$ the *fuzzy random renewal variable*.

For each $\omega \in \Omega$ and $\alpha \in (0, 1]$, the α-pessimistic and α-optimistic values of the fuzzy variables $\xi_i(\omega), S_n(\omega)$ and $N(t)(\omega)$ can be expressed by

$$\begin{aligned} \xi'_{i,\alpha}(\omega) &= \inf\{x \mid \mu_{\xi_i(\omega)}(x) \geq \alpha\}, \\ \xi''_{i,\alpha}(\omega) &= \sup\{x \mid \mu_{\xi_i(\omega)}(x) \geq \alpha\}, \\ S'_{n,\alpha}(\omega) &= \inf\{t \mid \mu_{S_n(\omega)}(t) \geq \alpha\}, \\ S''_{n,\alpha}(\omega) &= \sup\{t \mid \mu_{S_n(\omega)}(t) \geq \alpha\}, \\ N(t)'_\alpha(\omega) &= \inf\{n \mid \mu_{N(t)(\omega)}(n) \geq \alpha\}, \\ N(t)''_\alpha(\omega) &= \sup\{n \mid \mu_{N(t)(\omega)}(n) \geq \alpha\}. \end{aligned}$$

It is easy to prove that

$$S'_{n,\alpha}(\omega) = \sum_{i=1}^n \xi'_{i,\alpha}(\omega),$$
$$S''_{n,\alpha}(\omega) = \sum_{i=1}^n \xi''_{i,\alpha}(\omega)$$

when ξ_i are independent fuzzy random variables. Moreover, by Hwang [2], we have

$$N(t)'_\alpha(\omega) = \sup\{n \mid S''_{n,\alpha}(\omega) \leq t\},$$
$$N(t)''_\alpha(\omega) = \sup\{n \mid S'_{n,\alpha}(\omega) \leq t\}$$

for each $\alpha \in (0,1]$ and $\omega \in \Omega$. The following result is immediate from the classical law of large numbers and elementary renewal theory.

Theorem 4.1. [11] Let $\{\xi_n, n \geq 1\}$ be a sequence of iid positive fuzzy random interarrival times on probability space $(\Omega, \mathcal{A}, \Pr)$ and $N(t)$ the fuzzy random renewal variable. Then, for each $\alpha \in (0,1]$ and $\omega \in \Omega$, we have

$$\lim_{t\to\infty} \frac{N(t)'_\alpha(\omega)}{t} = \frac{1}{E\left[\xi''_{1,\alpha}\right]}, \qquad a.s.$$

$$\lim_{t\to\infty} \frac{N(t)''_\alpha(\omega)}{t} = \frac{1}{E\left[\xi'_{1,\alpha}\right]}, \qquad a.s.$$

Furthermore,

$$\lim_{t\to\infty} \frac{E\left[N(t)'_\alpha\right]}{t} = \frac{1}{E\left[\xi''_{1,\alpha}\right]},$$

$$\lim_{t\to\infty} \frac{E\left[N(t)''_\alpha\right]}{t} = \frac{1}{E\left[\xi'_{1,\alpha}\right]}.$$

Often, the fuzzy variables characterized by the α-pessimistic value $E\left[\xi'_{1,\alpha}(\omega)\right]$ and the α-optimistic value $E\left[\xi''_{1,\alpha}(\omega)\right]$ are not unique. We assume that ξ is one of the fuzzy variables with the α-pessimistic value $E\left[\xi'_{1,\alpha}(\omega)\right]$ and the α-optimistic value $E\left[\xi''_{1,\alpha}(\omega)\right]$. Zhao et al. [11] have the following theorem. But, here, we simplify the proof.

Theorem 4.2. (Fuzzy Random Elementary Renewal Theorem) Let $\{\xi_n\}$ be a sequence of iid positive fuzzy random interarrival times and $N(t)$ the fuzzy random renewal variable. If $E\left[\frac{1}{\xi}\right]$ are finite , then we have

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = E\left[\frac{1}{\xi}\right].$$

Proof. It follows from Definition 3.4 and Proposition 2.4 that

$$\begin{aligned}
&\frac{E[N(t)]}{t}\\
&= \int_\Omega \int_0^\infty \mathrm{Cr}\left\{\frac{N(t)(\omega)}{t} \geq r\right\} dr P(d\omega) \qquad (1)\\
&= \int_\Omega \int_0^1 \frac{1}{2}\left(\frac{N(t)'_\alpha(\omega)}{t} + \frac{N(t)''_\alpha(\omega)}{t}\right) d\alpha P(d\omega)\\
&= \int_0^1 \frac{1}{2}\left(\frac{E[N(t)'_\alpha]}{t} + \frac{E[N(t)''_\alpha]}{t}\right) d\alpha\\
&\qquad\qquad \text{(by Fubini's theorem)}
\end{aligned}$$

$$\begin{aligned}
E\left[\frac{1}{\xi}\right] &= \int_\Omega \int_0^\infty \mathrm{Cr}\left\{\frac{1}{\xi(\omega)} \geq r\right\} dr P(d\omega) \qquad (2)\\
&= \int_0^1 \frac{1}{2}\left(\frac{1}{E[\xi''_{1,\alpha}]} + \frac{1}{E[\xi'_{1,\alpha}]}\right) d\alpha.
\end{aligned}$$

From Theorem 4.1 we have that

$$\lim_{t\to\infty}\left(\frac{E[N(t)'_\alpha]}{t} + \frac{E[N(t)''_\alpha]}{t}\right) = \frac{1}{E[\xi''_{1,\alpha}]} + \frac{1}{E[\xi'_{1,\alpha}]}.$$

We note by Wald's equation in stochastic sense (see [7]) that since $E[N(t)'_\alpha] < \infty$, $E[N(t)''_\alpha] < \infty$ by classical renewal theory

$$E[\xi''_{1,\alpha}]E[N(t)'_\alpha + 1] = E[S_{N(t)''_\alpha+1}] \leq t + E[\xi''_{1,\alpha}]$$

$$E[\xi'_{1,\alpha}]E[N(t)''_\alpha + 1] = E[S_{N(t)'_\alpha+1}] \leq t + E[\xi'_{1,\alpha}]$$

and hence we have for each $\alpha \in (0,1]$

$$\frac{E[N(t)'_\alpha]}{t} \leq \frac{1}{E[\xi''_{1,\alpha}]}, \quad \frac{E[N(t)''_\alpha]}{t} \leq \frac{1}{E[\xi'_{1,\alpha}]}. \qquad (3)$$

Then applying the dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^1 \left(\frac{E[N(t)'_\alpha]}{t} + \frac{E[N(t)''_\alpha]}{t}\right) d\alpha\\
&\qquad = \int_0^1 \frac{1}{E[\xi''_{1,\alpha}]} + \frac{1}{E[\xi'_{1,\alpha}]} d\alpha.
\end{aligned}$$

Therefore the result follows immediate from (1) and (2). $\square$

5. Fuzzy random renewal reward process

Let $(\xi_1, \eta_1), (\xi_1, \eta_1), \cdots$ be a sequence of pairs of fuzzy random variables on the probability space $(\Omega, \mathcal{A}, \Pr)$. We shall interpret ξ_n as the interarrival time between the $(n-1)$th and nth event and η_n as the reward associated with the nth interarrival time $\xi_n, n = 1, 2, \cdots$, respectively.

Let $C(t)$ denote the total reward earned by time t. Then we have

$$C(t) = \sum_{i=1}^{N(t)} \eta_i,$$

where $N(t)$ is the fuzzy random renewal variable.

For each $\alpha \in (0,1]$ and $\omega \in \Omega$, when η_i are independent fuzzy random variables, it is easy to prove that

$$C(t)'_\alpha(\omega) = \sum_{i=1}^{N(t)'_\alpha(\omega)} \eta'_{1,\alpha}(\omega)$$

$$\text{and} \quad C(t)''_\alpha(\omega) = \sum_{i=1}^{N(t)''_\alpha(\omega)} \eta''_{1,\alpha}(\omega)$$

where $C(t)'_\alpha(\omega), C(t)''_\alpha(\omega), N(t)'_\alpha(\omega), N(t)''_\alpha(\omega), \eta'_{1,\alpha}(\omega)$, and $\eta''_{1,\alpha}(\omega)$ are the α-pessimistic and α-optimistic values of the fuzzy variables $C(t)(\omega), N(t)(\omega)$, and $\eta_i(\omega)$, respectively. Furthermore, Zhao et al. [11] have the following theorem.

Theorem 5.1. [11] Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \cdots$ be a sequence of pairs of iid positive fuzzy random variables, $N(t)$ the fuzzy random renewal variable and $C(t)$ the total reward earned by time t. Then we have

$$\lim_{t\to\infty} \frac{C(t)'_\alpha(\omega)}{t} = \frac{E[\eta'_{1,\alpha}]}{E[\xi''_{1,\alpha}]}, \quad a.s.$$
$$\lim_{t\to\infty} \frac{C(t)''_\alpha(\omega)}{t} = \frac{E[\eta''_{1,\alpha}]}{E[\xi'_{1,\alpha}]}, \quad a.s.$$
$$\lim_{t\to\infty} \frac{E[C(t)'_\alpha]}{t} = \frac{E[\eta'_{1,\alpha}]}{E[\xi''_{1,\alpha}]},$$
$$\lim_{t\to\infty} \frac{E[C(t)''_\alpha]}{t} = \frac{E[\eta''_{1,\alpha}]}{E[\xi'_{1,\alpha}]}.$$

Similar to the case in the fuzzy random renewal process, we assume that ξ is one of fuzzy variables with the α-pessimistic value $E\left[\xi'_{1,\alpha}\right]$ and the α-optimistic value $E\left[\xi''_{1,\alpha}\right]$ and η one of fuzzy variables with the α-pessimistic value $E\left[\eta'_{1,\alpha}\right]$ and the α-optimistic value $E\left[\eta''_{1,\alpha}\right]$. Zhao et al. [11] have the following theorem. But, here, we simplify the proof.

Theorem 5.2. (Fuzzy Random Renewal Reward Theorem) Let $(\xi_1, \eta_1), (\xi_1, \eta_1), \cdots$ be a sequence of pairs of iid positive fuzzy random variables, $N(t)$ the fuzzy random renewal variable and $C(t)$ the total reward earned by time t. If $E\left[\frac{\eta}{\xi}\right]$ are finite, then we have

$$\lim_{t\to\infty} \frac{E[C(t)]}{t} = E\left[\frac{\eta}{\xi}\right].$$

Proof. By Definition 3.4 and Proposition 2.4, we have

$$\begin{aligned} \frac{E[C(t)]}{t} &= \int_\Omega \int_0^\infty \mathrm{Cr}\left\{\frac{C(t)(\omega)}{t} \geq r\right\} dr P(d\omega) \\ &= \frac{1}{2}\int_0^1 \left(\frac{E[C(t)'_\alpha]}{t} + \frac{E[C(t)''_\alpha]}{t}\right) d\alpha \quad (4) \\ &\quad \text{(by Fubini's theorem)} \end{aligned}$$

$$E\left[\frac{\eta}{\xi}\right] = \frac{1}{2}\int_0^1 \left(\frac{E[\eta'_{1,\alpha}]}{E[\xi''_{1,\alpha}]} + \frac{E[\eta''_{1,\alpha}]}{E[\xi'_{1,\alpha}]}\right) d\alpha \quad (5)$$

From Theorem 5.1 we have that

$$\lim_{t\to\infty} \left(\frac{E[C(t)'_\alpha]}{t} + \frac{E[C(t)''_\alpha]}{t}\right) = \left(\frac{E[\eta'_{1,\alpha}]}{E[\xi''_{1,\alpha}]} + \frac{E[\eta''_{1,\alpha}]}{E[\xi'_{1,\alpha}]}\right).$$

We note by Wald's equation in stochastic sense (see Ross (1996)) that

$$E[\eta'_{1,\alpha}]E[N(t)'_\alpha + 1] = E[C(t)'_\alpha] + 1$$
$$E[\eta''_{1,\alpha}]E[N(t)''_\alpha + 1] = E[C(t)''_\alpha] + 1$$

and hence we have from (3), for each $\alpha \in (0, 1]$, for big t,

$$\frac{E[C(t)'_\alpha]}{t} \leq E[\eta'_{1,\alpha}]\frac{E[N(t)'_\alpha + 1]}{t} \leq \frac{2E[\eta'_{1,\alpha}]}{E[\xi''_{1,\alpha}]},$$

$$\frac{E[C(t)''_\alpha]}{t} \leq E[\eta''_{1,\alpha}]\frac{E[N(t)''_\alpha + 1]}{t} \leq \frac{2E[\eta''_{1,\alpha}]}{E[\xi'_{1,\alpha}]}.$$

Then applying the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{t\to\infty} \int_0^1 \left(\frac{E[C(t)'_\alpha]}{t} + \frac{E[C(t)''_\alpha]}{t}\right) d\alpha \\ = \int_0^1 \left(\frac{E[\eta'_{1,\alpha}]}{E[\xi''_{1,\alpha}]} + \frac{E[\eta''_{1,\alpha}]}{E[\xi'_{1,\alpha}]}\right) d\alpha. \end{aligned}$$

Therefore the result follows immediate from (4) and (5). □

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