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Interval-Valued Fuzzy Relations

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Abstract

By using the notion of interval-valued fuzzy relations, we forms the poset (IVFR $(X), \leq$) of interval-valued fuzzy relations on a given set X. In particular, we forms the subposet (IVFE $(X), \leq$) of interval-valued fuzzy equivalence relations on a given set X and prove that the poset (IVFE$(X), \leq$) is a complete lattice with the least element and greatest element.

Key words : interval-valued fuzzy set [relation, equivalence relation], (complete) lattice

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [11], several researchers were concerned about the generalizations of the notion of fuzzy sets, e.g., fuzzy set of type n [12], intuitionistic fuzzy sets [1] and interval-valued fuzzy sets [3]. The concept of interval-valued fuzzy sets was introduced by Gorzaxczany [3], and recently there has been progress in the study of such sets by several researchers (see [2], [4], [5], [6], [7], [8], [10]). In [5], the topology of interval-valued fuzzy sets(IVF) is defined, and some of its properties are discussed, and then Mondal et al. [6] studied the connectedness in the topology of interval-valued fuzzy sets. Using the concept of interval-valued fuzzy sets, Jun et al. [4] introduced the notions of IVF strongly semiopen (semiclosed) sets, IVF (strong) semi-interior (IVF (strong) semi-closure), IVF strongly semiopen (semiclosed) mapping, and IVF strongly semi-continuous mapping, and then they investigated several properties. In 1992, Roy et al. [9] introduced the concept of interval-valued fuzzy relation and obtained it's fundamental results.

In this paper, by using the notion of interval-valued fuzzy relations, we forms the poset (IVFR $(X), \leq$) of interval-valued fuzzy relations on a given set X. In particular, we forms the subposet (IVFE $(X), \leq$) of interval-valued fuzzy equivalence relations on a given set X and prove that the poset (IVFE$(X), \leq$) is a complete lattice with the least element and greatest element.

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2. Preliminaries

First we shall present the fundamental definitions given by [3-5, 7, 8]:

Let $D(I)$ be the set of all closed subintervals of the unit interval I. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and upper points respectively. Especially, we denote $\mathbf{0} = [0,0], \mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for every $a \in (0,1)$. We also note that

$(i)(\forall M, N \in D(I))\ (M = N \Leftrightarrow M^L = N^L, M^U = N^U)$.

$(ii)(\forall M, N \in D(I))(M \leq N \Leftrightarrow M^L \geq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the complement of M, denoted by M^C, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$.

Definition 2.1 [3]. Let X be a given nonempty set. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an *interval valued fuzzy set* (briefly, *IVFS*) in X, where A^L and A^U are fuzzy sets in X satisfying $A^L(x) \leq A^U(x)$ and $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$.

It is clear that every fuzzy set A in X is an IVFS of the form $A = [A, A]$. For any $[a, b] \in D(I)$, the IVFS whose value is the interval $[a, b]$ for all $x \in X$ is denoted by $[a, b]$. In particular, for any $a \in [0, 1]$, the IVFS whose value is $a = [a, a]$ for all $x \in X$ is denoted by simply $\tilde{a}$. For a point $p \in X$ and for $[a, b] \in D(I)$ with $b > 0$, the IVFS which takes the value $[a, b]$ at p and $\mathbf{0}$ elsewhere in X is called an *interval-valued fuzzy point* (briefly, an *IVF point*) and is denoted by $[a, b]_p$. In particular, if $b = a$, then it is also denoted by a_p. We will denote by $D(I)^X$ or IVF(X) and IVFp(X) the set of all IVFSs and the set of all IVF points in X respectively.

Notation. Let $X = \{x_1, x_2, \cdots, x_n\}$. Then $A = ([a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n])$ denotes an IVFS in X such that $A^L(x_i) = a_i$ and $A^U(x_i) = b_i$, for all $i = 1, 2, \cdots, n$.

Definition 2.2 [3,5]. Let X be a nonempty set and let $A, B \in D(I)^X$. Then :

(a) $A \subset B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.
(b) $A = B$ iff $A \subset B$ and $B \subset A$.
(c) The *complement* A^c of A is defined by $A^c(x) = [1 - A^U(x), 1- \quad A^L(x)]$ for all $x \in X$.
(d) If $\{A_i : i \in J\}$ is an arbitrary subset of $D(I)^X$, then

$$\bigcap A_i(x) = [\bigwedge_{i \in J} A_i^L(x), \bigwedge_{i \in J} A_i^U(x)],$$

$$\bigcup A_i(x) = [\bigvee_{i \in J} A_i^L(x), \bigvee_{i \in J} A_i^U(x)].$$

Definition 2.3 [5]. Let X be a nonempty set and let $A \in D(I)^X$. Then the set $\{x \in X | A^U(x) > 0\}$ is called the *support* of A and denoted by *supp*(A).

Definition 2.4 [5]. Let X be a nonempty set and let $A \in D(I)^X$. Then an IVF point M_x is said to *belong to* A, denoted by $M_x \tilde{\in} A$, if $A^L(x) \geq M^L$ and $A^U(x) \geq M^U$.

It is clear that $A = \bigcup\{M_x : M_x \tilde{\in} A\}$.

Result 2.A [5, Theorem 1]. *Let X be a nonempty set and let $A, B, C, A_i, B_i \in D(I)^X$. Then the following hold :*

(a) $\tilde{0} \subset A \subset \tilde{1}$.
(b) $A \cup B = B \cup A, A \cap B = B \cap A$.
(c) $A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$.
(d) $A, B \subset A \cup B, A \cap B \subset A, B$.
(e) $A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)$.
(f) $A \cup (\bigcap_i B_i) = \bigcap_i (A \cup B_i)$.
(g) $(\tilde{0})^c = \tilde{1}, (\tilde{1})^c = \tilde{0}$.
(h) $(A^c)^c = A$.
(i) $(\bigcup_i A_i)^c = \bigcap_i A_i^c$, $(\bigcap_i A_i)^c = \bigcup_i A_i^c$.

3. Interval-valued fuzzy relations

Definition 3.1 [9]. Let X and Y be nonempty sets. Then each $R = [R^L, R^U] \in D(I)^{X \times Y}$ is called an interval-valued fuzzy relation (briefly, IVFR) from X to Y. For each $(x, y) \in X \times Y$, $R(x, y)$ estimates the interval of the strength of the link between x and y. $R^L(x, y)$ and $R^U(x, y)$ are called the maximum strength and minimum strength of the link between x and y respectively. In particular, each member of $D(I)^{X \times X}$ is called an interval-valued fuzzy relation in X.

Definition 3.2 [9]. Let $R \in D(I)^{X \times Y}$. Then the inverse R^{-1} of R is an IVFR from Y to X such that $R^{-1}(y, x) = R(x, y)$ for each $(y, x) \in Y \times X$. It is clear that $R^{-1L}(y, x) = R^L(x, y)$ and $R^{-1U}(y, x) = R^U(x, y)$.

Definition 3.3. Let $R \in D(I)^{X \times Y}$ and $S \in D(I)^{Y \times Z}$. Then the *composition* $R \circ S$ of S and R is an IVFR from X to Z defined as follows : for each $(x, z) \in X \times Z$,

$R \circ S(x, z)$
$= [\bigvee_{y \in Y} (R^L(x, y) \wedge S^L(y, z),$
$\bigvee_{y \in Y} (R^U(x, y) \wedge S^U(y, z)]$.

The following is the immediate result of Definition 3.2 and 3.3.

Proposition 3.4. Let $R_1, R_2, R_3, Q_1, Q_2 \in D(I)^{X \times X}$. Then :

(a) $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$.
(b) If $R_1 \subset R_2$ and $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_2 \circ Q_2$. In particular, if $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_2 \circ Q_2$.
(c) $R_1 \circ (R_2 \cup R_3) = R_1 \circ R_2 \cup R_1 \circ R_3, R_1 \circ (R_2 \cap R_3) = R_1 \circ R_2 \cap R_1 \circ R_3$
(d) If $R_1 \subset R_2$, then $R_1^{-1} \subset R_2^{-1}$.
(e) $(R^{-1})^{-1} = R, (R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$.
(f) $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}, (R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$.

The following is the immediate result of proposition 3.4 (a).

Corollary 3.4. Let $R, S \in D(I)^{X \times X}$. If $R \circ S = S \circ R$, then

$$(R \circ S) \circ (R \circ S) = (S \circ S) \circ (R \circ R)$$

Definition 3.5 [9]. An IVFR R in X is called an *interval-valued fuzzy equivalence relation* (briefly, *IVFER*) in X if it satisfies the following conditions :

(a) it is *interval-valued fuzzy reflexive* , i.e., $R(x,x) = [1,1]$ for each $x \in X$,

(b) it is *interval-valued fuzzy symmetric* , i.e., $R^{-1} = R$,

(c) it is *interval-valued fuzzy transitive* , i.e., $R \circ R \subset R$.

We will denote the set of all IVFERs in X as IVFE(X). The following is the immediate result of Definition 3.5. and proposition 3.4.

Proposition 3.6. Let $R, S \in$ IVFE(X).

(a) If R is *interval-valued fuzzy reflexive [resp., symmetric, transitive]*, then so is R^{-1} .

(b) If R is *interval-valued fuzzy reflexive [resp., symmetric, transitive]*, then so is $R \circ R$.

(c) If R is *interval-valued fuzzy reflexive*, then $R \subset R \circ R$.

(d) If R is *interval-valued fuzzy symmetric*, then so are $R \cup R^{-1}$, $R \cap R^{-1}$ and $R \circ R^{-1} = R^{-1} \cap R$.

(e) If R and S are *interval-valued fuzzy reflexive [resp., symmetric, transitive]*, then so is $R \cap S$.

(f) If R and S are *interval-valued fuzzy symmetric*, then so is $R \cup S$.

From (a), (b) and (e) of Proposition 3.6, the proofs of the following result are obvious.

Corollary 3.6-1. If $R, S \in$ IVFE(X), then $R^{-1}, R \circ R, R \cap S \in$ IVFE (X).

The following is the immediate result of Definition 2.3 and Proposition 3.6(c).

Corollary 3.6-2. If $R \in$ IVFE(X), then $R \circ R = R$.

Theorem 3.7. *Let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a nonempty family of IVFERs in X. Then $\bigcap_{\alpha\in\Gamma} R_\alpha \in$ IVFE(X). However, in general, $\bigcup_{\alpha\in\Gamma} R_\alpha$ need not be an IVFER in X.*

Proof. Let $x \in X$ and let $R = \bigcap_{\alpha\in\Gamma} R_\alpha$. Then, since each R_α is interval-valued fuzzy reflexive,

$$R^L(x,x) = \bigwedge_{\alpha\in\Gamma} R^L_\alpha(x,x) = 1$$

and

$$R^U(x,x) = \bigwedge_{\alpha\in\Gamma} R^U_\alpha(x,x) = 1.$$

Thus $R(x,x) = [1,1]$. So R is interval-valued fuzzy reflexive. It is clear that R is interval-valued fuzzy symmetric. Now let $(x,z) \in X \times X$. Then

$$\begin{aligned}
&[R \circ R]^L(x,z) \\
&= \bigvee_{y\in Y} [R^L(x,y) \wedge R^L(y,z)] \\
&= \bigvee_{y\in Y} [(\bigwedge_{\alpha\in\Gamma} R^L_\alpha(x,y)) \wedge (\bigwedge_{\alpha\in\Gamma} R^L_\alpha(y,z))] \\
&\le \bigwedge_{\alpha\in\Gamma} (\bigvee_{y\in Y} [R^L_\alpha(x,y) \wedge R^L_\alpha(y,z)]) \\
&= \bigwedge_{\alpha\in\Gamma} (R_\alpha \circ R_\alpha)^L(x,z) \\
&\le \bigwedge_{\alpha\in\Gamma} R^L_\alpha(x,z) [\text{Since} \quad R_\alpha \circ R_\alpha \subset R_\alpha] \\
&= R^L(x,z).
\end{aligned}$$

Similarly, we can see that $(R \circ R)^U(x,z) \le R^U(x,z)$. Thus R is interval-valued transitive. Hence $R = \bigcap_{\alpha\in\Gamma} R_\alpha \in$ IVFE(X). □

Example 3.7. Let $X = \{a,b,c\}$ and let R and S be the IVFRs in X represented by the following matrices, respectively :

R	a	b	c
a	[1, 1]	[0.3, 0.8]	[0.4, 0.9]
b	[0.3, 0.8]	[1, 1]	[0.3, 0.8]
c	[0.4, 0.9]	[0.3, 0.8]	[1, 1]

S	a	b	c
a	[1, 1]	[0.4, 0.9]	[0.5, 0.7]
b	[0.4, 0.9]	[1, 1]	[0.4, 0.7]
c	[0.5, 0.7]	[0.4, 0.7]	[1, 1]

Then clearly $R, S \in$ IVFE(X) and $R \cup S$ is the IVFR in X represented by the following matrix :

$R \cup S$	a	b	c
a	[1, 1]	[0.4, 0.9]	[0.5, 0.9]
b	[0.4, 0.9]	[1, 1]	[0.4, 0.8]
c	[0.5, 0.9]	[0.4, 0.8]	[1, 1]

On the other hand,

$$[(R \cup S) \circ (R \cup S)]^U(b,c) = 0.9 > 0.8 = (R \cup S)^U(b,c).$$

Thus $(R \cup S) \circ (R \cup S) \not\subset R \cup S$. So $R \cup S$ is not interval-valued fuzzy transitive. Hence $R \cup S \notin$ IVFE(X). □

Proposition 3.8. Let R and S be interval-valued fuzzy reflexive relations in a set X. Then $R \circ S$ is interval-valued fuzzy reflexive.

Proof. Let $x \in X$. Then

$$\begin{aligned}(R \circ S)^L(x,x) &= \bigvee_{y \in X} [S^L(x,y) \wedge R^L(y,x)] \\ &\geq S^L(x,x) \wedge R^L(x,x) \text{ [By the hypotheses]} \\ &= 1.\end{aligned}$$

Similarly, we can see that $(R \circ S)^U(x,x) \geq 1$. Thus $(R \circ S)(x,x) = [1,1]$, for each $x \in X$. Hence this completes the proof. □

Proposition 3.9. Let $R, S \in$ IVFE(X). If $R \circ S = S \circ R$, then $R \circ S \in$ IVFE(X).

Proof. By Proposition 3.8, it is clear that $R \circ S$ is interval-valued fuzzy reflexive. Let $x, y \in X$. Then

$$\begin{aligned}(R \circ S)^L(x,y) &= \bigvee_{z \in X} [S^L(x,z) \wedge R^L(z,y)] \\ &= \bigvee_{z \in X} [R^L(y,z) \wedge S^L(z,x)] \\ &\text{[Since } R \text{ and } S \text{ are interval-valued fuzzy symmetric]} \\ &= (S \circ R)^L(y,x) = (R \circ S)^L(y,x). \\ &\text{[Since } R \circ S = S \circ R]\end{aligned}$$

Similarly, we can see that $(R \circ S)^U(x,y) = (R \circ S)^U(y,x)$. Thus $R \circ S$ is interval-valued fuzzy symmetric. On the other hand,

$$\begin{aligned}(R \circ S) \circ (R \circ S) &= (R \circ R) \circ (S \circ S) \quad \text{[By Corollary 3.4]} \\ &\subset R \circ S. \text{[By the hypothesis and Proposition 3.4(b)]}\end{aligned}$$

So $R \circ S$ is interval-valued fuzzy transitive. Hence $R \circ S \in$ IVFE(X). □

Let $R \in$ IVFE(X) and let $a \in X$. We define a mapping $R_a : X \to D(I)$ as follows : for each $x \in X$,

$$R_a(x) = R(a,x).$$

Then clearly R_a is an IVFS in X. In this case, R_a is called an *interval-valued fuzzy equivalence class of R containing a*. The set $\{R_a : a \in X\}$ is called the *interval-valued fuzzy quotient set of X by R* and denoted by X/R.

Theorem 3.10. *Let $R \in IVFE(X)$. Then :*

(a) $R_a = R_b$ if and only if $R(a,b) = [1,1]$ for any $a, b \in X$.

*(b) $R(a,b) = [0,0]$ if and only if $R_a \cap R_b =$ **0** for any $a, b \in X$.*

*(c) $\bigcup_{a \in X} R_a =$ **1**.*

(d) There exists the surjection $\pi : X \to X/R$ (called the natural mapping) defined by $\pi(x) = R_x$ for each $x \in X$.

Proof.(a)($\Rightarrow$) : Suppose $R_a = R_b$. Since R is interval-valued fuzzy reflexive, $R(a,b) = R_a(b) = R_b(b) = R(b,b) = [1,1]$. So $R(a,b) = [1,1]$.

($\Leftarrow$) : Suppose $R(a,b) = [1,1]$. Then $R^U(a,b) = 1$ and $R^L(a,b) = 1$. Let $x \in X$. Then

$$\begin{aligned}R_a^U(x) &= R^U(a,x) \\ &\geq \bigvee_{z \in X} [R^U(a,z) \wedge R^U(z,x)] \\ &\text{[Since R is interval-valued fuzzy transitive]} \\ &\geq R^U(a,b) \wedge R^U(b,x) = 1 \wedge R^U(b,x) \\ &= R^U(b,x) = R_b^U(x).\end{aligned}$$

Similarly, $R_a^L(x) \geq R_b^L(x)$. Thus $R_b \subset R_a$. By the similar arguments, $R_a \subset R_b$. Hence $R_a = R_b$. The proofs of (b), (c) and (d) are easy. This completes the proof. □

4. The interval-valued fuzzy equivalence relation generated by an IVFR

Definition 4.1. Let $R \in$ IVFR(X) and let $\{R_\alpha\}_{\alpha \in \Gamma}$ be the family of all the IVFERs in X containing R. Then $\bigcap_{\alpha \in \Gamma} R_\alpha$ is called the IVFER generated by R and denoted by R^e.

It is clear that R^e is the smallest IVFER containing R.

Definition 4.2. Let R be an IVFR in X. Then the interval-valued fuzzy transitive closure of R, denoted by $\hat{R}$, is defined as follows:

$$\hat{R} = \cup_{n \in \mathbb{N}} R^n, \qquad \text{where} R^n = R \circ R \circ \cdots \circ R,$$

in which R occurs n times.

The following is the immediate result of Definition 3.2.

Proposition 4.3. Let R be an IVFR in X. Then :

(a) $\hat{R}$ is an interval-valued fuzzy transitive relation in X.

(b) If there exists $n \in \mathbb{N}$ such that $R^{n+1} = R^n$, then $\hat{R} = R \cup R^2 \cup \cdots \cup R^n$.

R	a	b	c
a	[0.2, 0.6]	[0, 0.1]	[0.3, 0.7]
b	[0.1, 0.3]	[0.1, 0.5]	[0, 0]
c	[0.4, 0.8]	[0, 0]	[0.5, 0.8]

Then

R^2	a	b	c
a	[0.3, 0.7]	[0, 0.1]	[0.3, 0.7]
b	[0, 0]	[0.1, 0.5]	[0.1, 0.3]
c	[0.4, 0.8]	[0.4, 0.8]	[0.5, 0.8]

R^3	a	b	c
a	[0.3, 0.7]	[0, 0.1]	[0.3, 0.7]
b	[0.1, 0.3]	[0.1, 0.5]	[0.1, 0.3]
c	[0.4, 0.8]	[0.4, 0.8]	[0.5, 0.8]

Thus $R^2 = R^3$. So $\hat{R} = R \cup R^2$. Moreover $\hat{R} \circ \hat{R} \subset \hat{R}$.

$\hat{R}$	a	b	c
a	[0.3, 0.7]	[0, 0.1]	[0.3, 0.7]
b	[0.1, 0.3]	[0.1, 0.5]	[0.1, 0.3]
c	[0.4, 0.8]	[0.4, 0.8]	[0.5, 0.8]

$\hat{R} \circ \hat{R}$	a	b	c
a	[0.3, 0.7]	[0, 0.1]	[0.3, 0.7]
b	[0.1, 0.3]	[0.1, 0.5]	[0.1, 0.3]
c	[0.4, 0.8]	[0.4, 0.8]	[0.5, 0.8]

Hence $\hat{R} = R \cup R^2$ is interval-valued fuzzy transitive. □

The following is the immediate result of (b) and (f) in Proposition 2.6.

Proposition 4.4. If R is interval-valued fuzzy symmetric, then so is $\hat{R}$.

Proposition 4.5. Let R and S be IVFRs in X. Then :

(a) If $R \subset S$, then $\hat{R} \subset \hat{S}$.

(b) If $R \circ S = S \circ R$ and $R, S \in$ IVFE (X), then $\widehat{(R \circ S)} = R \circ S$.

Proof. (a) It is clear from proposition 3.4 (b).
(b) Suppose $R \circ S = S \circ R$ and $R, S \in$ IVFE (X). Then it is clear that $(R \circ S)^1 = R \circ S$. Now suppose $(R \circ S)^k = R \circ S$ for any $k \geq 2$. Then

$$\begin{aligned}(R \circ S)^{k+1} &= (R \circ S)^k \circ (R \circ s) = (R \circ S) \circ (R \circ S) \\ &= (R \circ R) \circ (S \circ S) = R \circ S.\end{aligned}$$

So $(R \circ S)^n = R \circ S$ for any $n \geq 1$. Hence $\widehat{R \circ S} = R \circ S$ □

Definition 4.6 We define two mappings $\triangle, \bigtriangledown : X \to D(I)$ as follows : for any $x, y \in X$,

$$\triangle(x,y) = \begin{cases} [1,1] & \text{if } x = y, \\ [0,0] & \text{if } x \neq y \end{cases}$$

and

$$\bigtriangledown(x,y) = [1,1].$$

It is clear that $\triangle, \bigtriangledown \in$ IVFE(X) and R is an interval-valued fuzzy reflexive relation in X if and only if $\triangle \subset R$.

Theorem 4.7. *If R is an IVFR in X, then $R^e = R \cup \widehat{R^{-1}} \cup 1$.*

Proof. Let $S = R \cup \widehat{R^{-1}} \cup \triangle$. Then clearly $R \subset S$. By Proposition 4.3 (a), S is interval-valued fuzzy transitive. Let $x \in X$. Since $\triangle \subset S$,

$$S^L(x,x) \geq \triangle^L(x,x) = 1$$

and

$$S^U(x,x) \geq \triangle^U(x,x) = 1.$$

Then $S(x,x) = [1.1]$. So S is interval-valued fuzzy reflexive. It is clear that $R \cup R^{-1} \cup \triangle$ is interval-valued fuzzy symmetric. Thus, by Proposition 4.4, S is interval-valued fuzzy symmetric. So $S \in$ IVFE (X). Now let $K \in$ IVFE (X) such that $R \subset K$. Then $\triangle \subset K$ and $R^{-1} \subset K^{-1} = K$ by Proposition 3.4(d). Thus $R \cup R^{-1} \cup \triangle \subset K$. By Proposition 3.4 (b), $[R \cup R^{-1} \cup \triangle]^n \subset K^n = K$ for any $n \geq 1$. So $S \subset K$. Hence $R^e = S$. This completes the proof. □

Proposition 4.8. Let $R, S \in$ IVFE(X). We define $R \vee S$ as follows :

$$R \vee S = \widehat{R \cup S}.$$

Then $R \vee S \in$ IVFE (X).

Proof. By Proposition 4.3 (a), $R \vee S$ is interval-valued fuzzy transitive. Since R and S are interval-valued fuzzy symmetric, $R \cup S$ is interval-valued fuzzy symmetric by Proposition 3.6 (f). Thus, by Proposition 4.4,$R \vee S = \widehat{R \cup S}$ is interval-valued fuzzy symmetric. Let $x \in X$. Then

$$\begin{aligned}&(R \vee S)(x,x) \\ &= [\bigvee_{n \in \mathbb{N}} [R^L(x,x) \vee S^L(x,x)]^n, \bigvee_{n \in \mathbb{N}} [R^U(x,x) \vee S^U(x,x)]^n] \\ &= [\bigvee_{n \in \mathbb{N}} (1 \vee 1)^n, \bigvee_{n \in \mathbb{N}} (1 \vee 1)^n] \\ &\quad [\text{Since } R \text{ and } S \text{ are interval-valued fuzzy reflexive }] \\ &= [1,1].\end{aligned}$$

So $R \vee S$ is interval-valued fuzzy reflexive. Hence $R \vee S \in$ IVFE (X). □

The following result gives another description for $R \vee S$ of two IVFERs R and S.

Proposition 4.9. Let $R, S \in$ IVFE (X). If $R \circ S \in$IVFE (X), then $R \circ S = R \vee S$, where $R \vee S$ denotes the least upper bound for $\{R, S\}$ with respect to the inclusion.

Proof. Let $x, y \in X$. Then

$$(R \circ S)^L(x,y) = \bigvee_{z \in X} [S^L(x,z) \wedge R^L(z,y)]$$
$$\geq S^L(x,y) \wedge R^L(y,y)$$
$$= S^L(x,y) \wedge 1$$
[Since$R$is interval-valued fuzzy reflexive]
$$= S^L(x,y).$$

Similarly, we can see that $(R \circ S)^U(x,y) \geq S^U(x,y)$. Thus $S \subset R \circ S$. Also, by the similar method, $R \subset R \circ S$. So $R \circ S$ is an upper bound for $\{R, S\}$ with respect to "$\subset$".

Now let $P \in$ IVFE(X) such that $R \subset P$ and $S \subset P$. Let $x, y \in X$. Then

$$(R \circ S)^L(x,y)$$
$$= \bigvee_{z \in X} [S^L(x,z) \wedge R^L(z,y)]$$
$$\leq \bigvee_{z \in X} [P^L(x,z) \wedge P^L(z,y)]$$
$$= (P \circ P)^L(x,y)$$
$$\leq P^L(x,y).$$
[Since$P$is interval-valued fuzzy transitive]

Similarly, we can see that $(R \circ S)^U(x,y) \leq P^U(x,y)$. Thus $R \circ S \subset P$. So $R \circ S$ is the least upper bound for $\{R, S\}$ with respect to "$\subset$".
Hence $R \circ S = R \vee S$.

Proposition 4.10. If $R, S \in$IVFE (X) such that $R \circ S = S \circ R$, then $R \vee S = \widehat{R \circ S}$.

Proof. Suppose $R, S \in$ IVFE (X). Then, by Theorem 4.7,

$$(R \cup S)^e = (R \cup S) \cup \widehat{(R \cup S)^{-1}} \cup \triangle.$$

Since $R, S \in$ IVFE (X), $(R \cup S) \cup (R \cup S)^{-1} \cup \triangle = R \cup S$. By Result 2. A(d), it is clear that $R \subset R \cup S$ and $S \subset R \cup S$. Thus

$$R \circ S \subset (R \cup S) \circ (R \cup S)$$ [By Proposition 3.4 (b)]
$$= R \cup S.$$
[By Proposition 3.4 (c) and the hypothesis]

By Proposition 4.5 (a), $\widehat{R \circ S} \subset \widehat{R \cup S}$. On the other hand, since $R, S \in$ IVFE (X), by Proposition 4.9, $R \subset R \circ S$ and $S \subset R \circ S$. Then $R \cup S \subset R \circ S$. Thus, by Proposition 3.5 (a), $\widehat{R \cup S} \subset \widehat{R \circ S}$. So $\widehat{R \circ S} = \widehat{R \cup S}$. Hence $R \vee S = (R \cup S)^e = \widehat{R \cup S} = \widehat{R \circ S}$. □

The following is the immediate result of Proposition 4.10 and Proposition 4.5 (b).

Corollary 4.10. If $R, S \in$ IVFE (X) such that $R \circ S = S \circ R$, then $R \vee S = R \circ S$.

For a set X, it is clear that IVFE(X) is a poset with respect to the inclusion "$\subset$". Moreover, for any $R, S \in$ IVFE (X), $R \cap S$ is the greatest lower bound for R and S in (IVFE (X), $\subset$).
Now, we define two binary operators $\vee$ and $\wedge$ an IVFE(X) as follows : for any $R, S \in$ IVFE (X),

$$R \wedge S = R \cap S \quad \text{and} \quad R \vee S = (R \cup S)^e.$$

Then we obtain the following result from Proposition 3.7, Definition 4.6, Proposition 4.8 and Theorem 4.10.

Theorem 4.11. *(IVFE (X), $\vee$, $\wedge$) is a complete lattice with the least element $\triangle$ and the greatest element $\bigtriangledown$.*

References

1. K. Atanassov, "Intuitionistic fuzzy Sets", *Fuzzy Sets and Systems*, 20, 87-96, 1986.
2. R. Biawas, "Rosenfeld's fuzzy subgroups with interval-valued membership functions" , *Fuzzy Sets and Systems*, 63, 87-90, 1994.
3. M.B Gorzalzany, "A method of inference in approximate reasoning based on interval-valued fuzzy sets", *Fuzzy Sets and Systems*, 21, 1-17, 1987.
4. Y. B. Jun, J. J. Bae, S. H. Cho and C. S. Kim, "Interval-valued fuzzy strong semi-openness and interval-valued fuzzy strong semi-continuity", *Honam Math. J.*, 28, no. 3, 417-431, 2006.
5. T. K. Mondal and S. K. Samanta, "Topology of interval fuzzy sets", *Indian J. Pure Appl. Math.*, 30, no. 1, 23-38, 1999.
6. T. K. Mondal and S. K. Samanta, "Connectedness in topology of interval-valued fuzzy sets", *Italian J. Pure Appl. Math.*, 18, 33-50, 2005.
7. J. H. Park, J. S. Park and Y. C. Kwun, "On fuzzy inclusion in the interval-valued sense", *FSKG 2005, LANI 3613, Springer-Verlag*, 1-10, 2005.

8. P. V. Ramakrishnan and V. Lakshmana Gomathi Nayagam, "Hausdorff interval-valued fuzzy filters", *J. Korean Math. Soc.*, 39, no. 1, 137-148, 2002.
9. M. K. Roy and R. Biswas, "I-v fuzzy relations and Sanchez's approach for medical diagnosis", *Fuzzy Sets and Systems*, 47, 35-38, 1992.
10. W. Zeng and Y. Shi, "Note on interval-valued fuzzy set", *FSKG 2005, LANI 3613, Springer-Verlag*, 20-25, 2005.
11. L. A. Zadeh, "Fuzzy sets", *Inform. Control*, 8, 338-353, 1965.
12. L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning I", *Inform, Sci.*, 8, 199-249, 1975.

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