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# 퍼지 일반위상 공간에 관한 연구

# Fuzzy Generalized Topological Spaces

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**요 약**

본 논문에서는 퍼지 일반-위상과 퍼지 일반-위상 공간의 개념을 소개한다. 퍼지 일반-위상은 smooth topology 와 Chang's fuzzy topology의 일반화된 개념이다. 퍼지 일반-위상의 일반적인 성질과 퍼지 일반-연속, 약 퍼지 일반-연속 함수의 개념과 성질을 조사한다.

**Abstract**

In this paper, we introduce the concept of fuzzy generalized topologies which are generalizations of smooth topologies and Chang's fuzzy topologies and obtain some basic properties of their structure. Also we introduce and study the concepts of fuzzy generalized continuity and weakly fuzzy generalized continuity.

**Key Words** : fuzzy generalized topological spaces, smooth topology, fuzzy generalized continuous, weakly fuzzy generalized continuous.

## 1. Introduction and Preliminaries

Let $X$ be a set and $I=[0,1]$. Let $I^X$ denote the set of all mapping $A: X\to I$. A member of $I^X$ is called a *fuzzy subset* [3] of $X$. $0_X$ and $1_X$ will denote the characteristic functions of $\varnothing$ and $X$, respectively. Let $X$ be a set and $I=[0,1]$. Let $I^X$ denote the set of all mapping $A: X\to I$. A member of $I^X$ is called a fuzzy subset of $X$. And unions and intersections of fuzzy sets are denoted by $\vee$ and $\wedge$, respectively, and defined by

$$\vee A_i = \sup\{A_i(x) \mid i\in J \text{ and } x\in X\},$$

$$\wedge A_i = \inf\{A_i(x) \mid i\in J \text{ and } x\in X\}.$$

A Chang's fuzzy topological space [1] is an ordered pair $(X, \tau)$ is a non-empty set $X$ and $\tau\subseteq I^X$ satisfying the following conditions:

(O1) $0_X, 1_X \in \tau$.
(O2) If $A, B \in \tau$, then $A\wedge B \in \tau$.
(O3) If $A_i \in \tau$, for all $i \in J$, then $\vee A_i \in \tau$.

$(X, \tau)$ is called a *fuzzy topological space*. Members of $\tau$ are called fuzzy open sets in $(X,\tau)$ and complement of a *fuzzy open set* is called a *fuzzy closed* set.

A *smooth topological space* [2] is an ordered pair $(X, \tau)$, where $X$ is a non-empty set and $\tau: I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X)=\tau(1_X)=1$.
(O2) $\tau(A_1\wedge A_2)\geq \tau(A_1)\wedge \tau(A_2)$ for $A_1, A_2\in I^X$.
(O3) $\tau(\vee A_i)\geq \wedge\, \tau(A_i)$ for $A_i\in I^X$.

Then $\tau: I^X \to I$ is called a *smooth topology* on $X$. The number $\tau(A)$ is called the *degree of openness* of $A$.

A mapping $\tau^*: I^X\to I$ is called a *smooth cotopology* [2] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X)=\tau^*(1_X)=1$.
(C2) $\tau^*(A_1\vee A_2)\geq \tau^*(A_1)\wedge\tau^*(A_2)$ for $A_1, A_2\in I^X$.
(C3) $\tau^*(\wedge A_i)\geq \wedge\,\tau^*(A_i)$ for $A_i\in I^X$.

Let $f$ be a mapping from a set $X$ into a set $Y$. Let $A$ and $B$ be respectively the fuzzy sets of $X$ and $Y$. Then $f(A)$ is a fuzzy set in $Y$, defined by

$$f(A)(y) = \begin{cases} \sup A(z)_{z\in f^{-1}(y)}, & \text{if } f^{-1}(y)\neq 0,\ y\in Y \\ 0, & \text{otherwise}, \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in $X$, defined by $f^{-1}(B)(x)=B(f(x)),\ x\in X$.

## 2. Fuzzy generalized topological spaces

**Definition 2.1.** A fuzzy generalized topological space (simply, FGTS) is an ordered pair $(X, \tau)$, where $X$ is a non-empty set and $\tau: I^X\to I$ is a mapping satisfying the following conditions:

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(GO1) $\tau(0_X)=1$.

(GO2) $\tau(\vee A_i) \geq \wedge \tau(A_i)$ for $A_i \in I^X$.

Then the mapping $\tau : I^X \to I$ is called a *fuzzy generalized topology* on $X$. The number $\tau(A)$ is called the *degree of generalized openness* of $A$.

Chang's fuzzy topology ⇒ smooth topology ⇒ fuzzy generalized topology

**Example 2.2.** Let $X=[0,1]$. For each $n \in N$, define a fuzzy set

$$A_n(x) = \frac{n}{n+1}x, \quad \text{for } x \in I ;$$
$$B(x) = x, \quad \text{for } x \in I .$$

Consider a fuzzy generalized topology $T : I^X \to I$ defined as follows

$$T(A)=\begin{cases} 1, & \text{if } A = 0_X, B, \\ \dfrac{n}{n+1}, & \text{if } A = A_n, \\ 0, & otherwise; \end{cases}$$

It is not a smooth topology.

**Definition 2.3.** A mapping $T^*: I^X \to I$ is called a *fuzzy generalized cotopology* if the following three conditions are satisfied:

(GC1) $T^*(1_X)=1$.

(GC2) $T^*(\wedge A_i) \geq \wedge T^*(A_i)$ for $A_i \in I^X$.

Then $T^*(A)$ is called the *degree of generalized closedness* of $A$.

**Theorem 2.4.** If $T$ is a fuzzy generalized topology on $X$, then the mapping $T^*: I^X \to I$ defined by $T^*(A)= T(1_X-A)$ is a fuzzy generalized cotopology on $X$.

Proof. From $T^*(1_X)=T(1_X-1_X)=T(0_X)=1$, we have the condition (GC1).

For every subfamily $\{A_i : i \in J\} \subseteq I^X$,

$$\begin{aligned} T^*(\wedge A_i) &= T(1_X-(\wedge A_i)) \\ &= T(\vee(1_X-A_i)) \\ &\geq \wedge T(1_X-A_i) \\ &= \wedge T^*(A_i). \end{aligned}$$

Thus we have (GC2).

Similarly, we have the next theorem:

**Theorem 2.5.** If $T^*$ is a fuzzy generalized cotopology on a nonempty set $X$, then the mapping $T: I^X \to I$ defined by $T(A)=T^*(1_X-A)$ is a fuzzy generalized topology on $X$.

**Definition 2.6.** Given a set $X$ and fuzzy generalized topologies $T_1$ and $T_2$ on $X$. We say that $T_1$ is *finer* than $T_2$ or $T_2$ is *coarser* than $T_1$ (denoted by $T_1 > T_2$) is $T_1(A) \geq T_2(A)$ for every $A \in I^X$.

**Theorem 2.7.** Let $\{T_k : k \in K\}$ be family of fuzzy generalized topologies on $X$. Then $T=\wedge T_k$ is a fuzzy generalized topology on $X$ for $k \in K$, where $(\wedge T_k)(A)=\wedge T_k(A)$.

Proof. (GO1) Obvious.

(GO2) For $j \in J$ and $k \in K$,

$$\begin{aligned} T(\vee A_j) &= \wedge T_k(\vee A_j) \\ &\geq \wedge(\wedge(T_k(A_j))) \\ &= \wedge(\wedge T_k(A_j)) \\ &= \wedge(T(A_j)). \end{aligned}$$

**Theorem 2.8.** Let $(X,T)$ be a fuzzy generalized topological space. Set $T_\alpha=\{A \in I^X : T(A) \geq \alpha\}$. Then

(1) $0_X \in T_\alpha$;

(2) If $A_i \in T_\alpha$ for each $i \in J$, then $\vee A_i \in T_\alpha$.

Proof. (1) Obvious.

(2) Let $A_i \in T_\alpha$ for each $i \in J$; then

$$T(\vee A_i) \geq \wedge T(A_i) \geq \alpha.$$

This implies that $\vee A_i \in T_\alpha$.

**Remark 2.9.** Let $(X,T)$ be a fuzzy generalized topological space. We call $T_\alpha=\{A \in I^X : T(A) \geq \alpha\}$ a fuzzy *α-level generalized topology* on $X$. From Theorem 2.8, it follows that every Chang's fuzzy topology is a fuzzy $\alpha$-level generalized topology $T_\alpha$ on $X$.

**Theorem 2.10.** Let $\{T_\alpha : \alpha \in (0,1]\}$ be a family of fuzzy $\alpha$-level generalized topologies on $X$ such that $\alpha \leq \beta$ implies $T_\alpha \geq T_\beta$. Then $T(A)=\vee\{\alpha : A \in T_\alpha\}$ is a fuzzy generalized topology on $X$.

Proof. (GO1) Obvious.

(GO2) For $i \in J$, from $\{\alpha : \vee A_i \in T_\alpha\} \supseteq \{\alpha : A_i \in T_\alpha\}$, it follows

$$\begin{aligned} T(\vee A_i) &= \vee\{\alpha : \vee A_i \in T_\alpha\} \\ &\geq \vee\{\alpha : A_i \in T_\alpha\} \\ &= T(A_i). \end{aligned}$$

This implies $T(\vee A_i) \geq \wedge T(A_i)$.

**Definition 2.11.** Let $f : (X, T_1) \to (Y, T_2)$ be a mapping on fuzzy generalized topological spaces. Then

(1) $f$ is said to be *fuzzy generalized continuous* if for every $A \in I^Y$, $T_1(f^{-1}(A)) \geq T_2(A)$.

(2) $f$ is said to be *weakly fuzzy generalized continuous* if for every $A \in I^Y$, we have

$$T_2(A)>0 \Rightarrow T_1(f^{-1}(A))>0.$$

**Remark 2.12.** We have the following implication but the converse may not be true as shown in the next example.

fuzzy generalized continuous
$\Rightarrow$ weakly fuzzy generalized continuous

**Example 2.13.** Let $X=Y=[0,1]$. For each $n\in N$, define a fuzzy set $A_n$ as follows

$$A_n(x)=\frac{n}{n+1}x,\quad \text{for } x\in I$$
$$B(x)=x,\quad \text{for } x\in I.$$

Consider a fuzzy generalized topology $T: I^X\to I$ defined as follows

$$T_1(A)=\begin{cases}1, & \text{if } A=0_X, B,\\ \frac{n}{n+3}, & \text{if } A=A_n,\\ 0, & otherwise;\end{cases}$$

and

$$T_2(A)=\begin{cases}1, & \text{if } A=0_X, B,\\ \frac{n}{n+2}, & \text{if } A=A_n,\\ 0, & otherwise;\end{cases}$$

Then the identity mapping $f:(X, T_1)\to(Y, T_2)$ is weakly fuzzy generalized continuous but it is not fuzzy generalized continuous.

**Theorem 2.14.** Let $f:(X, T_1)\to(Y, T_2)$ be a mapping on fuzzy generalized topological spaces. Then
(1) $f$ is fuzzy generalized continuous if and only if $T_2^*(\mathrm{A})\le T_1^*(f^{-1}(\mathrm{A}))$.
(2) $f$ is said to be weakly fuzzy generalized continuous if and only if

$$T_2^*(\mathrm{A})>0 \Rightarrow T_1^*(f^{-1}(\mathrm{A}))>0.$$

Proof. (1) From fuzzy generalized continuity, it follows

$$\begin{aligned}T_2^*(\mathrm{A})&=T_2(1_Y-\mathrm{A})\\ &\le T_1(f^{-1}(1_Y-\mathrm{A}))\\ &=T_1(1_Y-(f^{-1}(\mathrm{A})))\\ &=T_1^*(f^{-1}(\mathrm{A})).\end{aligned}$$

Similarly, the other implication is obvious.
(2) It is similar to (1).

## 3. 0-Closure operator ond 0-Interior operator on fuzzy generalized topological spaces

**Definition 3.1.** Let $(X,T)$ be a FGTS and $A\in I^X$. Then
(1) The *0-closure* of $A$, denoted by $A_-$, is defined by
$A_-=\wedge\{K\in I^X : T^*(K)>0,\ A\subseteq K\}$,
where $T^*(K)=T(1_X-K)$.
(2) The *0-interior* of $A$, denoted by $A_o$, is defined by $A_o=\vee\{K\in I^X : T(K)>0,\ K\subseteq A\}$.

**Theorem 3.2.** Let $(X,T)$ be a FGTS and $A, B\in I^X$. Then
(1) $(0_X)_o=0_X$,
(2) $A_o\subseteq A$,
(3) $A\subseteq B \Rightarrow A_o\subseteq B_o$,

Proof. Obvious.

Similarly, we have the next theorem.

**Theorem 3.3.** Let $(X,T)$ be a FGTS and $A,B\in I^X$. Then
(1) $(1_X)_-=1_X$,
(2) $A\subseteq A_-$,
(3) $A\subseteq B \Rightarrow A_-\subseteq B_-$,

**Theorem 3.4.** Let $(X,T)$ be a FGTS and $A\in I^X$. Then
(1) $1_X-A_-=(1_X-A)_o$.
(2) $1_X-A_o=(1_X-A)_-$.

Proof. (1) From Definition 3.1, we have

$$\begin{aligned}1_X-A_-&=1_X-\wedge\{K\in I^X : T^*(K)>0,\ A\subseteq K\}\\ &=\vee\{1_X-K: K\in I^X,\ T(1_X-K)=T^*(K)>0,\\ &\qquad 1_X-K\subseteq 1_X-A\}\\ &=\vee\{U\in I^X : T(U)>0,\ U\subseteq 1_X-A\}\\ &=(1_X-A)_o.\end{aligned}$$

(2) It is easily obtained from (1).

**Theorem 3.5.** Let $(X,T)$ be a FGTS and $A\in I^X$. Then
(1) If $T(A)>0$, then $A=A_o$.
(2) If $T^*(A)>0$, then $A=A_-$.

Proof. Obvious

In Theorem 3.5, the converses are not always true as shown in the ext example.

**Example 3.6.** Let $X=[0,1]$. For each $n\in N$, define a fuzzy set $A_n$ as follows

$$A_n(x)=\frac{n}{n+1}x,\quad \text{for } x\in I$$
$$B(x)=x,\quad \text{for } x\in I.$$

Consider a fuzzy generalized topology $T: I^X\to I$ defined as follows

$$T(A)=\begin{cases}1, & \text{if } A=0_X,\\ \dfrac{1}{n+1}, & \text{if } A=A_n,\\ 0, & otherwise.\end{cases}$$

Then $B=B_o$ but $T(B)=0$.

**Theorem 3.7.** Let $(X,T)$ be a FGTS and $A,B\in I^X$. Then
(1) $(A_o)_o=A_o$.
(2) $(A_-)_-=A_-$.

Proof. (1) In case $T(A_o)>0$: it follows from Theorem 3.5.

In case $T(A_o)=0$: Let $A_o=\vee\{K\in I^X: T(K)>0, K\subseteq A\}$; then for $K\subseteq A_o$ satisfying $T(K)>0$, from Theorem 3.2 and Theorem 3.5, it follows $K=K_o\subseteq(A_o)_o$. Hence we have $(A_o)_o=A_o$ from Theorem 3.2.

(2) It follows, from (1) and Theorem 3.4.

**Theorem 3.8.** Let $(X, T_1)$ and $(Y, T_2)$ be FGTS's. If $f: X\to Y$ is weakly fuzzy generalized continuous, then we have
(1) $T_2^*(B)>0 \Rightarrow T_1^*(f^{-1}(B))>0$ for $B\in I^Y$,
(2) $f(A_-)\subseteq f(A)_-$ for $A\in I^X$,
(3) $f^{-1}(B)_-\subseteq f^{-1}(B_-)$ for $B\in I^Y$,
(4) $f^{-1}(B_o)\subseteq f^{-1}(B)_o$ for $B\in I^Y$.

Proof. (1) Let $T_2^*(B)>0$ for $B\in I^Y$. Then $T_2(1_Y-B)>0$, and from the definition of weakly fuzzy generalized continuity, it follows $T_1(f^{-1}(1_Y-B))>0$. Hence we have $T_1^*(f^{-1}(B))>0$.
(2) Let $A\in I^X$; then we have

$$\begin{aligned} f^{-1}(f(A)_-)&=f^{-1}[\wedge\{U\in I^Y: T_2^*(U)>0 \text{ and } f(A)\subseteq U\}]\\ &=\wedge\{f^{-1}(U)\in I^X: T_1^*(f^{-1}(U))>0 \text{ and } A\subseteq f^{-1}(U)\}.\end{aligned}$$

From $T_1^*(f^{-1}(U))>0$ and Theorem 3.5, it follows $A_-\subseteq f^{-1}(U)_-=f^{-1}(U)$, and so

$$A_-\subseteq\wedge\{f^{-1}(U)\in I^X: T_1^*(f^{-1}(U))>0,\ A\subseteq f^{-1}(U)\}.$$

Consequently, we get $f(A_-)\subseteq f(A)_-$.
(3) It is similar to (2).
(4) It follows from (3) and Theorem 3.4.

**Theorem 3.9.** Let $(X, T_1)$ and $(Y, T_2)$ be FGTS's. If $f: X\to Y$ is fuzzy generalized continuous, then we have
(1) $f(A_-)\subseteq f(A)_-$ for $A\in I^X$,
(2) $f^{-1}(B)_-\subseteq f^{-1}(B_-)$ for $B\in I^Y$,
(3) $f^{-1}(B_o)\subseteq f^{-1}(B)_o$ for $B\in I^Y$.

Proof. It is obvious from Theorem 3.8.

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