한국지능시스템학회 논문지 2009, Vol. 19, No. 5, pp. 695-698

퍼지 r-일반 열린 집합과 퍼지 r-일반 연속성에 관한 연구

Fuzzy r-Generalized Open Sets and Fuzzy r-Generalized Continuity

민원근
Won Keun Min

강원대학교 수학과

요 약

본 논문에서는 퍼지 r-열린 집합을 일반화 시킨 퍼지 r-일반 열린 집합의 개념과 성질을 소개한다. 그리고 퍼지 r-일반 연속함수, 퍼지 r-일반 열린함수, 퍼지 r-일반 닫힌 함수의 개념과 특성을 연구한다.

Abstract

In this paper, we introduce the concept of fuzzy r-generalized open sets which are generalizations of fuzzy r-open sets defined by Lee and Lee [2] and obtain some basic properties of their structures. Also we introduce and study the concepts of fuzzy r-generalized continuous mapping, fuzzy r-generalized open mapping and fuzzy r-generalized closed mapping.

Key Words : fuzzy generalized topological space, fuzzy r-generalized open set, fuzzy r-generalized continuous, fuzzy r-generalized open mapping, fuzzy r-generalized closed mapping.

1. 서 론

Let X be a set and $I=[0,1]$. Let I^X denote the set of all mapping $A: X\to I$. A member of I^X is called a *fuzzy subset* [3] of X. 0_X and 1_X will denote the characteristic functions of $\emptyset$ and X, respectively. And unions and intersections of fuzzy sets are denoted by $\vee$ and $\wedge$, respectively, and defined by

$$\vee A_i = \sup\{A_i(x) \mid i\in J \text{ and } x\in X\},$$

$$\wedge A_i = \inf\{A_i(x) \mid i\in J \text{ and } x\in X\}.$$

A Chang's fuzzy topological space [1] is an ordered pair (X, T) is a non-empty set X and $T\subseteq I^X$ satisfying the following conditions:

(O1) $0_X, 1_X \in T$.
(O2) If $A, B \in T$, then $A\wedge B \in T$.
(03) If $A_i \in \tau$, for all $i\in J$, then $\vee A_i \in \tau$.

(X, T) is called a *fuzzy topological space*. Members of T are called fuzzy open sets in (X, T) and complement of a *fuzzy open set* is called a *fuzzy closed* set.

A *smooth topological space* [2] is an ordered pair (X, T), where X is a non-empty set and $T: I^X\to I$ is a mapping satisfying the following conditions:

(O1) $T(0_X)=T(1_X)=1$.
(O2) $T(A_1\wedge A_2)\geq T(A_1)\wedge T(A_2)$ for $A_1,\ A_2\in I^X$.
(O3) $T(\vee A_i)\geq \wedge T(A_i)$ for $A_i\in I^X$.

Then $T: I^X\to I$ is called a *smooth topology* on X. The number $T(A)$ is called the *degree of openness* of A.

A mapping $T^*: I^X\to I$ is called a *smooth cotopology* [2] iff the following three conditions are satisfied:

(C1) $T^*(0_X)=T^*(1_X)=1$.
(C2) $T^*(A_1\vee A_2)\geq T^*(A_1)\wedge T^*(A_2)$ for A_1, $A_2\in I^X$.
(C3) $T^*(\wedge A_i)\geq \wedge T^*(A_i)$ for $A_i\in I^X$.

Let f be a mapping from a set X into a set Y. Let A and B be the fuzzy sets of X and Y, respectively. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup A(z)_{z\in f^{-1}(y)}, & \text{if } f^{-1}(y)\neq 0,\ y\in Y \\ 0, & \text{otherwise}, \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x)=B(f(x)),\ x\in X$.

A fuzzy generalized topological space (simply, FGTS) is an ordered pair (X, T), where X is a non-empty set and $T: I^X\to I$ is a mapping satisfying the following conditions:

(GO1) $T(0_X)=1$.
(GO2) $T(\vee A_i)\geq \wedge T(A_i)$ for $A_i\in I^X$.

접수일자 : 2009년 7월 12일
완료일자 : 2009년 10월 1일

Then the mapping $T: I^X \to I$ is called a *fuzzy generalized topology* on X. The number $T(A)$ is called the *degree of generalized openness* of A.

A mapping $T^*: I^X \to I$ is called a *fuzzy generalized cotopology* if the following three conditions are satisfied:

(GO1) $T^*(1_X)=1$.
(GO2) $T^*(\wedge A_i) \geq \wedge T^*(A_i)$ for $A_i \in I^X$.

Then $T^*(A)$ is called the *degree of generalized closedness* of A.

Theorem 1.1 ([3]). (1) If T is a fuzzy generalized topology on X, then the mapping $T^*: I^X \to I$ defined by $T^*(A)=T(A^c)$, is a fuzzy generalized cotopology on X.
(2) If T^* is a fuzzy generalized cotopology on a non-empty set X, then the mapping $T: I^X \to I$ defined by $T(A)=T^*(A^c)$, is a fuzzy generalized topology on X.

2. Main Results

Definition 2.1. Let (X, T) be a FGTS and $A \in I^X$. Then
(1) The r-closure of A, denoted by $gCl_r(A)$, is defined by $gCl_r(A)=\cap\{K \in I^X : T^*(K) \geq r, A \subseteq K\}$, where $T^*(K)=T(K^c)$.
(2) The r-interior of A, denoted by $gInt_r(A)$, is defined by $gInt_r(A)=\cup\{K \in I^X : T(K) \geq r, K \subseteq A\}$.

We will call A a *fuzzy r-generalized open set* if $T(A) \geq r$, A a *fuzzy r-generalized closed set* if $T^*(A) \geq r$.

Theorem 2.2. Let (X, T) be a FGTS and A, $B \in I^X$. Then
(1) $gInt_r(0_X)=0_X$;
(2) $gInt_r(A) \subseteq A$;
(3) $A \subseteq B \Rightarrow gInt_r(A) \subseteq gInt_r(B)$.

Proof. Obvious.

Similarly, we have the next theorem:

Theorem 2.3. Let (X, T) be a FGTS and $A, B \in I^X$. Then
(1) $gCl_r(1_X)=1_X$;
(2) $A \subseteq gCl_r(A)$;
(3) $A \subseteq B \Rightarrow gCl_r(A) \subseteq gCl_r(B)$.

Theorem 2.4. Let (X, T) be a FGTS and $A \in I^X$. Then
(1) $(gCl_r(A))^c=gInt_r(A^c)$.
(2) $(gInt_r(A))^c=gCl_r(A^c)$.

Proof. (1) From Definition 3.1, we have

$$\begin{aligned}(gCl_r(A))^c &= (\cap\{K \in I^X: T^*(K) \geq \mathrm{r}, A \subseteq K\})^c \\ &= \cup\{K^c: K \in I^X, T(K^c)=T^*(K) \geq \mathrm{r}, K^c \subseteq A^c\} \\ &= \cup\{U \in I^X: T(U) \geq \mathrm{r}, U \subseteq A^c\} \\ &= gInt_r(A^c).\end{aligned}$$

(2) It is easily obtained from (1).

Lemma 2.5. Let (X, T) be a fuzzy generalized topological space. The statements are hold:
(1) If $T(A_i) \geq r$ for each $i \in J$, then $T(\cup_{i \in J} A_i) \geq r$.
(2) If $T^*(A_i) \geq r$ for each $i \in J$, $T^*(\cap_{i \in J} A_i) \geq r$

Proof. (1) For each $i \in J$, if $T(A_i) \geq r$ then from definition of fuzzy generalized topology,

$$T(\cup_{i \in J} A_i) \geq \wedge_{i \in J} T(A_i) \geq r$$

(2) It follows from definition of fuzzy generalized cotopology.

Theorem 2.6. Let (X, T) be a FGTS and $A \in I^X$. Then
(1) A is a fuzzy r-generalized open set iff $A=gInt_r(A)$.
(2) A is a fuzzy r-generalized closed set iff $A=gCl_r(A)$.

Proof. It follows from Lemma 2.5.

Theorem 2.7. Let (X, T) be a FGTS and $A, B \in I^X$. Then
(1) $gInt_r(gInt_r(A))=gInt_r(A)$.
(2) $gCl_r(gCl_r(A))=gCl_r(A)$.

Proof. It follows from Lemma 2.5 and Theorem 2.6.

Definition 2.8. Let $f: (X, T_1) \to (Y, T_2)$ be a mapping on fuzzy generalized topological spaces. Then f is said to be *fuzzy r-generalized continuous* if for every $A \in I^Y$, we have $T_2(A) \geq r \Rightarrow T_1(f^{-1}(A)) \geq r$.

Remark 2.9. Let $f: (X, T_1) \to (Y, T_2)$ be a mapping on fuzzy generalized topological spaces. Then f is said to be *fuzzy generalized continuous* in [3] if for every $A \in I^Y$, $T_1(f^{-1}(A)) \geq T_2(A)$. Thus we know that every fuzzy generalized continuous function is fuzzy r-generalized continuous but the converse is not true in general.

Example 2.10. Let $X=[0,1]$ and let us define a fuzzy set α, β as follows

$$\alpha(x)=x, \quad \text{if } 0 \leq x \leq 1;$$

$$\beta(x)=\frac{1}{2}x, \quad \text{if} \quad 0 \leq x \leq 1.$$

Consider fuzzy generalized topologies $T_1, T_2 : I^X \rightarrow I$ defined as

$$T_1(A)=\begin{cases} 1, & \text{if } A=0_X, \\ \frac{1}{2}, & \text{if } A=\alpha,\beta, \\ 0, & otherwise; \end{cases}$$

and

$$T_2(A)=\begin{cases} 1, & \text{if } A=0_X, \\ \frac{1}{3}, & \text{if } A=\alpha,\beta, \\ 0, & otherwise. \end{cases}$$

Then the identity mapping $f:(X, T_1) \rightarrow (X, T_2)$ is fuzzy $\frac{1}{3}$-generalized continuous but it is not fuzzy generalized continuous.

Theorem 2.11. Let (X, T_1) and (Y, T_2) be FGTS′s. Then the following are equivalent:

(1) f is fuzzy r-generalized continuous.

(2) For every fuzzy r-generalized open set A in Y, $f^{-1}(A)$ is fuzzy r-generalized open in X.

(3) $T_2^*(B) \geq r \Rightarrow T_1^*(f^{-1}(B)) \geq r$ for $B \in I^Y$.

(4) For every fuzzy r-generalized closed set A in Y, $f^{-1}(A)$ is fuzzy r-generalized closed in X.

(5) $f(gCl_r(A)) \subseteq gCl_r(f(A)$ for $A \in I^X$.

(6) $gCl_r(f^{-1}(B)) \subseteq f^{-1}(gCl_r(B))$ for $B \in I^Y$.

(7) $f^{-1}(gInt_r(B)) \subseteq gInt_r(f^{-1}(B))$ for $B \in I^Y$.

Proof. (1) $\Rightarrow$ (2) Let A be a fuzzy r-generalized open set. Then $T_2(A) \geq r$ and so by fuzzy r-generalized continuity, $T_1(f^{-1}(A)) \geq r$. Hence $f^{-1}(A)$ is fuzzy r-generalized open.

(2) $\Rightarrow$ (3) For $B \in I^Y$, let $T_2^*(B) \geq r$. Then $T_2(B^c) \geq r$, so B^c is fuzzy r-generalized open. By (2), $f^{-1}(B^c)$ is fuzzy r-generalized open, and so we have $T_1(f^{-1}(B^c))= T_1((f^{-1}(B))^c)= T_1^*(f^{-1}(B)) \geq r$. So $T_1^*(f^{-1}(B)) \geq r$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) Let $A \in I^X$; then we have

$$\begin{aligned} & f^{-1}(gCl_r(f(A))) \\ &= f^{-1}[\cap\{F \in I^Y : f(A) \subseteq F \text{ and } F \text{ is fuzzy } r\text{-generalized closed}\}] \\ &= \cap\{f^{-1}(F) \in I^X : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is fuzzy } r\text{-generalized closed}\}. \end{aligned}$$

Thus since $gCl_r(A)$ is the smallest fuzzy r-generalized closed set containing A, $gCl_r(A) \subseteq f^{-1}(gCl_r(f(A)))$. This implies $f(gCl_r(A)) \subseteq gCl_r(f(A))$.

(5) $\Rightarrow$ (6) Obvious.

(6) $\Rightarrow$ (7) Obvious.

(7) $\Rightarrow$ (1) For $B \in I^Y$, if $T_2(B) \geq r$ then B is fuzzy r-generalized open. So by Theorem 2.6 and (7), $f^{-1}(B)=f^{-1}(gInt_r(B)) \subseteq gInt_r(f^{-1}(B))$.

This implies $f^{-1}(B)$ is fuzzy r-generalized open, and so $T_1(f^{-1}(B)) \geq r$. Hence f is fuzzy r-generalized continuous.

Definition 2.12. Let $f:(X, T_1) \rightarrow (Y, T_2)$ be a mapping on fuzzy generalized topological spaces. Then f is said to be *fuzzy r-generalized open* if for every fuzzy r-generalized open set A in X, $f(A)$ is fuzzy r-generalized open in Y.

Theorem 2.13. Let $f:(X, T_1) \rightarrow (Y, T_2)$ be FGTS′s. Then the following are equivalent:

(1) f is fuzzy r-generalized open.

(2) For $A \in I^X$, $T_1(A) \geq r \Rightarrow T_2(f(A)) \geq r$.

(3) $f(gInt_r(A)) \subseteq gInt_r(f(A))$ for $A \in I^X$.

(4) $gInt_r(f^{-1}(B)) \subseteq f^{-1}(gInt_r(B))$ for $B \in I^Y$.

Proof. (1) $\Leftrightarrow$ (2) It is obvious from definition of fuzzy r-generalized open set.

(1) $\Rightarrow$ (3) For $A \in I^X$, $gInt_r(A)$ is fuzzy r-generalized open. Since f is fuzzy r-generalized open, $f(gInt_r(A))$ is fuzzy r-generalized open. So $f(gInt_r(\mathrm{A}))= gInt_r(f(gInt_r(\mathrm{A}))) \subseteq gInt_r(f(\mathrm{A}))$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) Let A be a fuzzy r-generalized open set. Then from (3), it follows $A=gInt_r(A) \subseteq gInt_r(f^{-1}(f(A))) \subseteq f^{-1}(gInt_r(f(A)))$. So $f(A) \subseteq gInt_r(f(A))$ and hence $f(A)$ is fuzzy r-generalized open.

Definition 2.14. Let $f:(X, T_1) \rightarrow (Y, T_2)$ be a mapping on fuzzy generalized topological spaces. Then f is said to be *fuzzy r-generalized closed* if for every fuzzy r-generalized closed set A in X, $f(A)$ is fuzzy r-generalized closed in Y.

Theorem 2.15. Let (X, T_1) and (Y, T_2) be FGTS′s. Then the following are equivalent:

(1) f is fuzzy r-generalized closed.

(2) For $A \in I^X$, $T_1^*(\mathrm{A}) \geq r \Rightarrow T_2^*(f(A)) \geq r$.

(3) $gCl_r(f(A)) \subseteq f(gCl_r(A))$ for $A \in I^X$.

Proof. (1) $\Leftrightarrow$ (2) Obvious.

(1) $\Rightarrow$ (3) For $A \in I^X$, $gCl_r(A)$ is fuzzy r-generalized closed. Since f is fuzzy r-generalized closed, $f(gCl_r(A))$ is fuzzy r-generalized closed. So

$$gCl_r(f(A)) \subseteq gCl_r(f(gCl_r(A)))=f(gCl_r(A)).$$

(3) $\Rightarrow$ (1) Let A be a fuzzy r-generalized closed set. Then from (2) and $gCl_r(A)=A$,

$$gCl_r(f(A)) \subseteq f(gCl_r(A)) = f(A).$$

Thus $f(A)$ is fuzzy r-generalized closed.

References

[1] C. L. Chang, "Fuzzy topological spaces", *J. Math. Anal. Appl.*, Vol. 24, pp. 182-190, 1968.

[2] S. J. Lee and E. P. Lee, "Fuzzy r-continuous and r-semicontinuous maps", *Int. J. Math. Math. Sci.,* vol. 27, no. 1, pp. 53-63, 2001.

[3] W. K. Min, "Fuzzy generalized topological spaces", *J. Fuzzy Logic and Intelligent Systems*, vol 19, no. 3, pp. 404-407, 2009.

[4] A. A. Ramadan, "Smooth topological spaces", *Fuzzy Sets and Systems*, vol. 48, pp. 371-375, 1992.

[5] L. A. Zadeh, "Fuzzy sets", *Inform. and Control*, vol. 8, pp. 338-353, 1965.

저 자 소 개

민원근(Won Keun Min)
1988년 ~ 현재 : 강원대학교 수학과 교수

관심분야 : 퍼지 위상, 퍼지 이론, 일반 위상
Phone : 033-250-8419
Fax : 033-252-7289
E-mail : wkmin@kangwon.ac.kr