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Almost Sure Convergence for Asymptotically Almost Negatively Associated Random Variable Sequences

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Abstract

We in this paper study the almost sure convergence for asymptotically almost negatively associated(AANA) random variable sequences and obtain some new results which extend and improve the result of Jamison *et al.* (1965) and Marcinkiewicz-Zygumnd strong law types in the form given by Baum and Katz (1965), three-series theorem.

Keywords: Asymptotically almost negatively associated random variable sequences, almost convergence, three-series theorem.

1. Introduction

Let $\{\Omega, F, P\}$ be a probability space and $\{X_1, \ldots, X_n\}$ a sequence of random variables defined on $\{\Omega, F, P\}$. A finite family $\{X_i \mid 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \longrightarrow R$ and $g : R^B \longrightarrow R$,

$$\mathrm{Cov}\Big(f(X_i, i \in A),\ g(X_j,\ j \in B)\Big) \leq 0.$$

Infinite family of random variables is negatively associated if every finite subfamily is negatively associated. Since the concept of negative association was introduced by Joag-Dev and Proschan (1983), its limit properties have aroused wide interest because of their numerous applications in multivariate statistical analysis, reliability theory and percolation theory. Moreover, primarily motivated by this, in order to enlarge the range of the sequence of NA random variables Chandra and Ghosal have introduced the following dependence condition.

Definition 1. (Chandra and Ghosal, 1996b) *A sequence $\{X_n \mid n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there is a nonnegative sequence $q(m) \longrightarrow 0$ such that*

$$Cov\,(f(X_m),\ g(X_{m+1}, \ldots, X_{m+k})) \leq q(m)\left[Var(f(X_m))\,Var(g(X_{m+1}, \ldots, X_{m+k}))\right]^{\frac{1}{2}} \tag{1.1}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the left side of (1.1) is finite.

The family of AANA sequences contains negatively associated (in particular, in dependent) sequences (with $q(m) = 0$ for all $m \geq 1$) and also some sequences of random variables which do not

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deviate much from being negatively associated. Condition (1.1) is clearly satisfied if the $R_{2,2}$-measure of dependence, see (1.1), between $\sigma(X_m)$ and $\sigma(X_{m+1}, X_{m+2}, \cdots)$ converges to zero. The following is a nontrivial example of an AANA sequence. It is possible to construct similar examples, but we shall not discuss this topic any more here.

Example 1. Let $\{Y_n\}$ be *i.i.d.* $N(0, 1)$ variables, and define $X_n = (1 + a_n^2)^{-1/2}(Y_n + a_n Y_{n+1})$ where $a_n > 0$ and $a_n \to 0$. Note that $\{X_n\}$ is not NA(indeed, it is associated and 1-dependent). We shall show that the correlation coefficient between $U = f(X_m)$ and $V = g(X_{m+1}, \ldots, X_{m+k})$ is dominated in absolute value a_m. It suffices to prove this under the additional hypotheses $EU = 0 = EV$, $EU^2 = 1 = EV^2$. Then

$$\begin{aligned}
(\mathrm{Cov}(U,\ V))^2 &\le [\mathrm{Cov}(U,\ E(U|X_{m+1}, \ldots, X_{m+k}))]^2 \\
&\le E(E(U|X_{m+1}, \ldots, X_{m+k}))^2 \\
&\le E(E(U|Y_{m+1}, \ldots, Y_{m+k+1}))^2 \\
&\le E(E(U|Y_{m+1}))^2 \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) \left\{ \frac{\psi_m(x,y)}{\phi(x)} - 1 \right\} \phi(x) dx \right]^2 \phi(y) dy,
\end{aligned}$$

where $\psi_m(x, y)$ is the conditional density of X_m given by $Y_{m+1} = y$ and $\phi(x)$ is the density of $N(0,\ 1)$. By the Cauchy-Schwarz inequality, the last integral is at most

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\psi_m(x,y)}{\phi(x)} - 1 \right)^2 \phi(x) dx\, \phi(y) dy = a_m^2.$$

See Chandra and Ghosal (1996a).

A random variable sequence $\{X_n | n \ge 1\}$ is said to be stochastically dominated by a nonnegative random variable X if the exists a positive constant C such that $P(|X_n| > x) \le CP(|X| > x)$ for all $n \ge 1$ and $x \ge 0$.

In this case we write $\{X_n\} < X$. Hereinafter C always stands for a positive constant which may differ from one place to another.

The following is the Jamison's Theorem 1 (Jamison *et al.*, 1965) which is established for independent identically distribution random variables.

Theorem 1. *Let $\{X_n | n \ge 1\}$ be a sequence of independent identically distribution random variables satisfying $E|X_1| < \infty$, $EX_1 = 0$ and let $\{a_i | i \ge 1\}$ be a sequence of positive numbers. If*

$$A_n = \sum_{i=1}^{n} a_i \uparrow \infty, \quad n \to \infty, \quad \sharp\{n | A_i/a_i \le n\} = O(n),\ n \ge 1,$$

then $T_n = \sum_{i=1}^{n} a_i X_i / A_n \longrightarrow 0$ almost surely as $n \to \infty$.

In this paper, we discuss the strong law of large numbers of AANA random variable sequences and try to obtain some new results. The main purpose of this paper is to extend and improve the Theorem A of Jamison's weighted sums under suitable conditions of AANA, and Marcinkiewicz-Zygumnd strong law types in the form given by Baum and Katz (1965) and the three-series theorem for AANA random variable sequences is also considered. In Section 2, we study some preliminary results and in Section 3, we derive the main results for AANA random variable sequences under suitable conditions.

2. Preliminaries

Lemma 1. **(Chandra and Ghosal, 1996a)** *Let $X_1, X_2, \dots, X_n$ be mean zero, square integrable random variable such that (1.1) holds for $1 \le m < k + m \le n$ and for all coordinatewise increasing continuous functions f and g whenever the left side of (1.1) is finite and let $A^2 = \sum_{m=1}^{n-1} q^2(m)$. Then*

$$P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| \ge \varepsilon\right) \le 2\varepsilon^{-2}\left(A + (1 + A^2)^{\frac{1}{2}}\right)^2 \sum_{k=1}^{n} EX_k^2.$$

Lemma 2. *Suppose that $\{X_n | n \ge 1\}$ be a sequence of AANA satisfying $EX_n = 0$, $EX_n^2 < \infty$ and let $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$ and $E(T_j(k))^2 \le \sum_{i=j+1}^{j+k} EX_i^2$, $j \ge 0$. If $\sum_{n=1}^{\infty} (\log n)^2 E(X_n - EX_n)^2 < \infty$, then $\sum_{n=1}^{\infty} (X_n - EX_n)$ converges almost surely.*

Proof: The proof will follow the classical line; (*cf.* Stout, 1974). Without loss of generality, we assume that for positive integers, $m \ge n \to \infty$ and $S_k = \sum_{i=1}^{k} X_i$. Then by assumptions,

$$E(S_m - S_n)^2 \le \sum_{l=n+1}^{m} EX_l^2 \longrightarrow 0.$$

So, $\{S_n | n \ge 1\}$ is a Cauchy sequence with respect to L^2. Hence, by the completeness of L^2, there exists a random variable $S \in L^2$ with $ES^2 < \infty$ and $E(S_n - S)^2 \longrightarrow 0$. By assumptions, we get

$$\sum_{l=1}^{\infty} P(|S_{2^l} - S| \ge \varepsilon) < \infty$$

and

$$\begin{aligned}
\sum_{l=1}^{\infty} P\left(\max_{2^{l-1}<i\le 2^l} |S_i - S_{2^{l-1}}| \ge \varepsilon\right) &\le 2\varepsilon^{-2}\left(A + (1 + A^2)^{\frac{1}{2}}\right)^2 \sum_{l=1}^{\infty} E\left(\max_{2^{l-1}<i\le 2^l} |S_i - S_{2^{l-1}}|^2\right) \\
&\le C \sum_{l=1}^{\infty} \left(\frac{\log 2^l}{\log 2}\right)^2 \sum_{i=2^{l-1}+1}^{2^l} EX_i^2 \\
&\le C \sum_{i=1}^{\infty} (\log i)^2 EX_i^2 < \infty.
\end{aligned}$$

Hence, by Borel-Cantelli Lemma, it follows that

$$\sum_{l=1}^{\infty} P\left(|S_{2^l} - S| \ge \varepsilon,\ i.o.\right) = 0 \tag{2.1}$$

and

$$\sum_{l=1}^{\infty} P\left(\max_{2^{l-1}<i\le 2^l} |S_i - S_{2^{l-1}}| \ge \varepsilon,\ i.o.\right) = 0. \tag{2.2}$$

From (2.1) and (2.2), we know $S_n \longrightarrow S$ almost surely and the proof is completed. □

3. Main Results

Theorem 2. *Let $\{X_n|n \geq 1\}$ be a sequence of AANA random variables satisfying $\{X_n\} < X$, $EX_n = 0$ and $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. Suppose that $\{a_n|n \geq 1\}$ be a sequence of positive numbers with $b_1 = A_1/a_1$, $b_n = A_n/(a_n \log n)$, $n \geq 2$ and $A_n = \sum_{i=1}^{n} a_i \uparrow \infty$, $n \to \infty$. Set $N(x) = Card\{n|x \geq b_n\}$, $x \in \mathbb{R}$. If*

1. $EN(X) < \infty$, (3.1)
2. $\int_0^\infty tP(|X| > t) \int_t^\infty N(y)y^{-3}dy\, dt < \infty$, (3.2)

then $\sum_{i=1}^{n} a_i X_i / A_n \longrightarrow 0$ almost surely.

Proof: Let $Y_i = -b_i I(X_i < -b_i) + X_i I(|X_i| \leq b_i) + b_i I(X_i > b_i)$, $i \geq 1$. Then

$$\sum_{i=1}^{n} \frac{a_i X_i}{A_n} = \sum_{i=1}^{n} \frac{a_i(X_i - Y_i)}{A_n} + \sum_{i=1}^{n} \frac{a_i(Y_i - EY_i)}{A_n} + \sum_{i=1}^{n} \frac{a_i EY_i}{A_n}$$
$$= I_1 + I_2 + I_3 \text{ (say)}.$$

We can easily get that

$$I_1 = \sum_{i=1}^{n} \frac{a_i(X_i - Y_i)}{A_n} \longrightarrow 0 \text{ almost surely.} \tag{3.3}$$

Next, we will show that

$$I_2 = \sum_{i=1}^{n} \frac{a_i(Y_i - EY_i)}{A_n} \longrightarrow 0 \text{ almost surely.}$$

Note that by the definition of AANA, we know that $\{a_n(Y_n - EY_n)/A_n \,|\, n \geq 1\}$ is still a sequence of AANA random variables. Thus, by Lemma 2, it suffices to show that $\sum_{n=1}^{\infty}(\log n)^2 V(a_n Y_n/A_n) < \infty$. This can be done by

$$\begin{aligned}
\sum_{n=1}^{\infty}(\log n)^2 V\left(\frac{a_n Y_n}{A_n}\right) &\leq \sum_{n=1}^{\infty} b_n^{-2} E|Y_n|^2 \\
&\leq C \sum_{n=1}^{\infty} P(|X| > b_n) + C \sum_{n=1}^{\infty} b_n^{-2} E|X|^2 I(|X| \leq b_n) \\
&\leq CEN(X) + 4C \int_0^\infty tP(|X| > t) \int_t^\infty N(y)y^{-3}dy\, dt \\
&< \infty.
\end{aligned}$$

So, by (3.1) and (3.2), we obtain that $\sum_{n=1}^{\infty}(\log n)^2 V(a_n Y_n/A_n) < \infty$ and by Lemma 2, $\sum_{n=1}^{\infty} a_n(Y_n - EY_n)/A_n$ converges almost surely. Hence, it follows that from Kronecker's Lemma

$$I_2 = \sum_{i=1}^{n} \frac{a_i(Y_i - EY_i)}{A_n} \longrightarrow 0 \quad \text{almost surely.} \tag{3.4}$$

Finally, by $EX_n = 0$, we get that

$$|EY_i| \le Eb_i I(|X_i| \ge b_i) + Eb_i I(|X_i| \ge b_i)$$
$$\le 2E|X|I(|X| \ge b_i) \longrightarrow 0, \quad \text{as } n \to \infty,$$

by $a_n/A_n > 0$, $\sum_{i=1}^n a_i/A_i = 1$ and Toeplitz Lemma, we get that

$$I_3 = \sum_{i=1}^{n} \frac{a_i EY_i}{A_n} \longrightarrow 0 \quad \text{almost surely.} \tag{3.5}$$

Therefore, by (3.3), (3.4) and (3.5), we know

$$\sum_{i=1}^{n} \frac{a_i X_i}{A_n} \longrightarrow 0 \quad \text{almost surely.}$$

The proof is completed. □

Theorem 3. *Let $\{X_n \,|\, n \ge 1\}$ be a sequence of AANA random variables satisfying $\{X_n\} < X$, $EX_n = 0$ and $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. Assume that $\{a_n \,|\, n \ge 1\}$ be a sequence of positive numbers with $b_1 = A_1/a_1$, $b_n = A_n/(a_n \log n)$, $n \ge 2$ and $A_n = \sum_{i=1}^n a_i \uparrow \infty$, $n \to \infty$. Set $N(x) = Card\{n \,|\, x \ge b_n\}$, $x \in \mathbb{R}$. If*

1. $EN(X) < \infty$, (3.6)
2. $\max_{1\le j\le n} b_j^2 \sum_{j=n}^{\infty} {b_j}^{-2} \le Cn$, (3.7)

then $\sum_{i=1}^n a_i X_i/A_n \longrightarrow 0$ almost surely.

Proof: By Lemma 2 and Theorem 2, we need only to prove that $\sum_{i=1}^n a_i(Y_i - EY_i)/A_n \longrightarrow 0$ almost surely. Let $\varepsilon_n = \max_{1\le j\le n} b_j$ and $\varepsilon_0 = 0$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} (\log n)^2 \frac{E\,|\,a_n(Y_n - EY_n)|^2}{{A_n}^2} &\le \sum_{n=1}^{\infty} {b_n}^{-2} E|Y_n|^2 \\ &\le C\sum_{n=1}^{\infty} P(|X| > b_n) + C\sum_{n=1}^{\infty} {b_n}^{-2} E|X|^2 I(|X| \le b_n) \\ &= I_4 + I_5 \quad \text{(say).}\end{aligned}$$

For I_4 and I_5,

$$I_4 = C\sum_{n=1}^{\infty} P(|X| > b_n) \le CEN(X) < \infty, \tag{3.8}$$

$$\begin{aligned} I_5 &= C\sum_{n=1}^{\infty} {b_n}^{-2} \sum_{j=1}^{n} E|X|^2 I\left(\varepsilon_{j-1} < |X| \le \varepsilon_j\right) \\ &\le C\sum_{j=1}^{\infty} jP\left(\varepsilon_{j-1} < |X| \le \varepsilon_j\right) \\ &\le C\left(1 + \sum_{n=1}^{\infty} P(|X| > b_n)\right) \\ &\le C\,(1 + EN(X)) < \infty. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) and Lemma 2, we have $\sum_{n=1}^{\infty} a_n(Y_n - EY_n)/A_n$ converges almost surely, and by Kronecker's Lemma, we know $\sum_{i=1}^{n}(Y_i - EY_i)/A_n \longrightarrow 0$ almost surely. □

Now, we extend the convergence for Marcinkiewicz-Zygmund types to the case of AANA random variable sequences.

Theorem 4. *Let $\{X_n|n \geq 1\}$ be a sequence of AANA random variables satisfying $\{X_n\} < X$, $EX_n = 0$ and $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. If*

1. $E|X|^p < \infty$, *for* $1 < p \leq 2$ (3.10)
2. $\gamma > 0$ and $\alpha \geq \max((1+\gamma)/p, 1)$, (3.11)

then $\sum_{n=1}^{\infty} n^{\alpha p - 2 - \gamma} P(\max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i| \geq \varepsilon n^{\alpha}) < \infty$, *for all* $\varepsilon > 0$.

Proof: Let $Y_{ni} = -n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \leq n^{\alpha}) + n^{\alpha} I(X_i > n^{\alpha})$, $i \geq 1$, $n \geq 1$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| \geq \varepsilon n^{\alpha}\right) &\leq \sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_{ni}\right| \geq \varepsilon n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} P\left(\max_{1\leq i\leq n} |X_i| > n^{\alpha}\right)\\ &= I_6 + I_7 \quad \text{(say)}.\end{aligned}$$

For I_7,

$$\begin{aligned}I_7 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} \sum_{i=1}^{n} P\left(|X_i| > n^{\alpha}\right)\\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\gamma} \sum_{j=n}^{\infty} P\left(j \leq |X|^{\frac{1}{\alpha}} < j+1\right)\\ &\leq C\sum_{j=1}^{\infty} j^{\alpha p-\gamma} P\left(j \leq |X|^{\frac{1}{\alpha}} < j+1\right)\\ &\leq CE|X|^p < \infty. \end{aligned} \tag{3.12}$$

Secondly, in order to prove that $I_6 < \infty$, we need to show that $n^{-\alpha}\max_{1\leq j\leq n}|\sum_{i=1}^{j} EY_{ni}| \longrightarrow 0$.

$$\begin{aligned}n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_{ni}\right| &\leq n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EX_i I(|X_i| \leq n^{\alpha})\right| + nP\left(|X| > n^{\alpha}\right)\\ &= I_8 + I_9 \quad \text{(say)}.\end{aligned}$$

As to I_8, by $EX_n = 0$, we have

$$\begin{aligned}I_8 &\leq Cn^{1-\alpha} E|X|I(|X| > n^{\alpha})\\ &\leq Cn^{1-\alpha}\sum_{j=n}^{\infty} j^{\alpha} P\left(j \leq |X|^{\frac{1}{\alpha}} < j+1\right)\\ &\leq C\sum_{j=n}^{\infty} P\left(j \leq |X|^{\frac{1}{\alpha}} < j+1\right) \longrightarrow 0. \end{aligned} \tag{3.13}$$

Now, as to I_9,

$$
\begin{aligned}
I_9 &= nP\left(|X| > n^{\alpha}\right) \\
&\le Cn \sum_{j=n}^{\infty} P\left(j \le |X|^{\frac{1}{\alpha}} < j+1\right) \longrightarrow 0.
\end{aligned}
\tag{3.14}
$$

Hence, by (3.13) and (3.14), we know that

$$
n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EY_{ni} \right| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.15}
$$

Therefore, it suffices to show that

$$
I_6^* = \sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| \ge \varepsilon n^{\alpha} \right) < \infty, \quad \text{for all} \ \ \varepsilon > 0.
$$

Since $\{Y_{ni} | 1 \le i \le n,\ n \ge 1\}$ is non-decreasing functions of X_i, $\{(Y_{ni} - EY_{ni}) | 1 \le i \le n,\ n \ge 1\}$ is still an AANA random variables. Thus we obtain that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| \ge \varepsilon n^{\alpha} \right) \\
&\qquad \le 2\varepsilon^{-2} \left(A + \left(1 + A^2\right)^{\frac{1}{2}} \right)^2 \sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} \sum_{i=1}^{n} E|Y_{ni}|^2 \\
&\qquad \le C \sum_{n=1}^{\infty} n^{\alpha p-1-\gamma-2\alpha} E|X|^2 I\left(|X| \le n^{\alpha}\right) + C \sum_{n=1}^{\infty} n^{\alpha p-1-\gamma} P\left(|X| > n^{\alpha}\right) \\
&\qquad = I_{10} + I_{11} \quad \text{(say)}.
\end{aligned}
$$

As to I_{10},

$$
\begin{aligned}
I_{10} &\le C \sum_{n=1}^{\infty} n^{\alpha(p-2)-1-\gamma} \sum_{j=1}^{n} j^{2\alpha} P\left(j < |X|^{\frac{1}{\alpha}} \le j+1\right) \\
&\le C \sum_{j=1}^{\infty} j^{2p-\gamma} P\left(j < |X|^{\frac{1}{\alpha}} \le j+1\right) \\
&\le CE|X|^p < \infty.
\end{aligned}
\tag{3.16}
$$

Finally, as to I_{11},

$$
\begin{aligned}
I_{11} &\le C \sum_{j=1}^{\infty} j^{\alpha p-\gamma} P\left(j < |X|^{\frac{1}{\alpha}} \le j+1\right) \\
&\le CE|X|^p < \infty,
\end{aligned}
\tag{3.17}
$$

so, by (3.16) and (3.17), we get that $I_6^* < \infty$.

Thus, by (3.12), (3.15), (3.16) and (3.17), we know that

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\gamma} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right| \ge \varepsilon n^{\alpha}\right) < \infty, \quad \text{for all } \varepsilon > 0.$$

The proof is completed. □

Taking $\alpha = 1$, $p = 2$ and $\gamma = 1$ in Theorem 4, we can get the following Corollaries.

Corollary 1. *Let $\{X_n|n \ge 1\}$ be a sequence of AANA random variables satisfying $\{X_n\} < X$, $EX_n = 0$ and $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. If $E|X|^2 < \infty$, then $\sum_{n=1}^{\infty} 1/nP(\max_{1\le j\le n} |\sum_{i=1}^{j} X_i| \ge \varepsilon n) < \infty$ for all $\varepsilon > 0$.*

Corollary 2. *Let $\{X_n|n \ge 1\}$ be a sequence of identically distributed AANA random variables satisfying $EX_1 = 0$ and $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. If $E|X_1|^2 < \infty$, then $\sum_{n=1}^{\infty} 1/nP(\max_{1\le j\le n} |\sum_{i=1}^{j} X_i| \ge \varepsilon n) < \infty$ for all $\varepsilon > 0$.*

Theorem 5. (Three series theorem) *Suppose that $\{X_n|n \ge 1\}$ be a sequence of AANA random variables satisfying $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. If there is a $C > 0$ such that*

1. $\sum_{n=1}^{\infty} P(|X_n| > C) < \infty,$ (3.18)
2. $\sum_{n=1}^{\infty} EX_n I(|X_n| \le C)$ converges, (3.19)
3. $\sum_{n=1}^{\infty} (\log n)^2 E|X_n I(|X_n| \le C)|^2 < \infty,$ (3.20)

then $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Proof: Applying the proof of Stout (1974), we can obtain the result of Theorem 5 in assumptions and Lemma 2. The proof is completed. □

Theorem 6. *Suppose that $\{X_n|n \ge 1\}$ be a sequence of AANA random variables satisfying $EX_n = 0$ and $A^2 = \sum_{m=1}^{\infty} q^2(m) < \infty$. Let $\{f_n(x)|n \ge 1\}$ be a sequence of even functions, positive and non-decreasing in the interval $x > 0$ and $\{b_n|n \ge 1\}$ is a sequence of positive numbers with $b_n \uparrow$. If for every $n \ge 1$,*

1. $X/f_n(x) \downarrow$ *and* $f_n(x)/x^2$, *as* $0 < x \uparrow$ (3.21)
2. $\sum_{n=1}^{\infty} (\log n)^2 E f_n(X_n)/f_n(b_n) < \infty,$ (3.22)

then $\sum_{n=1}^{\infty} X_n/b_n$ converges almost surely.

Proof: Let $Y_n = -b_n I(X_n < -b_n) + X_n I(|X_n| \le b_n) + b_n I(X_n > b_n)$, $n \ge 1$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{X_n}{b_n} &= \sum_{n=1}^{\infty} \frac{X_n - Y_n}{b_n} + \sum_{n=1}^{\infty} \frac{Y_n - EY_n}{b_n} + \sum_{n=1}^{\infty} \frac{EY_n}{b_n} \\ &= I_{12} + I_{13} + I_{14} \quad \text{(say).}\end{aligned}$$

Since $f_n(x) \uparrow$ as $x > 0$, $f_n(b_n) \le f_n(|X_n|)$ on $\{Y_n | n \ge 1\}$, by assumption $EX_n = 0$. Hence we have that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\frac{EY_n}{b_n}\right| &= \sum_{n=1}^{\infty}\left|\frac{E(-b_n I(X_n < -b_n) + X_n I(|X_n| \le b_n) + b_n I(X_n > b_n))}{b_n}\right| \\
&\le \sum_{n=1}^{\infty} P(|X_n| > b_n) + \sum_{n=1}^{\infty} E\left(\frac{|X_n|}{b_n}\right) I(|X_n| \le b_n) \\
&\le \sum_{n=1}^{\infty} \int \frac{f_n(|X_n|)}{f_n(b_n)} I\left(\frac{|X_n|}{b_n} > 1\right) dp + \sum_{n=1}^{\infty} E\left(\frac{f_n(|X_n|)}{f_n(b_n)}\right) I\left(\frac{|X_n|}{b_n} > 1\right) \\
&\le 2C \sum_{n=1}^{\infty} E\left(\frac{f_n(|X_n|)}{f_n(b_n)}\right) < \infty, \qquad (3.23)
\end{aligned}$$

so that I_{14} converges.

Next, if the condition (3.21) is satisfied, then in the interval $|x| \le b_n$, it follows from $f_n(x)/x^2 \downarrow$ that $|x^2|/{b_n}^2 \le {f_n}^2(x)/{f_n}^2(b_n) \le f_n(x)/f_n(b_n)$ and note that $\{(Y_n - EY_n)/b_n | n \ge 1\}$ is a still an AANA random variables.

Thus, by Lemma 2 and (3.22), we show that $\sum_{n=1}^{\infty}(Y_n - EY_n)/b_n$ converges almost surely.

$$\begin{aligned}
\sum_{n=1}^{\infty}(\log n)^2 E\left(\left|\frac{Y_n - EY_n}{b_n}\right|\right)^2 &\le \sum_{n=1}^{\infty}\frac{(\log n)^2}{{b_n}^2} E|Y_n|^2 \\
&= \sum_{n=1}^{\infty}\frac{(\log n)^2}{{b_n}^2} E\left|-b_n I(X_n < -b_n) + X_n I(|X_n| \le b_n) + b_n I(X_n > b_n)\right|^2 \\
&\le \sum_{n=1}^{\infty}(\log n)^2 P(|X_n| > b_n) + \sum_{n=1}^{\infty}(\log n)^2 E\left(\frac{|X_n|^2}{{b_n}^2}\right) I(|X_n| \le b_n) \\
&\le 2C \sum_{n=1}^{\infty}(\log n)^2 E\left(\frac{f_n(|X_n|)}{f_n(b_n)}\right) < \infty, \qquad (3.24)
\end{aligned}$$

so, I_{13} converges almost surely.

Finally, we estimate I_{12}.

$$\sum_{n=1}^{\infty} P(X_n \ne Y_n) \le \sum_{n=1}^{\infty} E\left(\frac{f_n(|X_n|)}{f_n(b_n)}\right) < \infty, \qquad (3.25)$$

so that $I_{12} < \infty$.

By Theorem 5, we know that from (3.23), (3.24) and (3.25), $\sum_{n=1}^{\infty} X_n/b_n$ converges almost surely. The proof is completed. □

From the Theorem 6 we can get the following corollary.

Corollary 3. *Let all the conditions except $EX_n = 0$ be satisfied and let $x/f_n(x) \uparrow$ as $0 < x \uparrow$, $n \ge 1$, then $\sum_{n=1}^{\infty} X_n/b_n$ converges almost surely.*

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