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# Comments on "A Goodness of Fit Approach to Major Life Testing Problems" by Ahmad et al. (2001)

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**Abstract.** We comment on a new testing procedure for testing exponentiality against different ageing classes. We show that the proposed test is *inappropriate* at least for two alternatives. We point out the subtle flaw in their argument.

**Key Words:** *Goodness of fit, life distributions, monotonic aging, scale invariant.*

## 1. INTRODUCTION

The assumption of exponentiality is widely used in the theory of reliability and life testing. Testing for exponentiality of the failure time is, in effect, the same as testing the Poisson assumption about the process producing the shock that causes failure. Many so-called omnibus tests for exponentiality exist. For a recent survey of such tests, see Henze and Meintanis (2005). However, the existence of some prior information may considerably reduce the space of alternative hypotheses.

Equally important in reliability theory is the concept of aging. No aging means the age of the component has no effect on the distribution of its residual life time. Positive (negative) aging means that age has an adverse (beneficial) effect, in some probabilistic sense, on the residual life time. These notions of aging are captured through the monotonic aging families like IFR, IFRA, NBU, NBUC, NBUE, HNBUE, DMRL and their duals. For definitions of these classes, see Lai and Xie (2006).

Since the closed form of a distribution function is more often than not unavailable in practical situations, it is of great importance to test statistically whether the population distribution of a given set of data belongs to a particular non-parametric family. Testing against exponentiality has been the subject of investigation for over four decades. An excellent quick reference is the book by Lai and Xie (2006). In this short note we comment on a novel test procedure proposed by Ahmad et al. (2001). Their approach is

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briefly described in Section 2, while the subtle drawback in at least two statistics is pointed out in Section 3.

## 2. AHMAD ET AL. (2001) SUGGESTED PROCEDURE

The null distribution for all the above mentioned aging classes is the exponential distribution. Hence, in general the hypothesis testing problem is of the following type:

$H_0$: The underlying distribution is exponential

versus

$H_1$: The underlying distribution belongs to a specific aging family

The general procedure in such testing problems is to define a measure of difference which under the null hypothesis of exponentiality will be zero and will be pronounced (large and significant) under the alternative $H_1$. This difference is then weighted suitably as a functional of $F$. Clearly this weighted difference will be zero under $H_0$ and large under $H_1$. Then a sample version of this measure is used as a test statistic and its properties are studied. It is estimated using the sample data. The knowledge of the null hypothesis being exponential is not utilized. In contrast in the case of goodness of fit problems the test statistic is based on a measure of departure from $H_0$ that depends on both $H_0$ and $H_1$. Ahmad et al. (2001) used the ingenious idea of incorporating $H_0$ into the measure of departure. They remark that this leads to simpler test statistics and enjoy the same properties and have equal or higher efficiency than the classical procedures. Note that they have incorporated both $H_0$ and $H_1$ in devising a test statistic.

## 3. THE SUBTLE FLAW

Ahmad et al. (2001) make a sweeping remark stating that the measures $\delta^{(i)}$, for $i = 1$ to $5$, are scale invariant and without loss of generality take $\mu = 1$ and hence $F_0 = 1 - e^{-x}$. Based on this choice of $F_0$, they develop the test statistics. Observe that this choice of $F_0$ makes their test statistics very simple. However, this choice of $F_0$ is valid only if the measure $\delta^{(i)}$ is scale invariant. Observe that the statement that the measure $\delta^{(i)}$ is scale invariant though made without any proof is very crucial. We, in fact, show in Propostions 1 and 2 below that this assumption is incorrect for at least two cases. This essentially means that the test proposed by Ahmad et al. (2001) is incorrect for the concerned alternatives. We begin with a definition.

**Definition 1:** A test statistic $T(X_1, X_2, \cdots, X_n)$ is said to be scale invariant if $T(X_1, X_2, \cdots, X_n) = T(Y_1, Y_2, \cdots, Y_n)$, where $Y_i = \lambda X_i$, for $\forall\, i = 1, \cdots, n$ and $\lambda > 0$.

**Proposition 1:** The measure of departure

$$\delta^{(4)} = \mu^2 \int_0^\infty \bar{F}(x)\,dF_0(x) - \mu \int_0^\infty \bar{\nu}(x)\,dF_0(x)$$

(given in Ahmed et al. (2001) as Equation 1.14) is not scale invariant.

**Proof:** We need to show that $\delta(y) \neq \delta(x)$ where $Y_i = bX$, for $\forall\, i = 1, \cdots, n$ and $b > 0$.

We have $\mu = E(X)$ and $\bar{\nu}(x) = \int_x^\infty \bar{F}(u)\,du$.

Now, assume $Y = bX,\, b > 0$. Let the cdf of $Y$ be $G(y)$ and the cdf of $X$ be $F(x)$. Then it is easy to show that $G(y) = F\left(\frac{y}{b}\right)$; hence $\bar{G}(y) = \bar{F}\left(\frac{y}{b}\right)$. Observe that

$$\bar{\nu}(y) = \int_y^\infty \bar{G}(u)du = \int_y^\infty \bar{F}(u/b)du = \int_{y/b}^\infty \bar{F}(z)dz = b\int_x^\infty \bar{F}(z)dz = b\bar{\nu}(x) = b\bar{\nu}(y/b).$$

Then

$$\begin{aligned}
\Delta_{AAM} &= \mu_Y^2 \int_0^\infty \bar{G}(y)\,dG_0(y) - \mu_Y \int_0^\infty \bar{\nu}(y)\,dG_0(y) \\
&= b^2\mu^2 \int_0^\infty \bar{F}(y/b)dF_0(y/b) - b\mu \int_0^\infty \left(b\int_{y/b}^\infty \bar{F}(u)du\right) dF_0(y/b) \\
&= b^2\mu^2 \int_0^\infty \bar{F}(x)dF_0(x) - b^2\mu \int_0^\infty \bar{\nu}(y/b)\,dF_0(x) \\
&= b^2\mu^2 \int_0^\infty \bar{F}(x)dF_0(x) - b^2\mu \int_0^\infty \bar{\nu}(x)\,dF_0(x) = b^2\delta^{(4)}
\end{aligned}$$

This proves Proposition 1. ■

**Proposition 2:** The measure of departure

$$\delta^{(5)} = \mu \int_0^\infty e^{-x/\mu}\,dF_0(x) - \int_0^\infty \bar{\nu}(x)dF_0(x)$$

(given in Ahmed et al. (2001) as Equation 1.17) is not scale invariant.

**Proof:** We need to show that $\delta(y) \neq \delta(x)$ where $Y_i = bX$, for $\forall\, i = 1, \cdots, n$ and $b > 0$.

Then , as before we have

$$\begin{aligned}\Delta_{AAM} &= \mu\int_0^\infty e^{-y/\mu_y}\,dG_0(y) - \int_0^\infty \bar{v}(y)\,d\,G_0(y)\\ &= b\mu\int_0^\infty e^{-y/b\mu}dF_0(y/b) - \int_0^\infty \left(b\int_{y/b}^\infty \bar{F}(u)\,du\right)dF_0(y/b)\\ &= b\mu\int_0^\infty e^{-x/\mu}dF_0(x) - b\int_0^\infty \bar{v}(y/b)\,dF_0(x)\\ &= b\mu\int_0^\infty e^{-x/\mu}dF_0(x) - b\int_0^\infty \bar{v}(x)\,dF_0(x) = b\delta^{(5)}\end{aligned}$$

This proves Proposition 2. ■

Since scale invariance of $\delta^{(4)}$ and $\delta^{(5)}$ is a key assumption and this assumption is in fact incorrect, it follows that the procedure developed in Ahmad et al. (2001) for testing exponentiality against NBUE and HNBUE alternatives is inappropriate.

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