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A Class of Bivariate Linear Failure Rate Distributions and Their Mixtures

Ammar M. Sarhan * †
Department of Mathematics & Statistics, Faculty of Science,
Dalhousie University, Halifax NS B3H 3J5, Canada

A. El-Gohary , **A. H. El-Bassiouny**
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt

N. Balakrishnan
Department of Mathematics and Statistics, McMaster University,
1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1

Abstract. A new bivariate linear failure rate distribution is introduced through a shock model. It is proved that the marginal distributions of this new bivariate distribution are linear failure rate distributions. The joint moment generating function of the bivariate distribution is derived. Mixtures of bivariate linear failure rate distributions are also discussed. Application to a real data is given.

Key Words : Mixture distributions, Bivariate distribution, Linear failure rate distribution, exponential distribution, joint moment generating function, marginal distributions

1. INTRODUCTION

The univariate linear failure rate distribution is widely used in reliability analysis and electronics; see, for example, Ahmad (2001), Pandey, Singh and Zimmer (1993), Sarhan (1996), Zacks (1991), Balakrishnan and Basu (1995), and Johnson, Kotz and Balakrishnan (1994).

Generally, shock models are used in reliability to describe different applications. Shocks can refer, for example, to damages caused to biological organs by illness or environmental causes of damage acting on a technical system; see A-Hameed and Proschan (1973).

Several basic multivariate parametric families of distributions such as multivariate exponential, Weibull, gamma, and normal distributions, and shock models that give rise to them have been discussed by Barlow and Proschan (1981). Earlier, Marshall and Olkin (1967a)

*Corresponding author.
E-mail address: asarhan@mathstat.dal.ca
†Home address: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

considered a shock model to derive a bivariate exponential distribution. Generalization of this bivariate exponential distribution was proposed by Marshall and Olkin (1967b). For a detailed review of all these developments, one may refer to Kotz, Balakrishnan and Johnson (2000).

Shock models have been used to estimate the reliability measures for series and parallel systems with nonindependent and nonidentical components by Grabski and Sarhan (1996), Sarhan (1996) and El-Gohary and Sarhan (2005). Also, Sarhan and Abouammoh (2001) used a shock model to derive the reliability function of a nonrepairable $k-$out$-$of$-n$ system with nonindependent and nonidentical components.

The main purpose of this paper is to introduce a new bivariate linear failure rate distribution (BLFRD). This distribution is derived as a distribution of the lifetimes of two nonindependent components each having a univariate linear failure rate distribution. Mixtures of BLFRD are also studied. This paper is organized as follows. Section 2 presents the shock model yielding the bivariate linear failure rate distribution. Some important properties such as the joint survival function, joint moment generating function, expectations, and marginal and conditional distributions are also discussed in this section. In Section 3, mixtures of these BLFRD's are discussed. Finally, in section 4, application to a real data set is given.

2. THE BLFRD

In this section, we introduce a new form of bivariate linear failure rate distribution. We start with the bivariate survival function from which the joint probability density function is derived. We then discuss the moment generating function, expectations, and the marginal and conditional distributions.

2.1 Shock model yielding the BLFRD

Suppose that there are three independent sources of shocks present in the environment of a system consisting of two components. A shock from Source 1 could destroy Component 1 and that it could occur at a random time U_1; a shock from Source 2 could destroy Component 2 and that it could occur at a random time U_2; finally, a shock from Source 3 could destroy both components and that it could occur at a random time U_3. Then, the random lifetime of Component 1, say X, is given by $X = \min(U_1, U_3)$, while the random lifetime of Component 2, say Y, is given by $Y = \min(U_2, U_3)$.

Let us assume that the random variables U_1 and U_2 have linear failure rate distributions with parameters (α_i, β_i), say LFRD(α_i, β_i), for $i = 1, 2$, respectively. Next, let U_3 have an exponential distribution with parameter θ_0. That is, the probability density function of U_i $(i = 1, 2)$ is given by

$$g_i(t) = (\alpha_i + \beta_i\, t) \exp\left\{-\left(\alpha_i\, t + \frac{\beta_i}{2}\, t^2\right)\right\}, \quad t \geq 0,\ \alpha_i,\ \beta_i > 0 \tag{2.1}$$

with the corresponding survival function as

$$\bar{G}_i(t) = \exp\left\{-\left(\alpha_i\, t + \frac{\beta_i}{2}\, t^2\right)\right\}, \quad t \geq 0. \tag{2.2}$$

The probability density function of U_3 is

$$g_3(t) = \theta_0 \exp\{-\theta_0 t\},\ t \geq 0,\ \theta_0 > 0 \tag{2.3}$$

with the corresponding survival function

$$\bar{G}_3(t) = \exp\{-\theta_0 t\}, \quad t \geq 0. \tag{2.4}$$

It is evident that the lifetimes of the system components, X and Y, are dependent because of the common source of Shock 3.

We now study the joint distribution of the random variables X and Y, which is called a *bivariate linear failure rate distribution.*

Lemma 2.1. The joint survival function of X and Y is

$$\bar{F}_{X,Y}(x,y) = \exp\left\{-\alpha_1 x - \alpha_2 y - \frac{1}{2}\beta_1 x^2 - \frac{1}{2}\beta_2 y^2 - \theta_0 \max(x,y)\right\}, \quad x, y > 0. \tag{2.5}$$

Proof. Since

$$\bar{F}_{X,Y}(x,y) = P\Big(X > x,\, Y > y\Big),$$

we have

$$\begin{aligned}
\bar{F}_{X,Y}(x,y) &= P\Big(\min(U_1, U_3) > x,\, \min(U_2, U_3) > y\Big) \\
&= P\Big(U_1 > x,\, U_3 > x,\, U_2 > y,\, U_3 > y\Big) \\
&= P\Big(U_1 > x,\, U_2 > y,\, U_3 > \max(x,\, y)\Big).
\end{aligned}$$

From the fact that U_1, U_2, U_3 are mutually independent with their survival functions as in (2.2) and (2.4), we get

$$\begin{aligned}
\bar{F}_{X,Y}(x,y) &= P(U_1 > x)\, P(U_2 > y)\, P\big(U_3 > \max(x,\, y)\big) \\
&= \bar{G}_1(x)\, \bar{G}_2(y)\, \bar{G}_3\big(\max(x,y)\big).
\end{aligned}$$

Upon substituting from (2.2) and (2.4) into the above equation, we readily obtain the expression in (2.5).

The following corollary gives the survival functions of the marginal distributions of X and Y.

Corollary 2.1. The marginal distributions of BLFRD are LFRD with the following marginal survival functions:

$$\bar{F}_i(t) = \exp\left\{-(\alpha_i + \theta_0)t - \frac{1}{2}\beta_i t^2\right\}, \quad t \geq 0, i = 1, 2. \tag{2.6}$$

The following corollary presents the joint distribution function of (X, Y).

Corollary 2.2. The joint distribution function of $(X,\, Y)$ is given by

$$\begin{aligned} F(x,y) &= 1-\exp\left\{-(\alpha_1+\theta_0)x-\frac{\beta_1}{2}\,x^2\right\}-\exp\left\{-(\alpha_2+\theta_0)y-\frac{\beta_2}{2}\,y^2\right\} \\ &\quad +\exp\left\{-\theta_0\max(x,\,y)-\alpha_1\,x-\alpha_2\,y-\frac{\beta_1}{2}\,x^2-\frac{\beta_2}{2}\,y^2\right\},\ \ x,y>0. \end{aligned} \tag{2.7}$$

Proof. The expression in (2.7) readily follows from the relation

$$F(t_1,t_2)=1-\bar{F}_1(t_1)-\bar{F}_2(t_2)+\bar{F}(t_1,\,t_2),\ \ t_1,t_2>0.$$

2.2 The joint, marginal and conditional density functions

In this subsection, we derive the joint probability density function of the BLFRD. We then derive the marginal pdf's of X and Y as well as the conditional pdf's of X given Y and of Y given X.

Theorem 2.1. If the joint survival function of (X,Y) is as in (2.5), the joint probability density function of (X,Y) is given by

$$f(x,y)=\begin{cases} f_1(x,y) & \text{if } \ x>y>0, \\ f_2(x,y) & \text{if } \ y>x>0, \\ f_0(x,y) & \text{if } \ x=y>0 \end{cases} \tag{2.8}$$

where

$$\begin{aligned} f_1(x,y) &= (\alpha_1+\theta_0+\beta_1 x)(\alpha_2+\beta_2 y)e^{-\alpha_1 x-\alpha_2 y-\frac{1}{2}\beta_1 x^2-\frac{1}{2}\beta_2 y^2-\theta_0 x}, \\ f_2(x,y) &= (\alpha_1+\beta_1 x)(\alpha_2+\theta_0+\beta_2 y)e^{-\alpha_1 x-\alpha_2 y-\frac{1}{2}\beta_1 x^2-\frac{1}{2}\beta_2 y^2-\theta_0 y}, \\ f_0(x,y) &= \theta_0 e^{-(\alpha_1+\alpha_2+\theta_0)x-\frac{1}{2}(\beta_1+\beta_2)x^2}. \end{aligned}$$

Proof. The forms of $f_1(x,y)$ and $f_2(x,y)$ can be readily obtained by differentiating $\bar{F}_{X,Y}(x,y)$ in (2.5) with respect to x and y. But, $f_0(x,x)$ can not be derived in a similar method. For this reason, we use the following identity to derive $f_0(x,x)$:

$$\int_0^\infty\int_0^x f_1(x,y)dydx+\int_0^\infty\int_0^y f_2(x,y)dxdy+\int_0^\infty f_0(x,x)dx=1. \tag{2.9}$$

One can find by direct integration that

$$\int_0^\infty\int_0^x f_1(x,y)dydx=1-\int_0^\infty(\theta_0+\alpha_1+\beta_1 x)e^{-(\alpha_1+\theta_0+\alpha_2)x-\frac{1}{2}\,(\beta_1+\beta_2)\,x^2}dx \tag{2.10}$$

and

$$\int_0^\infty\int_0^y f_2(x,y)dxdy=1-\int_0^\infty(\theta_0+\alpha_2+\beta_2 y)e^{-(\alpha_1 x+\alpha_2+\theta_0)y-\frac{1}{2}\,(\beta_1+\beta_2)y^2}dy. \tag{2.11}$$

From (2.10) and (2.11), we get

$$\int_0^\infty\int_0^x f_1(x,y)dydx+\int_0^\infty\int_0^y f_2(x,y)dxdy=1-\int_0^\infty\theta_0\,e^{-(\alpha_1+\beta_1+\theta_0)x-\frac{1}{2}\,(\alpha_2+\beta_2)x^2}\,dx. \tag{2.12}$$

Upon using (2.12) in (2.9), we immediately obtain

$$\int_0^\infty f_0(x,x)dx = \int_0^\infty \theta_0\, e^{-(\alpha_1+\beta_1+\theta_0)x+\frac{1}{2}\,(\alpha_2+\beta_2)x^2}\, dx$$

from which we get

$$f_0(x,x) = \theta_0 e^{-(\alpha_1+\beta_1+\theta_0)x-\frac{1}{2}\,(\alpha_2+\beta_2)x^2},\ x>0, \tag{2.13}$$

which completes the proof of the theorem.

The following corollary gives the marginal pdf's of X and Y.

Corollary 2.3. The marginal pdf's of X and Y are

$$f_X(x) = (\alpha_1+\theta_0+\beta_1 x)e^{-(\alpha_1+\theta_0)x-\frac{1}{2}\,\beta_1 x^2}\ ,\ x>0, \tag{2.14}$$

and

$$f_Y(y) = (\alpha_2+\theta_0+\beta_2 y)e^{-(\alpha_2+\theta_0)y-\frac{1}{2}\,\beta_2 y^2}\ ,\ y>0. \tag{2.15}$$

Proof. One can prove this corollary either by differentiating the marginal survival functions of X and Y given in (2.6) or by integrating the joint pdf of (X,Y) in (2.8) with respect to y and x, respectively.

Note that the marginal distributions of X and Y are also linear failure rate distributions.

Corollary 2.4. The conditional pdf's of X given Y and of Y given X are as follows:

$$f_{X|Y}(x|y) = \begin{cases} \frac{(\alpha_1+\theta_0+\beta_1 x)(\alpha_2+\beta_2 y)}{\alpha_2+\theta_0+\beta_2 y}\, e^{-\left\{\alpha_1 x+\theta_0(x-y)+\frac{1}{2}\beta_1 x^2\right\}} & \text{if } x>y>0\,, \\ (\alpha_1+\beta_1 x)\, e^{-(\alpha_1 x+\frac{1}{2}\beta_1 x^2)} & \text{if } y>x>0\,, \\ \frac{\theta_0}{\alpha_2+\theta_0+\beta_2 y}\, e^{-(\alpha_1 x+\frac{1}{2}\beta_1 x^2)} & \text{if } x=y>0, \end{cases} \tag{2.16}$$

and

$$f_{Y|X}(y|x) = \begin{cases} (\alpha_2+\beta_2 y)\, e^{-(\alpha_2 y+\frac{1}{2}\beta_2 y^2)} & \text{if } x>y>0\,, \\ \frac{(\alpha_2+\theta_0+\beta_2 y)(\alpha_1+\beta_1 x)}{(\alpha_1+\theta_0+\beta_1 x)}\, e^{-\left\{\theta_0(y-x)+\alpha_2 y+\frac{1}{2}\beta_2 y^2\right\}} & \text{if } y>x>0\,, \\ \frac{\theta_0}{\alpha_1+\theta_0+\beta_1 y}\, e^{-(\alpha_2 y+\frac{1}{2}\beta_2 y^2)} & \text{if } x=y>0\,. \end{cases} \tag{2.17}$$

Proof. With

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{and} \quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)},$$

upon substituting from (2.8), (2.14) and (2.15) in the above expressions and simplifying, we obtain the expressions of the conditional densities as in (2.16) and (2.17).

2.3 Moment generating functions and expectations

In this subsection, we derive the joint moment generating function of (X,Y) as well as the marginal moment generating functions of X and Y. These functions are then used to obtain the first- and second-order moments.

Theorem 2.2. The joint moment generating function of (X, Y) is

$$\begin{aligned} M_{X,Y}(t_1,t_2) &= 1 + t_1 \sqrt{\frac{\pi}{2\beta_1}}\, e^{\frac{(\alpha_1+\theta_0-t_1)^2}{2\beta_1}} \left\{1 - \Psi\left(\frac{\alpha_1+\theta_0-t_1}{\sqrt{2\beta_1}}\right)\right\} \\ &\quad + t_2 \sqrt{\frac{\pi}{2\beta_2}}\, e^{\frac{(\alpha_2+\theta_0-t_2)^2}{2\beta_2}} \left\{1 - \Psi\left(\frac{\alpha_2+\theta_0-t_2}{\sqrt{2\beta_2}}\right)\right\} \\ &\quad + t_1 t_2 \left[\sqrt{\frac{\pi}{2\beta_1}}\, e^{\frac{(\alpha_1+\theta_0-t_1)^2}{2\beta_1}}\, I_1(t_1,t_2) + \sqrt{\frac{\pi}{2\beta_2}}\, e^{\frac{(\alpha_2+\theta_0-t_2)^2}{2\beta_2}}\, I_2(t_1,t_2)\right], \end{aligned} \tag{2.18}$$

where

$$I_1(t_1,t_2) = \int_0^\infty \left\{1 - \Psi\left(\frac{\alpha_1+\theta_0-t_1+\beta_1 u}{\sqrt{2\beta_1}}\right)\right\} e^{-\left[(\alpha_2-t_2)u+\frac{1}{2}\beta_2 u^2\right]} du\,, \tag{2.19}$$

$$I_2(t_1,t_2) = \int_0^\infty \left\{1 - \Psi\left(\frac{\alpha_2+\theta_0-t_2+\beta_2 u}{\sqrt{2\beta_2}}\right)\right\} e^{-\left[(\alpha_1-t_1)u+\frac{1}{2}\beta_1 u^2\right]} du\,, \tag{2.20}$$

and $\Psi(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-u^2}\, du$.

Proof. With the moment generating function of (X, Y) being

$$M_{X,Y}(t_1,t_2) = E\left[e^{t_1 X + t_2 Y}\right],$$

we can write from (2.8) that

$$\begin{aligned} M_{X,Y}(t_1,t_2) &= \int_0^\infty \int_0^x e^{t_1 x + t_2 y}\, f_1(x,y)\, dy\, dx + \int_0^\infty \int_0^y e^{t_1 x + t_2 y} f_2(x,y)\, dx\, dy \\ &\quad + \int_0^\infty e^{(t_1+t_2)x}\, f_0(x,x) dx. \end{aligned}$$

Substituting the expressions of f_0, f_1 and f_2, carrying out the integrations, and simplifying the resulting expression, we obtain (2.18).

The following corollary presents the marginal moment generating functions of X and Y.

Corollary 2.5. The marginal moment generating functions of X and Y are given by

$$M_X(t_1) = 1 + \sqrt{\frac{\pi}{2\beta_1}}\, t_1 \left\{1 - \Psi\left(\frac{\alpha_1+\theta_0-t_1}{\sqrt{2\beta_i}}\right)\right\} \exp\left\{\frac{(\alpha_1+\theta_0-t_1)^2}{2\beta_1}\right\} \tag{2.21}$$

and

$$M_Y(t_2) = 1 + \sqrt{\frac{\pi}{2\beta_2}}\, t_2 \left\{1 - \Psi\left(\frac{\alpha_2+\theta_0-t_2}{\sqrt{2\beta_2}}\right)\right\} \exp\left\{\frac{(\alpha_2+\theta_0-t_2)^2}{2\beta_2}\right\}. \tag{2.22}$$

Corollary 2.6. From (2.18), (2.21) and (2.22), we obtain:

$$E[X] = \sqrt{\frac{\pi}{2\beta_1}} \left\{1 - \Psi\left(\frac{\alpha_1+\theta_0}{\sqrt{2\beta_1}}\right)\right\} \exp\left\{\frac{(\alpha_1+\theta_0)^2}{2\beta_1}\right\}, \tag{2.23}$$

$$
\begin{aligned}
E[Y] &= \sqrt{\frac{\pi}{2\beta_2}}\left\{1-\Psi\left(\frac{\alpha_2+\theta_0}{\sqrt{2\beta_2}}\right)\right\}\exp\left\{\frac{(\alpha_2+\theta_0)^2}{2\beta_2}\right\}, && (2.24)\\
E[X^2] &= \frac{2}{\beta_1}+\frac{\sqrt{2\pi}\,(\alpha_1+\theta_0)}{\beta_1\sqrt{\beta_1}}\left\{1-\Psi\left(\frac{\alpha_1+\theta_0}{\sqrt{2\beta_1}}\right)\right\}\exp\left\{\frac{(\alpha_1+\theta_0)^2}{2\beta_1}\right\}, && (2.25)\\
E[Y^2] &= \frac{2}{\beta_2}+\frac{\sqrt{2\pi}\,(\alpha_2+\theta_0)}{\beta_2\sqrt{\beta_2}}\left\{1-\Psi\left(\frac{\alpha_2+\theta_0}{\sqrt{2\beta_2}}\right)\right\}\exp\left\{\frac{(\alpha_2+\theta_0)^2}{2\beta_2}\right\}, && (2.26)\\
E[X\,Y] &= \sqrt{\frac{\pi}{2\beta_1}}\,I_1(0,0)\exp\left\{\frac{(\alpha_1+\theta_0)^2}{2\beta_1}\right\}+\sqrt{\frac{\pi}{2\beta_2}}\,I_2(0,0)\exp\left\{\frac{(\alpha_2+\theta_0)^2}{2\beta_2}\right\}. && (2.27)
\end{aligned}
$$

The expressions in (2.23) – (2.27) can be used to derive the variances of X and Y and the covariance and correlation coefficient between X and Y.

Note that, by setting $\beta_1=\beta_2=0$ in the above results, we get the corresponding formulae for the bivariate exponential distribution. For example, we have the following results.

Corollary 2.7. The moment generating function of the bivariate exponential distribution is

$$M_{X,Y}(t_1,t_2)=\frac{1}{(\alpha_1+\alpha_2+\theta_0-t_1-t_2)}\left\{\frac{\alpha_1(\alpha_2+\theta_0)}{(\alpha_2+\theta_0-t_2)}+\frac{\alpha_2(\alpha_1+\theta_0)}{(\alpha_1+\theta_0-t_1)}+\theta_0\right\}. \quad (2.28)$$

From (2.28), we readily get the following:

$$E(X)=\frac{1}{\alpha_1+\theta_0},\quad E(Y)=\frac{1}{\alpha_2+\theta_0},$$

and

$$E(XY)=\frac{\alpha_1+2\alpha_2+2\theta_0}{(\alpha_2+\theta_0)(\alpha_1+\alpha_2+\theta_0)^2}+\frac{(\alpha_1+\alpha_2)(\alpha_1+\theta_0)+2\alpha_2(\alpha_1+\alpha_2+\theta_0)}{(\alpha_2+\theta_0)^2(\alpha_1+\alpha_2+\theta_0)^2}. \quad (2.29)$$

3. MIXTURES OF BLFRD

In this section, we first discuss the mixture of independent linear failure rate distributions. We then derive a mixture of bivariate linear failure rate distributions where the dependence among the components is characterized by a latent random variable independently distributed of the individual components.

Consider a system of two components where the lifetime of Component i ($i=1,2$), say X_i, is a mixture of two independent linear failure rate distributions as follows:

$$X_1 \sim a_1\,\mathrm{LFRD}(\alpha_{11},\beta_{11})+(1-a_1)\,\mathrm{LFRD}(\alpha_{12},\beta_{12}),\ 0\le a_1\le 1,$$

and

$$X_2 \sim a_2\,\mathrm{LFRD}(\alpha_{21},\beta_{21})+(1-a_2)\,\mathrm{LFRD}(\alpha_{22},\beta_{22}),\ 0\le a_2\le 1.$$

Here, the notation LFRD$(\alpha_{ij}, \beta_{ij})$ means a random variable, say X_{ij}, having a linear failure rate distribution with parameters $(\alpha_{ij}, \beta_{ij})$ and density function

$$f_{X_{ij}}(x) = (\alpha_{ij} + \beta_{ij}x)e^{-(\alpha_{ij}x+\frac{1}{2}\beta_{ij}x^2)},\ x > 0,\ \alpha_{ij} > 0,\ \beta_{ij} > 0\ \forall\, i, j\,.$$

Consider also an exponentially distributed random variable, say Z, with parameter θ_0 which is independent of X_{ij} for all i, j. This random variable Z will be used as a latent variable in order to introduce dependence among $X's$. The density function of Z is

$$f_Z(z) = \theta_0 e^{-\theta_0 z},\ \ z > 0,\ \ \theta_0 > 0\,.$$

Because of the independence assumption made above, we see that Z is also independent of X_1 and X_2.

Define $S_i = \text{Min}(X_i, Z)$ for $i = 1, 2$. Then, the vector (S_1, S_2) has a bivariate distribution with S_1 and S_2 being obviously dependent as they commonly share the influence of the latent random variable Z.

Now, we derive the joint survival function of $(S_1,\ S_2)$ from which we also derive the joint density function of $(S_1,\ S_2)$.

Corollary 3.1. The joint survival function of (S_1, S_2) is given by

$$\begin{aligned}\bar{F}(s_1, s_2) \;\;=\;\; & p_{11}\, e^{-\{\alpha_{11}s_1+\alpha_{21}s_2+\frac{1}{2}(\beta_{11}s_1^2+\beta_{21}s_2^2)+\theta_0 s_0\}}\\ & +p_{12}\, e^{-\{\alpha_{11}s_1+\alpha_{22}s_2)+\frac{1}{2}(\beta_{11}s_1^2+\beta_{22}s_2^2)+\theta_0 s_0\}}\\ & +p_{21}\, e^{-\{\alpha_{12}s_1+\alpha_{21}s_2+\frac{1}{2}(\beta_{12}s_1^2+\beta_{21}s_2^2)+\theta_0 s_0\}}\\ & +p_{22}\, e^{-\{\alpha_{12}s_1+\alpha_{22}s_2+\frac{1}{2}(\beta_{12}s_1^2+\beta_{22}s_2^2)+\theta_0 s_0\}},\ s_1, s_2 > 0, \end{aligned} \tag{3.1}$$

where $s_0 = max(s_1, s_2) > 0$, and for $i, j \in \{1, 2\}$

$$p_{ij} = a_1^{2-i}\, a_2^{2-j}\, (1-a_1)^{i-1}\, (1-a_2)^{j-1}\,.$$

Proof. From

$$\bar{F}(s_1, s_2) = P(S_1 > s_1, S_2 > s_2),$$

upon using the definitions of S_1 and S_2, we readily find

$$\begin{aligned}\bar{F}(s_1, s_2) &= P(X_1 > s_1)\, P(X_2 > s_2)\, P(Z > s_0)\\ &= e^{-\theta_0 s_0} \prod_{i=1}^{2} \left[a_i\, e^{-(\alpha_{i1}s_i+\frac{1}{2}\beta_{i1}s_i^2)} + (1-a_i)\, e^{-(\alpha_{i2}s_i+\frac{1}{2}\beta_{i2}s_i^2)}\right]\end{aligned}$$

which can be rewritten as in (3.1).

Note that:

1. For $i, j \in \{1, 2\}$, $p_{ij} \geq 0$ and $p_{11} + p_{12} + p_{21} + p_{22} = 1$.
2. Each function on the right hand side of (3.1) is a survival function of a bivariate linear failure rate distribution of the form given in (2.5).

This implies that the joint survival function given in (3.1) is indeed a joint survival function of a mixture of four bivariate linear failure rate distributions.

The following theorem gives the joint pdf of S_1 and S_2.

Theorem 3.1. The joint pdf of (S_1, S_2) is given by

$$f(s_1,s_2)=\begin{cases} f_1(s_1,s_2) & \text{if } s_1>s_2\,,\\ f_2(s_1,s_2) & \text{if } s_2>s_1\,,\\ f_0(s_1,s_1) & \text{if } s_1=s_2\,,\end{cases} \tag{3.2}$$

where

$$\begin{aligned} f_1(s_1,s_2)= \;& p_{11}(\theta_0+\alpha_{11}+\beta_{11}s_1)(\alpha_{21}+\beta_{21}s_2)e^{-\left\{(\alpha_{11}+\theta_0)s_1+\alpha_{21}s_2+\frac{1}{2}(\beta_{11}s_1^2+\beta_{21}s_2^2)\right\}}\\ &+p_{12}(\theta_0+\alpha_{11}+\beta_{11}s_1)(\alpha_{22}+\beta_{22}s_2)e^{-\left\{(\alpha_{11}+\theta_0)s_1+\alpha_{22}s_2+\frac{1}{2}(\beta_{11}s_1^2+\beta_{22}s_2^2)\right\}}\\ &+p_{21}(\theta_0+\alpha_{12}+\beta_{12}s_1)(\alpha_{21}+\beta_{21}s_2)e^{-\left\{(\alpha_{12}+\theta_0)s_1+\alpha_{21}s_2+\frac{1}{2}(\beta_{12}s_1^2+\beta_{21}s_2^2)\right\}}\\ &+p_{22}(\theta_0+\alpha_{12}+\beta_{12}s_1)(\alpha_{22}+\beta_{22}s_2)e^{-\left\{(\alpha_{12}+\theta_0)s_1+\alpha_{22}s_2+\frac{1}{2}(\beta_{12}s_1^2+\beta_{22}s_2^2)\right\}}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} f_2(s_1,s_2)= \;& p_{11}(\alpha_{11}+\beta_{11}s_1)(\theta_0+\alpha_{21}+\beta_{21}s_2)e^{-\left\{\alpha_{11}s_1+(\theta_0+\alpha_{21})s_2+\frac{1}{2}(\beta_{11}s_1^2+\beta_{21}s_2^2)\right\}}\\ &+p_{12}(\alpha_{11}+\beta_{11}s_1)(\theta_0+\alpha_{22}+\beta_{22}s_2)e^{-\left\{\alpha_{11}s_1+(\theta_0+\alpha_{22})s_2+\frac{1}{2}(\beta_{11}s_1^2+\beta_{22}s_2^2)\right\}}\\ &+p_{21}(\alpha_{12}+\beta_{12}s_1)(\theta_0+\alpha_{21}+\beta_{21}s_2)e^{-\left\{\alpha_{12}s_1+(\theta_0+\alpha_{21})s_2+\frac{1}{2}(\beta_{12}s_1^2+\beta_{21}s_2^2)\right\}}\\ &+p_{22}(\alpha_{12}+\beta_{12}s_1)(\theta_0+\alpha_{22}+\beta_{22}s_2)e^{-\left\{\alpha_{12}s_1+(\theta_0+\alpha_{22}s_2)+\frac{1}{2}(\beta_{12}s_1^2+\beta_{22}s_2^2)\right\}}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} f_0(s_0,s_0)= \;& p_{11}\theta_0 e^{-\left\{(\alpha_{11}+\theta_0+\alpha_{21})s_0+\frac{1}{2}(\beta_{11}+\beta_{21})s_0^2\right\}}\\ &+p_{12}e^{-\left\{(\alpha_{11}+\theta_0+\alpha_{22})s_0+\frac{1}{2}(\beta_{11}+\beta_{22})s_0^2\right\}}\\ &+p_{21}e^{-\left\{(\alpha_{12}+\theta_0+\alpha_{21})s_0+\frac{1}{2}(\beta_{12}+\beta_{21})s_0^2\right\}}\\ &+p_{22}e^{-\left\{(\alpha_{12}+\theta_0+\alpha_{22})s_0+\frac{1}{2}(\beta_{12}+\beta_{22})s_0^2\right\}}. \end{aligned} \tag{3.5}$$

Corollary 3.2. The marginal pdf's of S_1 and S_2 are given by

$$\begin{aligned} f_{S_1}(s_1)= \;& a_1(\alpha_{11}+\theta_0+\beta_{11}s_1)e^{-\{(\alpha_{11}+\theta_0)s_1+\frac{1}{2}\beta_{11}s_1^2\}}\\ &+(1-a_1)(\alpha_{12}+\theta_0+\beta_{12}s_1)e^{-\{(\alpha_{12}+\theta_0)s_1+\frac{1}{2}\beta_{12}s_1^2\}}\,,\; s_1>0\,, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f_{S_2}(s_2)= \;& a_2(\alpha_{21}+\theta_0+\beta_{21}s_2)e^{-\{(\alpha_{21}+\theta_0)s_2+\frac{1}{2}\beta_{21}s_2^2\}}\\ &+(1-a_2)(\alpha_{22}+\theta_0+\beta_{22}s_2)e^{-\{(\alpha_{22}+\theta_0)s_2+\frac{1}{2}\beta_{22}s_2^2\}}\,,\; s_2>0\,. \end{aligned} \tag{3.7}$$

3.1 Moment generating functions and expectations

In this subsection, we derive the joint moment generating function of (S_1, S_2) and the marginal moment generating functions of S_1 and S_2, using which we derive the first- and second-order moments of S_1 and S_2.

Theorem 3.2. The joint moment generating function of $(S_1,\, S_2)$ is given by

$$M_{S_1,S_2}(t_1,t_2) \;=\; 1+\sqrt{\pi}\,t_1\left[\frac{a_1}{\sqrt{2\beta_{11}}}\left\{1-\Psi\left(\frac{\alpha_{11}+\theta_0-t_1}{\sqrt{2\beta_{11}}}\right)\right\}\exp\left\{\frac{(\alpha_{11}+\theta_0-t_1)^2}{2\beta_{11}}\right\}\right.$$

$$
\begin{aligned}
&+\frac{(1-a_1)}{\sqrt{2\beta_{12}}}\left\{1-\Psi\left(\frac{\alpha_{12}+\theta_0-t_1}{\sqrt{2\beta_{12}}}\right)\right\}\exp\left\{\frac{(\alpha_{12}+\theta_0-t_1)^2}{2\beta_{12}}\right\}\Bigg]\\
&+\sqrt{\pi}\,t_2\left[\frac{a_2}{\sqrt{2\beta_{21}}}\left\{1-\Psi\left(\frac{\alpha_{21}+\theta_0-t_2}{\sqrt{2\beta_{21}}}\right)\right\}\exp\left\{\frac{(\alpha_{21}+\theta_0-t_2)^2}{2\beta_{21}}\right\}\right.\\
&+\frac{(1-a_2)}{\sqrt{2\beta_{22}}}\left\{1-\Psi\left(\frac{\alpha_{22}+\theta_0-t_2}{\sqrt{2\beta_{22}}}\right)\right\}\exp\left\{\frac{(\alpha_{22}+\theta_0-t_2^2}{2\beta_{22}}\right\}\Bigg]\\
&+t_1t_2\sqrt{\pi}\left[\frac{1}{\sqrt{2\,\beta_{11}}}\left\{p_{11}\,I_1^{(11)}(t_1,t_2)+p_{12}\,I_1^{(12)}(t_1,t_2)\right\}\right.\\
&\qquad\qquad\times\exp\left\{\frac{(\alpha_{11}+\theta_0-t_1)^2}{2\beta_{11}}\right\}\\
&+\frac{1}{\sqrt{2\,\beta_{12}}}\left\{p_{21}\,I_1^{(21)}(t_1,t_2)+p_{22}\,I_1^{(22)}(t_1,t_2)\right\}\exp\left\{\frac{(\alpha_{12}+\theta_0-t_1)^2}{2\beta_{12}}\right\}\\
&+\frac{1}{\sqrt{2\,\beta_{21}}}\left\{p_{11}\,I_2^{(11)}(t_1,t_2)+p_{21}\,I_2^{(12)}(t_1,t_2)\right\}\exp\left\{\frac{(\alpha_{21}+\theta_0-t_2)^2}{2\beta_{21}}\right\}\\
&+\frac{1}{\sqrt{2\,\beta_{22}}}\left\{p_{12}\,I_2^{(12)}(t_1,t_2)+p_{22}\,I_2^{(22)}(t_1,t_2)\right\}\exp\left\{\frac{(\alpha_{22}+\theta_0-t_2)^2}{2\beta_{22}}\right\}\Bigg],
\end{aligned}
\tag{3.8}
$$

where $(I_1^{(11)},\, I_2^{(11)})$, $(I_1^{(12)},\, I_2^{(12)})$, $(I_1^{(21)},\, I_2^{(21)})$ and $(I_1^{(22)},\, I_2^{(22)})$ can be obtained from (2.19) (2.10), respectively, by replacing $(\alpha_1,\, \alpha_2,\, \beta_1,\, \beta_2)$ with $(\alpha_{11},\, \alpha_{12},\, \beta_{11},\, \beta_{12})$, $(\alpha_{11},\, \alpha_{22},\, \beta_{11},\, \beta_{22})$, $(\alpha_{12},\, \alpha_{21},\, \beta_{12},\, \beta_{21})$ and $(\alpha_{12},\, \alpha_{22},\, \beta_{12},\, \beta_{22})$.
Proof. From

$$M_{S_1,S_2}(t_1,t_2)=E\left(e^{t_1\,S_1+t_2\,S_2}\right)=\int_0^\infty\int_0^\infty e^{t_1s_1+t_2s_2}\,f(s_1,s_2)ds_1ds_2\,,$$

we can express from (3.2)

$$
\begin{aligned}
M_{S_1,S_2}(t_1,t_2) &= \int_0^\infty e^{(t_1+t_2)s_0}f_0(s_0,s_0)ds_0\\
&\quad+\int_0^\infty\int_0^{s_1} e^{(t_1s_1+t_2s_2)}f_1(s_1,s_2)ds_2ds_1\\
&\quad+\int_0^\infty\int_0^{s_2} e^{(t_1s_1+t_2s_2)}f_2(s_1,s_2)ds_1ds_2.
\end{aligned}
$$

Now, upon substituting from (3.3) – (3.5), carrying out the integrations, and simplifying the resulting expression, we obtain (3.8).

Corollary 3.3. The marginal moment generating functions of S_1 and S_2 are given by

$$
\begin{aligned}
M_{S_1}(t_1) &= 1+a_1\,t_1\sqrt{\frac{\pi}{2\beta_{11}}}\left\{1-\Psi\left(\frac{\alpha_{11}+\theta_0-t_1}{\sqrt{2\beta_{11}}}\right)\right\}\exp\left\{\frac{(\alpha_{11}+\theta_0-t_1)^2}{2\beta_{11}}\right\}\\
&\quad+(1-a_1)\,t_1\sqrt{\frac{\pi}{2\beta_{12}}}\left\{1-\Psi\left(\frac{\alpha_{12}+\theta_0-t_1}{\sqrt{2\beta_{12}}}\right)\right\}\exp\left\{\frac{(\alpha_{12}+\theta_0-t_1)^2}{2\beta_{12}}\right\}
\end{aligned}
\tag{3.9}
$$

and

$$\begin{aligned} M_{S_2}(t_2) &= 1 + a_2\, t_2 \sqrt{\frac{\pi}{2\beta_{21}}} \left\{1 - \Psi\left(\frac{\alpha_{21}+\theta_0-t_2}{\sqrt{2\beta_{21}}}\right)\right\} \exp\left\{\frac{(\alpha_{21}+\theta_0-t_2)^2}{2\beta_{21}}\right\} \\ &\quad +(1-a_2)\, t_2 \sqrt{\frac{\pi}{2\beta_{22}}} \left\{1 - \Psi\left(\frac{\alpha_{22}+\theta_0-t_2}{\sqrt{2\beta_{22}}}\right)\right\} \exp\left\{\frac{(\alpha_{22}+\theta_0-t_2)^2}{2\beta_{22}}\right\}. \end{aligned} \tag{3.10}$$

Corollary 3.4. From (3.8) – (3.10), we obtain:

$$\begin{aligned} E[S_1] &= a_1 \sqrt{\frac{\pi}{2\beta_{11}}} \left\{1 - \Psi\left(\frac{\alpha_{11}+\theta_0}{\sqrt{2\beta_{11}}}\right)\right\} \exp\left\{\frac{(\alpha_{11}+\theta_0)^2}{2\beta_{11}}\right\} \\ &\quad +(1-a_1) \sqrt{\frac{\pi}{2\beta_{12}}} \left\{1 - \Psi\left(\frac{\alpha_{12}+\theta_0}{\sqrt{2\beta_{12}}}\right)\right\} \exp\left\{\frac{(\alpha_{12}+\theta_0)^2}{2\beta_{12}}\right\}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} E[S_2] &= a_2 \sqrt{\frac{\pi}{2\beta_{21}}} \left\{1 - \Psi\left(\frac{\alpha_{21}+\theta_0}{\sqrt{2\beta_{21}}}\right)\right\} \exp\left\{\frac{(\alpha_{21}+\theta_0)^2}{2\beta_{21}}\right\} \\ &\quad +(1-a_2) \sqrt{\frac{\pi}{2\beta_{22}}} \left\{1 - \Psi\left(\frac{\alpha_{22}+\theta_0}{\sqrt{2\beta_{22}}}\right)\right\} \exp\left\{\frac{(\alpha_{22}+\theta_0)^2}{2\beta_{22}}\right\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} E[S_1^2] &= a_1 \left[\frac{2}{\beta_{11}} + \frac{\sqrt{2\pi}\,(\alpha_{11}+\theta_0)}{\beta_{11}\sqrt{\beta_{11}}} \left\{1 - \Psi\left(\frac{\alpha_{11}+\theta_0}{\sqrt{2\beta_{11}}}\right)\right\} \exp\left\{\frac{(\alpha_{11}+\theta_0)^2}{2\beta_{11}}\right\}\right] \\ &\quad +(1-a_1) \left[\frac{2}{\beta_{12}} + \frac{\sqrt{2\pi}\,(\alpha_{12}+\theta_0)}{\beta_{12}\sqrt{\beta_{12}}} \left\{1 - \Psi\left(\frac{\alpha_{12}+\theta_0}{\sqrt{2\beta_{12}}}\right)\right\} \exp\left\{\frac{(\alpha_{12}+\theta_0)^2}{2\beta_{12}}\right\}\right], \end{aligned} \tag{3.13}$$

$$\begin{aligned} E[S_2^2] &= a_2 \left[\frac{2}{\beta_{21}} + \frac{\sqrt{2\pi}\,(\alpha_{21}+\theta_0)}{\beta_{21}\sqrt{\beta_{21}}} \left\{1 - \Psi\left(\frac{\alpha_{21}+\theta_0}{\sqrt{2\beta_{21}}}\right)\right\} \exp\left\{\frac{(\alpha_{21}+\theta_0)^2}{2\beta_{21}}\right\}\right] \\ &\quad +(1-a_2) \left[\frac{2}{\beta_{22}} + \frac{\sqrt{2\pi}\,(\alpha_{22}+\theta_0)}{\beta_{22}\sqrt{\beta_{22}}} \left\{1 - \Psi\left(\frac{\alpha_{22}+\theta_0}{\sqrt{2\beta_{22}}}\right)\right\} \exp\left\{\frac{(\alpha_{22}+\theta_0)^2}{2\beta_{22}}\right\}\right], \end{aligned} \tag{3.14}$$

$$\begin{aligned} E[S_1S_2] &= \sqrt{\frac{\pi}{2\,\beta_{11}}} \left\{p_{11} I_1^{(11)}(0,0) + p_{12} I_1^{(12)}(0,0)\right\} \exp\left\{\frac{(\alpha_{11}+\theta_0)^2}{2\beta_{11}}\right\} \\ &\quad + \sqrt{\frac{\pi}{2\,\beta_{12}}} \left\{p_{21} I_1^{(21)}(0,0) + p_{22} I_1^{(22)}(0,0)\right\} \exp\left\{\frac{(\alpha_{12}+\theta_0)^2}{2\beta_{12}}\right\} \\ &\quad + \sqrt{\frac{\pi}{2\,\beta_{21}}} \left\{p_{11} I_2^{(11)}(0,0) + p_{21} I_2^{(21)}(0,0)\right\} \exp\left\{\frac{(\alpha_{21}+\theta_0)^2}{2\beta_{21}}\right\} \\ &\quad + \sqrt{\frac{\pi}{2\,\beta_{22}}} \left\{p_{12} I_2^{(12)}(0,0) + p_{22} I_2^{(22)}(0,0)\right\} \exp\left\{\frac{(\alpha_{22}+\theta_0)^2}{2\beta_{22}}\right\}. \end{aligned} \tag{3.15}$$

From the expressions in (3.11) – (3.15), we can derive the variances of S_1 and S_2 as well as the covariance and correlation coefficient between S_1 and S_2.

Finally, we note that the mixture of bivariate exponential distributions and its properties can all be derived from the mixture of bivariate linear failure rate distributions presented above by setting all β's as zero.

4. APPLICATION TO A REAL DATA

In this section we illustrate how the proposed bivariate distribution fits a real data from a Diabetic retinopathy study. The 197 patients in this data set were a 50% random sample of the patients with high risk diabetic retinopathy as defined as Diabetic Retinopathy Study (DRS), see Rojo and Ghebremichael (2006). Diabetic retinopathy is a complication associated with diabetic mellitus consisting of abnormalities in the microvasculature within the retina of the eye. In patients under 60 years of age in the United States, it is the leading cause of new cause of blindness. It is the major cause of visual loss elsewhere in many industrialized countries. The DES begun in 1971 to study the effectiveness of laser photocoagulation in delaying the onset of blindness in patients with diabetic retinopathy. Patients with diabetic retinopathy in both eyes and visual acuity of 20/100 or better were eligible for the study. One eye of each patient was randomly selected for the treatment and the other eye was observed without treatment. For each eye, the event for interest was the time from initiation of the treatment to the time when visual acuity dropped below 5/200 two visits in a row (call it "blindness"). Thus there is a built-in lag time of approximately 6 months (visits were every 3 months). Survival times in this data set are therefore the actual time to blindness in months, minus the minimum possible time to event (6.5 months). Diabetes are classified into two general groups by the age at the onset: juvenile (< 20 years) and adult diabetes. In the DRS study censoring was caused by death, dropout, or the end of the study. In the data sets used here, there is no censoring. For each uncensored case i, the survival times of the treated (X_i) and untreated (Y_i) eyes are given in Tables 4.1 and 4.2.

Table 4.1. Survival times (months) for adults

Patient, i	1	2	3	4	5	6	7
X_i	38.57	1.33	21.90	13.87	48.30	9.90	8.30
Y_i	30.83	5.77	25.63	25.80	5.73	9.90	8.30
Patient, i	8	9	10	11	12	13	14
X_i	12.20	33.63	27.60	1.63	1.57	4.97	9.87
Y_i	4.10	33.63	63.33	38.47	13.83	12.93	24.43

Table 4.2. Survival times (months) for juvenile

Patient, i	1	2	3	4	5	6	7	8
X_i	6.90	1.63	13.83	35.53	14.8	6.20	22.0	1.7
Y_i	20.17	10.27	5.67	5.90	33.9	1.73	30.2	1.7
Patient, i	9	10	11	12	13	14	15	16
X_i	43.03	6.53	42.17	48.43	9.60	7.60	1.80	9.90
Y_i	1.77	18.70	42.17	14.30	13.33	14.27	34.57	21.57
Patient, i	17	18	19	20	21	22	23	24
X_i	13.77	0.83	1.97	11.3	30.40	19.0	5.43	46.63
Y_i	13.77	10.33	11.07	2.1	13.97	13.8	13.57	42.43

Let $\hat{\bar{F}}(x, y)$ be the empirical bivariate survival function. Also, let $\bar{F}_{BLFRD}(x, y)$ and $\bar{F}_{BED}(x, y)$ be the bivariate survival functions of the bivariate linear failure rate distribution and bivariate exponential distribution, respectively.
For each data set we do the following:

1. Compute and plot the empirical bivariate survival function $\hat{\bar{F}}$. Tables 4 and 5 give $\hat{\bar{F}}$ computed by using data sets 1 and 2, respectively. Figures 1.a and 1.b show $\hat{\bar{F}}$ for the data sets 1 and 2, respectively.
2. Estimate, using the maximum likelihood approach, the parameters included in the bivariate exponential distribution (BED) and bivariate linear failure rate distribution (BLFRD). Table 4.3 gives estimations of the parameters.
3. Based on the estimations of the parameters, we compute the survival functions of both distributions. We denote by Tables 4.4 and 4.5 give the values of these functions at each observation of sets 1 and 2, respectively.
4. Compute the absolute difference, called absolute error, between $\hat{\bar{F}}$ and both $\bar{F}_{BLFRD}$ and $\bar{F}_{BED}$ at each observation, according to the following relations

$$d_{BLFRD}^{(i)} = \left|\hat{\bar{F}}(x_i, y_i) - \bar{F}_{BLFRD}(x_i, y_i)\right|, \quad d_{BED}^{(i)} = \left|\hat{\bar{F}}(x_i, y_i) - \bar{F}_{BED}(x_i, y_i)\right|$$

   Tables 4.4 and 4.5 gives the values of $d_{BLFRD}^{(i)}, d_{BED}^{(i)}$ for the sets 1 and 2, respectively.
5. Compute the average of the absolute errors associated with each distribution.
6. Plot the functions $\bar{F}_{BLFRD}(x, y)$ and $\bar{F}_{BED}(x, y)$ when the unknown parameters included are replaced by their estimates given in Table 4.3. Figures (a) and (b) of Figure 4.2 show $\bar{F}_{BLFRD}(x, y)$ for the data sets 1 and 2, respectively. Figures (a) and (b) of Figure 4.3 show $\bar{F}_{BED}(x, y)$ for the data sets 1 and 2, respectively. Figures

The mean of the absolute errors associated with both BLFRD and BED are computed and given respectively by 0.052, 0.055, using set 1 and 0.043, 0.045, using set 2.

From the results obtained and Figures 4.1, 4.2 and 4.3, it seems that the BLFRD fits both two data sets better than the BED.

Table 4.3. The parameter estimations.

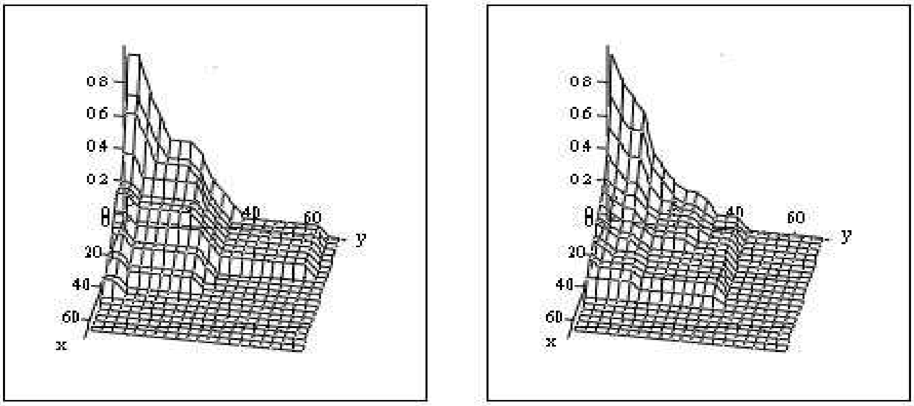
set	BLFRD					BED		
	α_1	α_2	β_1	β_2	θ_0	α_1	α_2	θ_0
1	0.042	0.014	$\frac{1.036}{10^4}$	$\frac{7.51}{10^4}$	0.019	0.042	$\frac{9.927}{10^3}$	0.028
2	0.048	0.029	$\frac{5.43}{10^5}$	$\frac{1.453}{10^3}$	0.014	0.045	0.023	0.022

Table 4.4. The values of $\hat{\bar{F}}$, $\bar{F}_{BLFRD}$, $\bar{F}_{BED}$, d_{BLFRD} and d_{BED} for the data in Table 4.1.

i	$\hat{\bar{F}}$	$\bar{F}_{BLFRD}$	$\bar{F}_{BED}$	$d^{(i)}_{BLFRD}$	$d^{(i)}_{BED}$
1	0.000	0.04000	0.050	0.04000	0.050000
2	0.786	0.77200	0.760	0.01300	0.026000
3	0.214	0.13100	0.151	0.08300	0.063000
4	0.214	0.18500	0.210	0.03000	0.004137
5	0.000	0.04200	0.032	0.04200	0.032000
6	0.357	0.45800	0.454	0.10100	0.096000
7	0.500	0.52200	0.515	0.02200	0.015000
8	0.429	0.44300	0.409	0.01400	0.019000
9	0.000	0.05000	0.068	0.05000	0.068000
10	0.000	0.00836	0.028	0.00836	0.028000
11	0.071	0.15100	0.217	0.07900	0.146000
12	0.500	0.55200	0.554	0.05200	0.054000
13	0.429	0.49800	0.497	0.06900	0.069000
14	0.357	0.23500	0.262	0.12200	0.095000

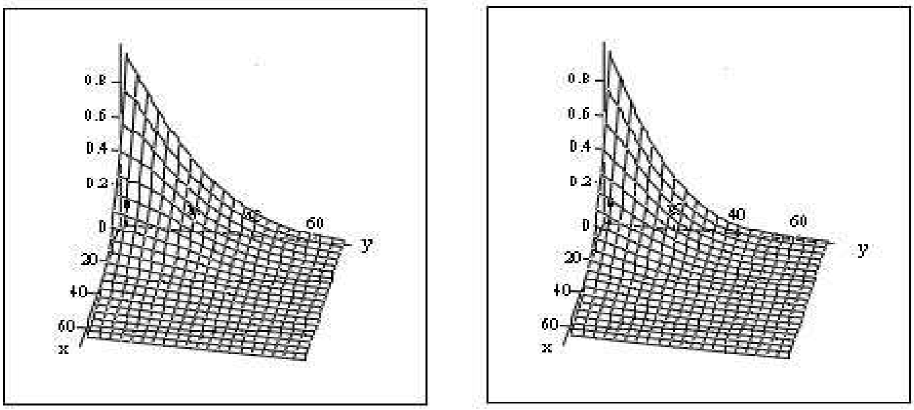
Table 4.5. The values of $\hat{\bar{F}}$, $\bar{F}_{BLFRD}$, $\bar{F}_{BED}$, d_{BLFRD} and d_{BED} for the data in Table 4.2.

i	$\hat{\bar{F}}$	$\bar{F}_{BLFRD}$	$\bar{F}_{BED}$	$d^{(i)}_{BLFRD}$	$d^{(i)}_{BED}$
1	0.208	0.223000	0.298	0.015000	0.090000
2	0.667	0.550000	0.588	0.116000	0.079000
3	0.333	0.349000	0.349	0.015000	0.016000
4	0.125	0.087000	0.082	0.038000	0.043000
5	0.083	0.049000	0.113	0.034000	0.030000
6	0.708	0.645000	0.636	0.064000	0.073000
7	0.083	0.048000	0.096	0.036000	0.013000
8	0.875	0.854000	0.859	0.021000	0.016000
9	0.083	0.062000	0.055	0.021000	0.029000
10	0.250	0.253000	0.324	0.002573	0.074000
11	0.042	0.005532	0.023	0.036000	0.019000
12	0.000	0.026000	0.029	0.026000	0.029000
13	0.375	0.310000	0.358	0.065000	0.017000
14	0.250	0.323000	0.376	0.073000	0.126000
15	0.083	0.087000	0.198	0.003565	0.114000
16	0.167	0.174000	0.245	0.007415	0.078000
17	0.292	0.246000	0.291	0.046000	0.000794
18	0.667	0.570000	0.608	0.097000	0.059000
19	0.583	0.516000	0.559	0.067000	0.025000
20	0.417	0.463000	0.449	0.046000	0.032000
21	0.125	0.085000	0.096	0.040000	0.029000
22	0.208	0.177000	0.205	0.032000	0.003190
23	0.500	0.375000	0.428	0.125000	0.072000
24	0.000	0.004053	0.017	0.004053	0.017000



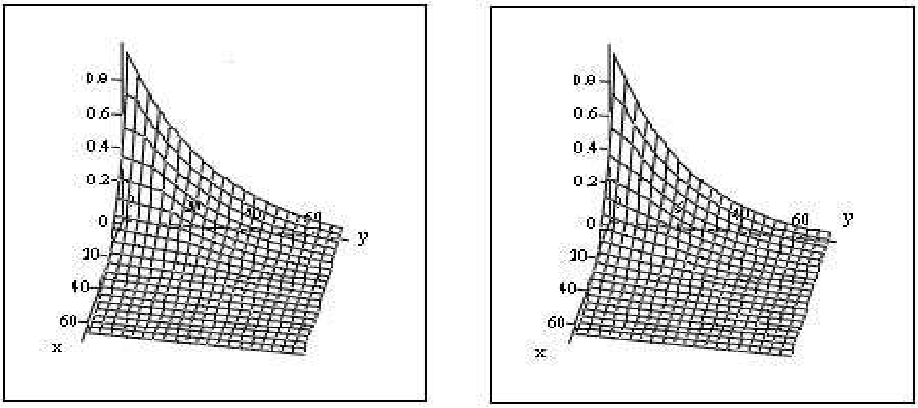
a) Data set 1 b) Data set 2

Figure 4.1. The empirical bivariate survival function.



a) Data set 1 b) Data set 2

Figure 4.2. The bivariate survival function of BLFRD.



a) Data set 1 b) Data set 2

Figure 4.3. The bivariate survival function of BED.

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