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ON THE STRONG LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF ARRAYS OF ROWWISE NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. Let $\{X_{ni} \,|\, 1 \le i \le n,\, n \ge 1\}$ be an array of rowwise negatively dependent (ND) random variables. We in this paper discuss the conditions of $\sum_{i=1}^{n} a_{ni} X_{ni} \to 0$ completely as $n \to \infty$ under not necessarily identically distributed setting and the strong law of large numbers for weighted sums of arrays of rowwise negatively dependent random variables is also considered.

1. Introduction

Let $\{X_n \,|\, n \ge 1\}$ be a sequence of random variables. Hsu and Robbins [4] introduced the concept of complete convergence of $\{X_n \,|\, n \ge 1\}$. A sequence $\{X_n \,|\, n \ge 1\}$ of random variables converges to a constant c completely if

$$\sum_{n=1}^{\infty} P\left(|X_n - c| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

If $X_n \to c$ completely, then the Borel-Cantelli lemma implies that $X_n \to c$ almost surely, but the converse is not true in general.

Let $\{X_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ be an array of random variables with $EX_{ni} = 0$ for all n and i. Many authors studied the complete convergence of $n^{-1/p} \sum_{i=1}^{n} X_{ni}$ which is defined

$$\sum_{n=1}^{\infty} P\left(|n^{-1/p} \sum_{i=1}^{n} X_{ni}| > \epsilon\right) \text{ for all } \epsilon > 0, \tag{1.1}$$

where $0 < p < 2$.

In particular, Erdös [4] showed that for an array of independent identically distributed ($i.i.d.$) random variables $\{X_{ni}|1 \le i \le n, n \ge 1\}$, (1.1) holds if and only if $E|X_{11}|^{2p} < \infty$. Hu et al. [6] showed that Erdös' result could be obtained by replacing the $i.i.d.$ condition by the uniformly bounded condition.

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We recall that any away $\{X_{ni}|1 \leq i \leq n, n \geq 1\}$ of random variables is said to be uniformly bounded by a random variable X if for all i, n and $x \geq 0$,

$$\sup P(|X_{ni}| \geq x) \leq P(|X| > x). \tag{1.2}$$

Hu et al. [5] had obtained the following result in complete convergence and they had established (1.3) for non identcally random variable when no assumption of independence between rows of the array is made.

Theorem A. *Let $\{X_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ be an array of rowwise independent random variables with $EX_{ni} = 0$. Suppose that $\{X_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ are uniformly bounded by some random variable X. If $E|X|^{2p} < \infty$ for some $1 \leq p \leq 2$, then*

$$n^{-1/p} \sum_{i=1}^{n} X_{ni} \longrightarrow 0 \textit{ completely as } n \to \infty \tag{1.3}$$

if and only if $E|X_{11}|^{2p} < \infty$.

In this paper, we discuss the strong law of large numbers for weighted sums of arrays of rowwise ND random variables. The main purpose of this paper is to extend and generalize Theorem A to rowwise ND random variables which satisfy suitable conditions. Further, the last two properties of this paper show that neither ND nor $EX_{ni} = 0$ are needed to obtain the corresponding strong law of large numbers when $0 < p < 1/2$ or a weak law of large numbers when $1/2 \leq p < 1$.

2. Preliminaries

This section will list some background materials which will be used in obtaining the main results in the next section and we define $a^+ = \max(0, a)$, $a^- = \max(0, -a)$.

Definition 2.1 (Ebrahimi et al. [2])**.** Random variables X and Y are negatively dependent(ND) if

$$P[X \leq x,\ Y \leq y] \leq P[X \leq x]P[Y \leq y] \tag{2.1}$$

for all $x, y \in R$. A collection of random variables is said to be pairwise ND if every pair of random variables in the collection satisfies (2.1).

It is important to note that Definition 2.1 implies

$$P[X > x,\ Y > y] \leq P[X > x]P[Y > y] \tag{2.2}$$

for the $x, y \in R$. Moreover, it follows that (2.2) implies (2.1), and hence, they are equivalent for pairwise ND. Ebrahimi and Ghosh [2] showed that (2.1) and (2.2) are not equivalent for $n \geq 3$. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 2.2. The random variables $X_1, X_2, \ldots$ are said to be

(a) lower negatively dependent(LND) if for each n,

$$P[X_1 \le x_1, \ldots, X_n \le x_n] \le \prod_{i=1}^{n} P[X_i \le x_i] \tag{2.3}$$

for all $x_1, \ldots, x_n \in R$,

(b) upper negatively dependent(UND) if for each n,

$$P[X_1 > x_1, \ldots, X_n > x_n] \le \prod_{i=1}^{n} P[X_i > x_i] \tag{2.4}$$

for all $x_1, \ldots, x_n \in R$,

(c) negatively dependent (ND) if both (2.3) and (2.4) hold.

The following properties are listed for reference in obtaining the main result in the next section and detailed proofs can be found in their paper.

Lemma 2.1 (Ebrahimi et al. [2]). *Let $\{X_n \mid n \ge 1\}$ be a sequence of ND random variables and $\{f_n \mid n \ge 1\}$ be a sequence of monotone increasing (decreasing) Borel functions. Then $\{f_n(X_n) \mid n \ge 1\}$ is a sequence of ND random variables.*

Lemma 2.2 (Taylor et al. [7]). (a) *Let $X_1, X_2, \ldots, X_n$ be nonnegative random variables which are upper negatively dependent. Then*

$$E\left(\prod_{i=1}^{n} X_i\right) \le \prod_{i=1}^{n} EX_i.$$

(b) *Let $X_1, X_2, \ldots, X_n$ be a pairwise ND random variables. Then*

$$\text{(i)} \quad EX_iX_j \le EX_iEX_j \text{ for } i \ne j$$
$$\text{(ii)} \quad \operatorname{Cov}(X_i, X_j) \le 0 \text{ for } i \ne j.$$

(c) *Let $X_1, X_2, \ldots, X_n$ be a ND random variables. Then for any real numbers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ such that $a_i \le b_i$, $1 \le i \le n$, $Y_i = X_iI\ (a_i \le X_i \le b_i) + b_iI(X_i > b_i) + a_iI(X_i < a_i)$ are ND random variables.*

Lemma 2.3 (Hu et al. [6]). *For any $r \ge 1$ and $p > 0$,*

(a) *$E|X|^r$ if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| > \epsilon n) < \infty \text{ for all } \epsilon > 0,$$

(b) $E|X|^r I(|X| \le n^{1/p}) \le r \int_0^{n^{1/p}} t^{r-1} P(|X| > t) dt,$

(c) $r2^{-r} \sum_{n=1}^{\infty} n^{r-1} P(|X| > n) \le E|X|^r \le 1 + r2^r \sum_{n=1}^{\infty} n^{r-1} P(|X| > n).$

Lemma 2.4 (Bozorgnia et al. [1]). *Let X be a random variable such that $EX = 0$ and $|X| \le M <$ a.e. Then for all constant M,*

$$1 \le Ee^{tx} \le e^{t^2 EX^2} \text{ for all } |t| \le \frac{1}{M}.$$

3. Strong law of large numbers for weighted sums

Theorem 3.1 is to extend and generalize Theorem A to rowwise ND random variables. Note that the range $0 < p < 2$ is allowed in Theorem 3.1 and Theorem 3.2 whereas previous results usually addressed the important subset $1 \le p \le 2$. Throughout the proof c will represent positive constants whose value may change from one to another.

Theorem 3.1. *Suppose that $0 < p < 2$ and let $\{X_{ni} \mid 1 \le i \le n, n \ge 1\}$ be an array of rowwise ND random variables with $EX_{ni} = 0$. Suppose that $\sup P(|X_{ni}| > x) \le P(|X| > x)$ for all i, n and $x \ge 0$. Assume that $\{a_{ni} \mid 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying*

$$\text{(i)} \max_{1 \le x \le n} |a_{ni}| = O(n^{-1/p}), \qquad \text{(ii)} \sum_{i=1}^{n} a_{ni}^2 = o\left(\frac{1}{\log n}\right).$$

If $E|X|^{2p} < \infty$, then

$$\sum_{i=1}^{n} a_{ni} X_{ni} \longrightarrow 0 \text{ completely as } n \to \infty.$$

Proof. Since $a_{ni} = a_{ni}^{+} - a_{ai}^{-}$, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}\right| > \epsilon\right) < \infty \text{ for all } \epsilon > 0, \tag{1}$$

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}^{-} X_{ni}\right| > \epsilon\right) < \infty \text{ for all } \epsilon > 0. \tag{2}$$

We prove only (1), the proof of (2) is analogous.

To prove (1), we need only prove that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \epsilon\right) < \infty \text{ for all } \epsilon > 0, \tag{3}$$

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} < -\epsilon\right) < \infty \text{ for all } \epsilon > 0. \tag{4}$$

Without loss of generality, we can assume that $0 < a_{ni}^{+} \le n^{-1/p}$ for all $1 \le i \le n$, $n \ge 1$ and let q be the constant such that $0 < p < q < 2p < 2$ and for $\alpha > 0$, $\alpha = 1/p - 1/q$.

Note that

$$\left(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} \geq \epsilon\right) \subset \left(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q}) \geq \epsilon/2\right)$$
$$\bigcup (a_{ni}^{+} X_{ni} \geq \epsilon/2 \text{ for some } i,\ 1 \leq i \leq n)$$
$$\bigcup (X_{ni} > n^{1/q} \text{ for at least two values of } i,\ 1 \leq i \leq n).$$

Thus,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}{}^{+} X_{ni} \geq \epsilon\right) \\
\leq &\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}{}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q}) \geq \epsilon/2\right) \\
&+ \sum_{n=1}^{\infty} P(a_{ni}{}^{+} X_{ni} \geq \epsilon/2 \text{ for some } i,\ 1 \leq i \leq n) \\
&+ \sum_{n=1}^{\infty} P(X_{ni} > n^{1/q} \text{ for at least two values of } i,\ 1 \leq i \leq n) \\
= &\ I_1 + I_2 + I_3 (\text{say}).
\end{aligned} \tag{5}$$

To prove $I_1 < \infty$, we first define that

$$Y_{ni} = X_{ni} I(|X_{ni}| \leq n^{1/q}) + n^{1/q} I(X_{ni} > n^{1/q}) - n^{1/q} I(X_{ni} < -n^{1/q}).$$

So that $\{Y_{ni} | 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise ND random variables by definition.

Next,

$$\begin{aligned}
&\sum_{i=1}^{n} a_{ni}{}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q}) \\
= &\sum_{i=1}^{n} a_{ni}{}^{+} (Y_{ni} - EY_{ni}) \\
&- n^{1/q} \sum_{n=1}^{n} a_{ni}{}^{+} (I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q})) \\
&+ n^{1/q} \sum_{n=1}^{n} a_{ni}{}^{+} (I(X_{ni} < -n^{1/q}) - P(X_{ni} < -n^{1/q})) \\
&+ \sum_{i=1}^{n} a_{ni}{}^{+} E X_{ni} I(|X_{ni}| \leq n^{1/q}) \\
= &\ I_4 + I_5 + I_6 + I_7 \text{ (say)}.
\end{aligned} \tag{6}$$

As to I_4, we consider two cases of (a) $p \geq 1$ and (b) $0 < p < 1$, and note that $\{{a_{ni}}^+(Y_{ni} - EY_{ni}) \mid 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise ND random variables by definition and $|{a_{ni}}^+(Y_{ni} - EY_{ni})| \leq 2n^{-1/p\,+\,1/q} = 2n^{-\alpha}$.

(a) when $p \geq 1$, note that

$$\begin{aligned} E|Y_{ni}|^2 &\leq E|X_{ni}|^2 I(|X_{ni}| \leq n^{1/q}) + n^{2/q} P(|X_{ni}| > n^{1/q}) \\ &\leq E|X|^2 I(|X| \leq n^{1/q}) + n^{2/q} P(|X| > n^{1/q}) \\ &\leq 2E|X|^2 < \infty \end{aligned}$$

which implies that $E|Y_{ni}|^2 < \infty$ since $E|X|^{2p} < \infty$ implies $E|X|^2 < \infty$. Hence, by using Lemma 2.2 and Lemma 2.4 and taking $t = 2\log n/\epsilon$, we get

$$\begin{aligned} I_4 &= P\left(\sum_{i=1}^{n} {a_{ni}}^+(Y_{ni} - EY_{ni}) > \epsilon\right) \\ &\leq e^{-\epsilon t} \textstyle\prod_{i=1}^{n} E e^{t{a_{ni}}^+(Y_{ni} - EY_{ni})} \\ &\leq e^{-2\log n} \textstyle\prod_{i=1}^{n} e^{t{a_{ni}}^+(Y_{ni} - EY_{ni})} \\ &\leq e^{-2\log n} \textstyle\prod_{i=1}^{n} e^{t^2({a_{ni}}^+)^2 E{Y_{ni}}^2} \\ &\leq e^{-2\log n} \textstyle\prod_{i=1}^{n} e^{ct^2({a_{ni}}^+)^2 E|X|^2} \\ &\leq e^{-2\log n} e^{c(\log n)^2 \sum_{i=1}^{n} ({a_{ni}}^+)^2} \\ &\leq e^{-2\log n} e^{c(\log n)^2 \cdot (\frac{1}{\log n})} \\ &\leq ce^{-c\log n} \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} {a_{ni}}^+(Y_{ni} - EY_{ni}) > \epsilon\right) < \infty.$$

(b) when $0 < p < 1$, taking $t = n^{\alpha}/2$, we get

$$\begin{aligned} I_4 &= P\left(\sum_{i=1}^{n} {a_{ni}}^+(Y_{ni} - EY_{ni}) > \epsilon\right) \\ &\leq e^{-\epsilon t} e^{t^2 \sum_{i=1}^{n} ({a_{ni}}^+)^2 EY_{ni}^2} \\ &\leq e^{-\epsilon t} e^{cn^{2\alpha} \sum_{i=1}^{n} n^{-2/p} E{X_{ni}}^2 I(|X_{ni}| \leq n^{1/q}) + n^{2/q} P(|X_{ni}| > n^{1/q})} \\ &\leq e^{-\epsilon t} e^{cn^{(2/p - 2/q)} n^{1-2/p} c(1 + n^{2(1-p)/q})} \\ &\leq e^{-cn^{\alpha}} e^{cn^{1-2/q+1-2p/q}} \end{aligned}$$

which is summable since $\alpha > 0$ and $0 < p < q < 2p < 2$ implies that $1 - 2/q < 0$ and $1 - 2p/q < 0$. Hence, by (a) and (b), for all $0 < p < 2$,

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} {a_{ni}}^+(Y_{ni} - EY_{ni}) > \epsilon\right) < \infty. \tag{7}$$

As to I_5, let $Z_{ni} = n^{1/q}{a_{ni}}^+(I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q}))$ and noticing that $|Z_{ni}| = |n^{1/q}{a_{ni}}^+(I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q}))| \le n^{1/q-1/p} \le n^{-\alpha}$ and

$$\begin{aligned} E{Z_{ni}}^2 &\le n^{2/q}({a_{ni}}^+)^2 P(X_{ni} > n^{1/q}) + P(X_{ni} > n^{1/q}) \\ &\le cn^{2/q}n^{-2/p}E|X|^{2p}/n^{2p/q}, \end{aligned}$$

we get that

$$\begin{aligned} I_5 &= P\left(\sum_{i=1}^n Z_{ni} > \epsilon\right) \\ &= P\left(n^\alpha \sum_{i=1}^n Z_{ni} > n^\alpha \epsilon\right) \\ &\le e^{-\epsilon n^\alpha} E e^{n^\alpha \sum_{i=1}^n Z_{ni}} \\ &\le e^{-\epsilon n^\alpha} \prod_{i=1}^n e^{n^{2\alpha} E(Z_{ni})^2} \\ &\le e^{-\epsilon n^\alpha} \prod_{i=1}^n e^{n(2/p-2/q)} c n^{2/q-2/p} E|X|^{2p}/n^{2p/q} \\ &\le e^{-\epsilon n^\alpha} e^{cn(1-2p/q)} \\ &\le e^{-\epsilon n^\alpha + cn^{1-2p/q}} \end{aligned}$$

which is summable since $\alpha > 0$ and $1 - 2p/q < 0$. Hence,

$$\sum_{n=1}^\infty P\left(n^{1/q}\sum_{i=1}^n {a_{ni}}^+(I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q})) > \epsilon\right) < \infty. \tag{8}$$

(9) As to I_6, the proof of I_6 is similar to I_5.

Next, as to I_7, we consider three cases of (c) $p > 1/2$, (d) $p = 1/2$, and (e) $0 < p < 1/2$. Since $EX_{ni} = 0$, it follows that

$$EX_{ni}I(|X_{ni}| \le n^{1/q}) = |-EX_{ni}I(|X_{ni}| > n^{1/q})|.$$

(c) when $p > 1/2$,

$$\begin{aligned} |I_7| &\le \sum_{i=1}^n |{a_{ni}}^+ EX_{ni}I(|X_{ni}| \le n^{1/q})| \\ &\le \sum_{i=1}^n n^{-1/p}E|X_{ni}|I(|X_{ni}| > n^{1/q}) \\ &\le \sum_{i=1}^n n^{-1/p}(n^{1/q}P(|X_{ni}| > n^{1/q}) + \int_{n^{1/q}}^\infty P(|X_{ni}| > t)dt) \\ &\le n^{1-1/p+1/q}P(|X| > n^{1/q}) + n^{1-1/p}\int_{n^{1/q}}^\infty P(|X| > t)dt \end{aligned}$$

$$\begin{aligned} &\le n^{1-1/p+1/q-2p/q} E|X|^{2p} + n^{1-1/p+1/q-2p/q} E|X|^{2p} \\ &\le cn^{1-1/p+1/q(1-2p)} \end{aligned}$$

which implies that

$$\sum_{i=1}^{n} a_{ni}{}^{+} EX_{ni} I(|X_{ni}| \le n^{1/q}) \longrightarrow 0 \text{ as } n \to \infty$$

since $1 - 1/p + 1/q(1-2p) < 0$.

(d) when $p = 1/2$,

$$\begin{aligned} |I_7| &\le \sum_{i=1}^{n} |a_{ni}{}^{+} EX_{ni} I(|X_{ni}| \le n^{1/q})| \\ &\le \sum_{i=1}^{n} n^{-1/p} E|X| I(|X| \le n^{1/q}) \\ &\le n^{1-1/p} E|X| \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

(e) when $0 < p < 1/2$, note that $E|X|^{2p} < \infty$ implies that $P(|X| > t) \le t^{-2t}$, where $t \ge A$ for some constant A. Hence, for $n^{1/q} \ge A$,

$$\begin{aligned} |I_7| &\le \sum_{i=1}^{n} |a_{ni}{}^{+} EX_{ni} I(|X_{ni}| \le n^{1/q})| \\ &\le \sum_{i=1}^{n} n^{-1/p} E|X_{ni}| I(|X_{ni}| \le n^{1/q}) \\ &\le n^{1-1/p} E|X| I(|X| \le n^{1/q}) \\ &\le n^{1-1/p} \int_0^{n^{1/q}} P(|X| > t) dt \\ &= n^{1-1/p} \left(\int_0^{A} P(|X| > t) dt + \int_A^{n^{1/q}} P(|X| > t) dt \right) \\ &\le n^{1-1/p} \left(A + \int_A^{n^{1/q}} P(|X| > t) dt \right) \\ &\le n^{1-1/p} \left(A + \frac{1}{1-2p} n^{1/q(1-2p)} \right) \\ &\le c \, \frac{1}{1-2p} n^{1-1/p \; + \; 1/q(1-2p)} \end{aligned}$$

which implies that

$$\sum_{i=1}^{n} a_{ni}{}^{+} EX_{ni} I(|X_{ni}| \le n^{1/q}) \longrightarrow 0 \text{ as } n \to \infty$$

since $1/q(1-2p) < 0$ and $1 - 1/p < 0$. Hence, by (c), (d), and (e), we get

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}{}^{+} E X_{ni} I(|X_{ni}| \leq n^{1/q})\right) < \infty. \tag{10}$$

Thus, by (7), (8), (9), and (10), we have

$$I_1 = \sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}{}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q}) \geq \epsilon/2) < \infty. \tag{11}$$

As to I_2, we get that

$$\begin{aligned}
& P(a_{ni}{}^{+} X_{ni} \geq \epsilon/2 \text{ for some } i,\ 1 \leq i \leq n) \\
\leq\ & P\left(\bigcup_{i=1}^{n} |a_{ni}{}^{+} X_{ni}| > \epsilon/2\right) \\
\leq\ & \sum_{i=1}^{n} P(|a_{ni}{}^{+} X_{ni}| > \epsilon/2) \\
\leq\ & \sum_{i=1}^{n} P(n^{-1/p} |X_{ni}| > \epsilon/2) \\
\leq\ & \sum_{i=1}^{n} P(|X| > n^{1/p} \epsilon/2) \\
\leq\ & n^{-1} \left(\frac{2}{\epsilon}\right)^{2p} E|X|^{2p}
\end{aligned}$$

which implies that

$$n^{-1} E|X|^{2p} \longrightarrow 0 \text{ as } n \to \infty.$$

Hence,

$$I_2 = \sum_{n=1}^{\infty} P(a_{ni}{}^{+} X_{ni} \geq \epsilon/2 \text{ for some } i,\ 1 \leq i \leq n) < \infty. \tag{12}$$

Finally, we get that

$$\begin{aligned}
& P(X_{ni} > n^{1/q} \text{ for at least two values of } 1 \leq i \leq n) \\
\leq\ & P\left(\bigcup_{i \neq j} X_{ni} > n^{1/q}, X_{nj} > n^{1/q}\right) \\
\leq\ & \sum_{i \neq j} P(|X_{ni}| > n^{1/q}, |X_{nj}| > n^{1/q}) \\
\leq\ & \sum_{i \neq j} P(|X| > n^{1/q}) P(|X| > n^{1/q})
\end{aligned}$$

$$\begin{aligned} &\leq \sum_{i \neq j} E|X|^{2p}(n^{-1/q})^{2p} E|X|^{2p}(n^{-1/q})^{2p} \\ &\leq cn(n-1)n^{-4q/p} \\ &\leq cn^{-2(2p/q-1)} \end{aligned}$$

which implies that

$$n^{-2(2p/q-1)} \longrightarrow 0 \text{ as } n \to \infty$$

since $2p/q - 1 > 0$. Hence,

$$I_3 = \sum_{n=1}^{\infty} P(X_{ni} > n^{1/q} \text{ for at least two values of } i, 1 \leq i \leq n) < \infty. \tag{13}$$

Thus, by (11), (12), and (13), we have

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}\right| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

By replacing X_{ni} by $(-X_{ni})$ from (3) and noticing $\{a_{ni}^{+}(-X_{ni}) \mid 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise ND random variables by definition, we also know that

$$\sum_{n=1}^{\infty} P\left(\sum_{n=1}^{n} a_{ni}^{+}(-X_{ni}) > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

The proof is complete. □

Corollary 3.1. *Let* $0 < p < 2$ *and let* $\{X_{ni} \mid 1 \leq i \leq n, n \geq 1\}$ *be an array of rowwise* ND *random variables with* $EX_{ni} = 0$. *Suppose that there is a random variables* X *such that* $\sup P(|X_{ni}| > x) \leq P(|X| > x)$ *for all* i, n *and* $x \geq 0$. *Assume that* $\{b_{ni} | 1 \leq i \leq n, n > 1|\}$ *is an array of constants satisfying* $\lim_{n\to\infty} \sup \sum_{i=1}^{n} b_{ni}^2 < \infty$. *If* $E|X|^{2p} < \infty$, *then*

$$\sum_{i=1}^{n} b_{ni} X_{ni} / \log n \longrightarrow 0 \textit{ completely as } n \to \infty.$$

Proof. Let $a_{ni} = b_{ni}/\log n$. Then we can obtain the result by Theorem 3.1. The proof is complete. □

Theorem 3.2. *Let* $0 < p < 2$ *and let* $\{X_{ni} | 1 \leq i \leq n, n \geq 1\}$ *be an array of rowwise* ND *random variables with* $EX_{ni} = 0$. *Assume that* $\{a_{ni} \mid 1 \leq i \leq n, n \geq 1\}$ *is an array of real numbers satisfying* $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-1/p})$.

If $|X_{ni}| \leq M$, *then*

$$\sum_{i=1}^{n} a_{ni} X_{ni} \longrightarrow 0 \textit{ completely as } n \to \infty.$$

Proof. Without loss of generality, we assume that

$$0 < a_{ni}^{+} \leq n^{-1/p} \text{ for } 1 \leq i \leq n,\ n \geq 1.$$

As for the proof of Theorem 3.1, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

We also know that $\{a_{ni}^{+} X_{ni} \mid 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise ND random variables by definition and $|a_{ni}^{+} X_{ni}| \leq n^{-1/p} M$ and $E a_{ni}^{+} X_{ni} = 0$. Hence,

$$\begin{aligned}
& \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \epsilon\right) \\
= & \sum_{n=1}^{\infty} P\left(e^{\epsilon n^{1/p-1/2}/M} \sum_{i=1}^{n} a_{ni}^{+} X_{ni} > e^{\epsilon^2 n^{1/p-1/2}/M}\right) \\
\leq & \sum_{n=1}^{\infty} e^{-\epsilon^2 n^{1/p-1/2}/M} \prod_{i=1}^{n} E e^{\epsilon n^{1/p-1/2}/M} (a_{ni}^{+} X_{ni}) \\
\leq & \sum_{n=1}^{\infty} e^{-\epsilon^2 n^{1/p-1/2}/M} \prod_{i=1}^{n} e^{(\epsilon n^{1/p-1/2}/M)^2} (a_{ni}^{+})^2 E(X_{ni})^2 \\
\leq & \ c \sum_{n=1}^{\infty} e^{-\epsilon^2 n^{1/p-1/2}/M} \prod_{i=1}^{n} e^{\epsilon^2/n} \\
\leq & \ c \sum_{n=1}^{\infty} e^{-\epsilon^2 n^{1/p-1/2}/M} < \infty
\end{aligned}$$

since $0 < p < 2$ and $1/p - 1/2 > 0$. The proof is complete. □

Proposition 3.1. *Let $\{X_{ni} | 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise random variables. Suppose that there is a random variable X such that $\sup P(|X_{ni}| > x) \leq P(|X| > x)$ for all i, n and $x \geq 0$ and let $E|X|^{2p} < \infty$ for some $0 < p < 1$. Assume that $\{a_{ni} | 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-1/p})$ for some $0 < p < 1/2$. If $0 < p < 1/2$, then*

$$\sum_{i=1}^{n} a_{ni} X_{ni} \longrightarrow 0 \textit{ completely as } n \to \infty.$$

Proof. Without loss of generality, we assume that $0 < a_{ni}^{+} \leq n^{-1/p}$ for some $0 < p < 1/2$ and $1 \leq i \leq n,\ n \geq 1$. As for the proof of Theorem 3.1, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}\right| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

Then,

$$
\begin{aligned}
& P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}\right| > \epsilon\right) \\
\leq\ & P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/p})\right| > \epsilon/2\right) \\
& + P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| > n^{1/p})\right| > \epsilon/2\right) \\
=\ & I_8 + I_9 \text{ (say)}.
\end{aligned}
$$

As to I_8,

$$
\begin{aligned}
I_8 &= P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/p})\right| > \epsilon/2\right) \\
&\leq 2/\epsilon\ E\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/p})\right| \\
&\leq 2/\epsilon\ n^{-1/p} \sum_{i=1}^{n} E|X| I(|X| \leq n^{1/p}) \\
&\leq cn^{1-1/p} \int_0^{n^{1/p}} P(|X| > t) dt \quad \text{taking } t = n^{1/p} r \\
&= cn \int_0^1 P(|X| > n^{1/p} r) dr \\
&\leq cn^{-1} E|X|^{2p} \int_0^1 r^{-2p} dr
\end{aligned}
$$

which implies that

$$
n^{-1} E|X|^{2p} \int_0^1 r^{-2p} dr \longrightarrow 0 \text{ as } n \to \infty \tag{14}
$$

since $0 < p < 1/2$. As to I_9,

$$
\begin{aligned}
I_9 &= P\left(\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| > n^{1/p})\right| > \epsilon/2\right) \\
&\leq P\left(\sum_{i=1}^{n} n^{-1/p} |X_{ni} I(|X_{ni}| > n^{1/p}|) > \epsilon/2\right) \\
&\leq P\left(\bigcup_{i=1}^{n} |X_{ni}| > n^{1/p}\right) \qquad (15)
\end{aligned}
$$

$$\begin{aligned}
&\le \sum_{i=1}^{n} P(|X_{ni}| > n^{1/p}) \\
&\le \ nP(|X|^p > n) \\
&\le n^{-1}E|X|^{2p} \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, by (14) and (15), $\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0$. The proof is complete. □

Proposition 3.2. *Let $\{X_{ni}|1 \le i \le n, n \ge 1\}$ be an array of rowwise random variables. Suppose that there is a random variable X such that* $\sup P(|X_{ni}| > x) \le P(|X| > x)$ *for all i, n and $x \ge 0$ and let $E|X|^{2p} < \infty$ for some $0 < p < 1$. Assume that $\{a_{ni}|1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |a_{ni}| = O(n^{-1/p})$ for some $1/2 \le p < 1$. If $1/2 \le p < 1$, then*

$$\sum_{i=1}^{n} a_{ni}X_{ni} \longrightarrow 0 \textit{ in probability.}$$

Proof. It suffices to show that

$$\sum_{i=1}^{n} E|a_{ni}X_{ni}| \to 0$$

which implies the weak law of large numbers. By $\sup P(|X_{ni}| > x) \le P(|X| > x)$, note that $p \ge 1/2$ yield $\sup E|X_{ni}| < E|X| < \infty$. Hence,

$$\sum_{i=1}^{n} E|a_{ni}X_{ni}| \le \sum_{i=1}^{n} n^{-1/p}E|X_{ni}| \le n^{1-1/p}E|X| \longrightarrow 0 \text{ as } n \to \infty$$

since $1/2 \le p < 1$. Hence $\sum_{i=1}^{n} E|a_{ni}X_{ni}| \to 0$. The proof is complete. □

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