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CONDITIONS IMPLYING CONTINUITY OF MAPS

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Abstract. In this paper, we generalize the notions of preserving and strongly preserving maps to arbitrary set based topological categories. Further, we obtain characterizations of each of these concepts as well as interprete analogues and generalizations of theorems of Gerlits at al [20] in the categories of filter and local filter convergence spaces.

1. Introduction

Recall [20] that, a function f from a topological space X into a space Y is called preserving if the image of every compact subspace of X is compact in Y and the image of every connected subspace of X is connected in Y. It is well known that any continuous function is preserving. The converse is also true for real functions. However, the converse is not true, in general. McMillan [26] proved if X is Hausdorff, locally connected, and Frechet, Y is Hausdorff, then any preserving function from X into Y is continuous. Gerlits at al [20] proved that if X and Y are T_1 spaces, then any preserving function from X into Y is continuous.

The following facts are well known:

(1) A topological space X is compact if and only if the projection $\pi_2\colon X \times Y \to Y$ is closed for each topological space Y,
(2) For a topological space X, the followings are equivalent:
  (a) X is connected.
  (b) $\emptyset$ and X are the only subsets of X which are both closed and open.
  (c) Every continuous function from X to any discrete space is constant.
(3) A topological space X is locally connected if and only if the components of each open set in X are open.

The facts (1) and (2)(c) are used by several authors (see, [6], [14], [15], [22], [23], [25], [29] and [31]) to motivate a closer look at analogous situations in a more general categorical setting. A categorical notion of compactness with

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respect to a factorization structure was defined in the case of a general category by Manes [25] and Herrlich, Salicrup, and Strecker [22]. A categorical study of these notions with respect to an appropriate notion of "closedness" based on closure operators (in the sense of [17]) was done in [12], [13], [14] (for abstract categories), [18], [22], [25]. Baran in [2] and [4] introduced the notion of "closedness" and "strong closedness" in set-based topological categories and used these notions in [6] and [10] to generalize each of the notions of compactness and connectedness (2)(b) to arbitrary set-based topological categories.

Let X be a topological space and $\mathrm{Pr}(X, T_i)$ $(i = 1, 2, 3)$ denote the following statement: Every preserving map from X into any T_i space $(i = 1, 2, 3)$ is continuous.

Proposition 1.1 ([20]). *For topological spaces X and Y, the followings are known*:

(a) *If $q : X \to Y$ is a quotient mapping of X onto Y, then for any $i = 1, 2, 3$* $\mathrm{Pr}(X, T_i)$ *implies* $\mathrm{Pr}(Y, T_i)$.

(b) *For a T_1 space X, the following conditions are equivalent*:
(i) *If Y is T_1 and $f : X \to Y$ is a strong connectedness preserving map, then f is continuous.*
(ii) *If Y is T_1 and $f : X \to Y$ is a preserving map, then f is continuous* (*i.e.,* $\mathrm{Pr}(X, T_1)$ *holds*).

(c) *If* $\mathrm{Pr}(X, T_1)$ *holds for a T_1 space X, then every closed subspace of X is the topological sum of its components.*

(d) *If* $\mathrm{Pr}(X, T_1)$ *holds for a T_3 space X, then every closed subspace of X is locally connected.*

(e) *If* $\mathrm{Pr}(X, T_1)$ *holds for a T_3 space X, then X is discrete.*

The organization of the paper is as follows: In Section 2 we give some basic definitions and some technical results that are closely related to the definitions of (strong) connectedness, (strong) compactness, and (strong) locally connectedness. In Section 3, we recall the notion of (strong) compactness and (strong) connectedness. In Section 4, we introduce the notions of locally connected and strongly locally connected objects in a set-based topological category. In Section 5, we introduce the notions of a preserving map and a strongly preserving map in a set-based topological category and characterize each of these concepts as well as interprete analogues of Proposition 1.1 in categories of filter and local filter convergence spaces.

2. Preliminaries

Let $\mathcal{E}$ be a category and SET be the category of sets. The functor $U : \mathcal{E} \to$ SET is said to be topological or the category $\mathcal{E}$ is said to be topological over SET if U is concrete (i.e., faithful and amnestic (i.e., if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., sets) fibers, and if every U-source has an initial lift or, equivalently, if every U-sink has a final lift [1, 19, 21, 28].

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. M is called a subspace of X if the inclusion map $i : M \to X$ is an initial lift (i.e., an embedding) and we denote it by $M \subset X$.

Let B be a set and $p \in B$. The infinite wedge product $\bigvee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times \cdots$ be the countable cartesian product of B. Define $A_p^\infty : \bigvee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \ldots, x, p, p, \ldots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, \ldots, x, p, p, \ldots)$ and $\bigtriangledown_p^\infty : \bigvee_p^\infty B \to B$ by $\bigtriangledown_p^\infty(x_i) = x$ for all i, [2] or [4].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, p, \ldots, p, id, p, \ldots) : B \to B^\infty$, where the identity map, id, is in the j-th place.

Definition 2.1 (cf. [2, p. 335] or [4, p. 386])**.** Let $U : \mathcal{E} \to$ SET be topological and X an object in $\mathcal{E}$ with $UX = B$. Let M be a nonempty subset of B. We denote by X/M the final lift of the epi U-sink $q : U(X) = B \to B/M = (B \backslash M) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash M$ and identifying M with a point $*$. Let p be a point in B.

(1) X is T_1 at p if and only if the initial lift of the U-source $\{S_p : B \bigvee_p B \to U(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.

(2) p is closed if and only if the initial lift of the U-source $\{A_p^\infty : \bigvee_p^\infty B \to B^\infty = U(X^\infty)$ and $\bigtriangledown_p^\infty : \bigvee_p^\infty B \to UD(B) = B\}$ is discrete.

(3) $M \subset X$ is strongly closed if and only if X/M is T_1 at $*$ or $M = \emptyset$.

(4) $M \subset X$ is closed if and only if $*$, the image of M, is closed in X/M or $M = \emptyset$.

(5) If $B = M = \emptyset$, then we define M to be both closed and strongly closed.

(6) $M \subset X$ is open if and only if M^c, the complement of M, is closed in X.

(7) $M \subset X$ is strongly open if and only if M^c, the complement of M, is strongly closed in X.

In TOP, the category of topological spaces, the notion of closedness and openness coincides with the usual ones [2] and M is strongly closed if and only if M is closed and for each $x \notin M$ there exists a neighbourhood of M missing x [2]. If a topological space is T_1, then the notions of openness (closedness) and strong openness (resp., closedness) coincide [2].

Let A be a set and δ be a filter on A. The filter δ is said to be proper (improper) if and only if δ does not contain (resp., δ contains) the empty set, $\emptyset$.

A function L on A that assigns to each point x of A a set of filters (the "filters converging to x") is called a convergence structure on A ((A, L) a filter convergence space) if and only if it satisfies the following two conditions:

(1) $[x] = [\{x\}] \in L(x)$ for each $x \in A$ (where $[M] = \{B \subset A : M \subset B\}$).

(2) $\beta \supset \alpha \in L(x)$ implies $\beta \in L(x)$ for any filter β on A.

A map $f : (A, L) \to (B, S)$ between filter convergence spaces is called continuous if and only if $\alpha \in L(x)$ implies $f(\alpha) \in S(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$). The category of filter convergence spaces and continuous maps is denoted by FCO (see [16] or [30]). A filter convergence space (A, L) is said to be a local filter convergence space (in [28], it is called a convergence space) if $\alpha \cap [x] \in L(x)$ whenever $\alpha \in L(x)$ (see [27] or [28]). These spaces are the objects of the full subcategory LFCO (in [28] Conv) of FCO.

For filters α and β we denote by $\alpha \cup \beta$ the smallest filter containing both α and β.

Note that (A, L) is a discrete object in FCO (resp., LFCO) if and only if $L(a) = \{[a], [\emptyset]\}$ for all a in A [4].

Note that both FCO and LFCO are topological categories over SET.

More on these categories can be found in [1, 16, 24, 27, 28], and [30].

Theorem 2.2 ([4], Theorems 3.1 and 3.2)**.** *Let (B, L) be in* FCO (*resp.,* LFCO).

(a) *$\emptyset \neq M \subset X$ is closed if and only if for any $a \notin M$, if there exist $\alpha \in L(a)$ such that $\alpha \cup [M]$ is proper, then $[a] \notin L(c)$ for all $c \in M$.*

(b) *$\emptyset \neq M \subset X$ is strongly closed if and only if for any $a \in B$, if $a \notin M$, then $[a] \notin L(c)$ for all $c \in M$ and if $\alpha \in L(a)$, then $\alpha \cup [M]$ is improper.*

Theorem 2.3 ([10], Theorem 2.5)**.** *Let (B, L) be in* FCO (*resp.,* LFCO).

(a) *$\emptyset \neq M \subset B$ is open if and only if for any $a \in M$, if there exists $\alpha \in L(a)$ such that $\alpha \cup [M^c]$ is proper, then $[a] \notin L(c)$ for all $c \notin M$.*

(b) *$\emptyset \neq M \subset B$ is strongly open if and only if for any $a \in B$, if $a \in M$, then $[a] \notin L(c)$ for all $c \notin M$ and if $\alpha \in L(a)$, then $\alpha \cup [M^c]$ is improper.*

We give the following useful lemmas which will be needed later.

Lemma 2.4 (cf. [3], Lemmas 3.16 and 3.19)**.** (1) *For $a \in B$ with $a \notin M$, $q(\alpha) \subset [a]$ if and only if $\alpha \subset [a]$.*

(2) *$q(\alpha) \subset [*]$ if and only if $\alpha \cup [M]$ is proper.*

(3) *If $\alpha \cup [M]$ is improper, then $q(\sigma) \subset q(\alpha)$ if and only if $\sigma \subset \alpha$.*

(4) *If $\alpha \cup [M]$ is proper, then $q(\sigma) \subset q(\alpha)$ if and only if $\sigma \cap [M] \subset \alpha$ and $\sigma \cup [M]$ is proper.*

Lemma 2.5 (cf. [8], Lemma 3.2)**.** *Let $f : A \to B$ be a map.*

(1) *If α and β are proper filters on A, then $f(\alpha) \cup f(\beta) \subset f(\alpha \cup \beta)$.*

(2) *If δ is proper filter on B, then $\delta \subset ff^{-1}(\delta)$, where $f^{-1}(\delta)$ is the proper filter generated by $\{f^{-1}(D) : D \in \delta\}$.*

Lemma 2.6 (cf. [7], Lemma 1.4)**.** *Let α and β be proper filters on B. Then $q(\alpha) \cup q(\beta)$ is proper if and only if either $\alpha \cup \beta$ is proper or $\alpha \cup [M]$ and $\beta \cup [M]$ are proper.*

Let B be a set and $B^2 \bigvee_{\triangle} B^2$ be the wedge product of B^2, i.e., two disjoint copies of B^2 identified along the diagonal, $\triangle$. A point (x, y) in $B^2 \bigvee_{\triangle} B^2$

will be denoted by $(x,y)_1$ (resp. $(x,y)_2$) if (x,y) is in the first (resp. second) component of $B^2 \bigvee_\triangle B^2$ [2].

Recall that the principal axis map $A : B^2 \bigvee_\triangle B^2 \to B^3$ is given by $A(x,y)_1 = (x,y,x)$ and $A(x,y)_2 = (x,x,y)$. The skewed axis map $S : B^2 \bigvee_\triangle B^2 \to B^3$ is given by $S(x,y)_1 = (x,y,y)$ and $S(x,y)_2 = (x,x,y)$ and the fold map, $\nabla : B^2 \bigvee_\triangle B^2 \to B^2$ is given by $\nabla(x,y)_i = (x,y)$ for $i = 1,2$ [2].

Definition 2.7 (cf. [2], [5], or [7])**.** Let $U : \mathcal{E} \to \mathrm{SET}$ be topological and X an object in $\mathcal{E}$ with $U(X) = B$. Let M be a nonempty subset of B.

(1) X is T_0' if and only if the initial lift of the U-source $\{id : B^2 \bigvee_\triangle B^2 \to U(B^2 \bigvee_\triangle B^2)' = B^2 \bigvee_\triangle B^2$ and $\nabla : B^2 \bigvee_\triangle B^2 \to UD(B^2) = B^2\}$ is discrete, where $(B^2 \bigvee_\triangle B^2)'$ is the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 \bigvee_\triangle B^2\}$ and $D(B^2)$ is the discrete structure on B^2. Here, i_1 and i_2 are the canonical injections.

(2) X is T_1 if and only if the initial lift of the U-source $\{S : B^2 \bigvee_\triangle B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \bigvee_\triangle B^2 \to UD(B^2) = B^2\}$ is discrete.

(3) X is PreT_2' if and only if the initial lift of the U-source $\{S : B^2 \bigvee_\triangle B^2 \to U(X^3) = B^3\}$ and the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 \bigvee_\triangle B^2\}$ coincide, where i_1 and i_2 are the canonical injections.

(4) X is Pre$\bar{T}_2$ if and only if the initial lifts of the U-sources $\{A : B^2 \bigvee_\triangle B^2 \to U(X^3) = B^3\}$ and $\{S : B^2 \bigvee_\triangle B^2 \to U(X^3) = B^3\}$ coincide.

(5) X is T_2' if and only if X is T_0' and $\mathrm{Pr}\,eT_2'$.

(6) X is T_3' if and only if X is T_1 and X/M is PreT_2' for all closed $M \neq \emptyset$ in $U(X)$.

(7) X is $S\bar{T}_3$ if and only if X is T_1 and X/M is Pre$\bar{T}_2$ for all strongly closed $M \neq \emptyset$ in $U(X)$.

Note that for the category TOP of topological spaces, T_0', T_1, PreT_2', Pre$\bar{T}_2$, T_2' and both of the T_3' and $S\bar{T}_3$ reduce to the usual T_0, T_1, PreT_2, T_2 and T_3 separation axioms, respectively ([2], [5] and [7], where a topological space is called PreT_2 [2] if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neigbourhoods.

Theorem 2.8 ([5])**.** *Let (B,L) be in* FCO *(resp.,* LFCO*) and $\emptyset \neq M \subset B$.*

(1) *(B,L) is T_1 if and only if for each distinct pair of points x and y in B, $[x] \notin L(y)$.*

(2) *All objects (B,L) in* FCO *(resp.,* LFCO*) are T_0'.*

(3) *(B,L) is* PreT_2' *(T_2') if and only if (B,L) is discrete, i.e., for all x in B, $L(x) = \{[\emptyset], [x]\}$.*

Theorem 2.9. *(B,L) in* FCO *(resp.,* LFCO*) is $S\bar{T}_3$ if and only if conditions (1), (2), and (3) hold, where the conditions are:*

(1) *for all $a \neq b$ in B, $L(a) \cap L(b) = \{[\emptyset]\}$;*

(2) *for any nonempty strongly closed subset M of B, $a \in B$, and any proper filters α, $\delta \in L(a)$:*

(i) *if $a \notin M$ and $\alpha \cup \delta$ is proper, then there exists a filter $\beta \in L(a)$ such that $\beta \subset \alpha \cap \delta$;*

(ii) *if $a \in M$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper, then $\exists d \in M$ and a filter $\beta \in L(d)$ such that either $\beta \subset \alpha \cap \delta$ or $\beta \cap [M] \subset \alpha \cap \delta$ and $\beta \cup [M]$ is proper;*

(3) *for any nonempty strongly closed subset M of B and any proper filters $\alpha \in L(c)$ and $\delta \in L(d)$ with $c, d \in M$, if both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper, then there exist $e \in M$ and a filter $\beta \in L(e)$ such that $\beta \cap [M] \subset \alpha \cap \delta$ and $\beta \cup [M]$ is proper.*

Proof. Suppose (B, L) is $S\bar{T}_3$. Let $\alpha \in (L(a) \cap L(b))$ with $a \neq b$ in B. Then $q(\alpha) \in L^{'}(q(a)) \cap L^{'}(q(b))$, where $L^{'}$ is the quotient structure on B/M induced by the map $q : B \to B/M$ that identifies M to a point $*$. (B, L) is $S\bar{T}_3$, in particular, $(B/M, L^{'})/M$ is $\bar{T}_2$ and so, by Theorem 2.7 of [5], $q(\alpha) = [\emptyset]$. Hence $\alpha = [\emptyset]$. This shows condition (1) holds.

Suppose that for any nonempty strongly closed subset M of B, $a \in B$, and any proper filters $\alpha, \delta \in L(a)$.

Suppose $a \notin M$ and $\alpha \cup \delta$ is proper. Then, by Lemma 2.6, $q(\alpha) \cup q(\delta)$ is proper. Note that $q(\alpha)$, $q(\delta) \in L^{'}(q(a))$. Since $(B/M, L^{'})/M$ is Pre$\bar{T}_2$, by Theorem 2.6 of [5], $q(\alpha \cap \delta) = q(\alpha) \cap q(\delta) \in L^{'}(q(a))$. It follows from definition of $L^{'}$ that there exists $\beta \in L(a)$ such that $q(\beta) \subset q(\alpha \cap \delta)$. If $(\alpha \cap \delta) \cup [M]$ is proper, then, by Lemma 2.4(2), $q(\alpha \cap \delta) \subset [*]$ and thus, $[*] \in L^{'}(q(a))$, a contradiction. Therefore, $(\alpha \cap \delta) \cup [M]$ must be improper and by Lemma 2.4(3), $\beta \subset \alpha \cap \delta$. Suppose $a \in M$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper. It follows from definition of $L^{'}$ that there exist $d \in M$ and $\beta \in L(d)$ such that $q(\beta) \subset q(\alpha \cap \delta)$ and $q(d) = * = q(a)$. If $(\alpha \cap \delta) \cup [M]$ is improper, then, by Lemma 2.4(3), $\beta \subset \alpha \cap \delta$. If $(\alpha \cap \delta) \cup [M]$ is proper, then, by 2.4(4), $\beta \cap [M] \subset \alpha \cap \delta$ and $\beta \cup [M]$ is proper. So, condition (2) also holds.

Suppose that for any nonempty strongly closed subset M of B, any proper filters $\alpha \in L(c)$ and $\delta \in L(d)$ with $c, d \in M$, $\alpha \cup [M]$ and $\delta \cup [M]$ are proper. Then, by Lemma 2.6, $q(\alpha) \cup q(\delta)$ is proper. Note that $q(\alpha), q(\delta) \in L^{'}(*)$. Since $(B/M, L^{'})/M$ is Pre$\bar{T}_2$, by Theorem 2.6 of [5], $q(\alpha \cap \delta) = q(\alpha) \cap q(\delta) \in L^{'}(*)$. It follows that there exist $e \in M$ and a filter $\beta \in L(e)$ such that $q(\beta) \subset q(\alpha \cap \delta)$ and $q(e) = *$. Since $(\alpha \cap \delta) \cup [M]$ is proper, by Lemma 2.4(4), $\beta \cap [M] \subset \alpha \cap \delta$ and $\beta \cup [M]$ is proper.

Conversely, suppose that the conditions hold. By (1) and Theorem 2.2, (B, L) is T_1. Suppose M is strongly closed subset of B. Note, by Theorem 2.8, that $(B/M, L^{'})/M$ is T_1. Hence, it is sufficient to show that $(B/M, L^{'})/M$ is $\bar{T}_2$ for any nonempty strongly closed subset M of B. Let $x \neq y$ in B/M and $\sigma \in L^{'}(x) \cap L^{'}(y)$. If $\sigma = [\emptyset]$, then we are done. Suppose $\sigma \neq [\emptyset]$. It follows that there exist $\alpha \in L(a)$ and $\delta \in L(b)$ such that $q(\alpha) \subset \sigma$, $q(\delta) \subset \sigma$ and $q(a) = x$,

$q(b) = y$. Notice that $q(\alpha) \cup q(\delta)$ is proper, and so, by Lemma 2.6, either $\alpha \cup \delta$ is proper or both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper. By the assumption (1), the first case can not occur. The second case can not happen either, since M is strongly closed subset of B (by Theorem 2.8, we may assume that $a \notin M$). Hence, we must have $\sigma = [\emptyset]$.

It remains to show that for any proper filters $\sigma, \gamma \in L^{'}(x)$ with $\alpha \cup \gamma$ proper, $\alpha \cap \gamma \in L^{'}(x)$. Let $x \neq *$. If $\sigma, \gamma \in L^{'}(x)$, then there exist $\alpha, \delta \in L(a)$ such that $q(\alpha) \subset \sigma$, $q(\delta) \subset \gamma$ and $q(a) = a = x$. It follows that $q(\alpha) \cup q(\delta)$ is proper, and so, by Lemma 2.6, either $\alpha \cup \delta$ is proper or both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper. The second case can not occur since M is strongly closed subset of B (by Theorem 2.8). Hence, we must have $\alpha \cup \delta$ is proper. By the assumption (2), there exists $\beta \in L(a)$ such that $\beta \subset \alpha \cap \delta$. Note that $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ and consequently, $\sigma \cap \gamma$ is in $L^{'}(x)$.

Suppose $x = *$ and $\sigma, \gamma \in L^{'}(*)$. Then there exist $c, d \in M$ and $\alpha \in L(c)$, $\delta \in L(d)$ such that $q(\alpha) \subset \sigma$, $q(\delta) \subset \gamma$ and $q(c) = * = q(d)$. It follows that $q(\alpha) \cup q(\delta)$ is proper, and so, by Lemma 2.6, either $\alpha \cup \delta$ is proper or both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper.

If $c \neq d$, then the first case can not hold since $\alpha \cup \delta \in L(c) \cap L(d)$. Thus, the second must hold. By the assumption (3), there exist $e \in M$ and $\beta \in L(e)$ such that $\beta \cap [M] \subset \alpha \cap \delta$ and $\beta \cup [M]$ is proper. Hence, $q(\beta) = q(\beta \cap [M])$ $= q(\beta) \cap [*] \subset \sigma \cap \gamma$ and consequently $\sigma \cap \gamma \in L^{'}(*)$, since by Lemma 2.4(2), $\beta \cup [M]$ is proper if and only if $q(\beta) \subset [*]$.

Suppose $c = d$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [M]$ and $\delta \cup [M]$ are proper. Then, by the assumption (2), there exist $e \in M$ and $\beta \in L(e)$ such that $\beta \subset \alpha \cap \delta$ or $\beta \cap [M] \subset \alpha \cap \delta$ and $\beta \cup [M]$ is proper. If the first case holds, then $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ and consequently, $\sigma \cap \gamma \in L^{'}(*)$. If the second case holds, then $q(\beta) = q(\beta \cap [M]) = q(\beta) \cap [*] \subset \sigma \cap \gamma$ and consequently, $\sigma \cap \gamma \in L^{'}(*)$, since by Lemma 2.4(2), $\beta \cup [M]$ is proper if and only if $q(\beta) \subset [*]$. Hence, by Theorem 2.7 of [5], $(B/M, L^{'})/M$ is $\bar{T}_2$ and thus, (B, L) is $S\bar{T}_3$. □

Theorem 2.10. *Let* (B, L) *in* FCO (*resp.,* LFCO). (B, L) *is* T_3' *if and only if for all* $x \neq y$ *in* M, $[x] \notin L(y)$ *for any* $x \in B$ *and for any proper filter* $\alpha \in L(x)$ *either* $\alpha = [x]$ *or* $M \in \alpha$ *for any nonempty subset* M *of* B.

Proof. Suppose (B, L) is T_3'. Since (B, L) is T_1, by Theorem 2.8, in particular, for all $x \neq y$ in M, $[x] \notin L(y)$. If $\alpha \in L(x)$, where $x \in B$, then $q(\alpha) \in L^{'}(qx)$. Since $(B/M, L^{'})/M$ is PreT_2', (M is nonempty subset of B) by Theorem 2.8, $q(\alpha) = [qx]$ (since α is proper). If $x \notin M$, then it is easy to see that $[x] = q^{-1}(x) = q^{-1}q(\alpha) \subset \alpha$ and consequently $\alpha = [x]$. If $x \in M$, it follows easily that $q(\alpha) = [*]$ if and only if $M \in \alpha$.

Conversely, suppose the conditions hold. By Theorem 2.8, clearly, (B, L) is T_1. We now show that $(B/M, L^{'})/M$ is PreT_2' for all nonempty subset M of B. If $x \in B/M$ and $\alpha \in L^{'}(x)$, it follows that there exists $\beta \in L(a)$ such that $q(\beta) \subset \alpha$ and $qa = x$. If β is improper, then so is α. If β is proper, then by assumption $\beta = [a]$ or $M \in \beta$. If the first case holds, then $[qa] = q(\beta) \subset \alpha$ and thus $\alpha = [qa]$. If the second case holds, then $* = q(M) \in q(\beta) \subset \alpha$ and consequently $\alpha = [*]$. Hence, by Theorem 2.8, $(B/M, L^{'})/M$ is PreT_2' and consequently, (B, L) is T_3'. □

3. Compact and connected objects

Recall that the notions of each of (strongly) closed morphisms and (strongly) compact objects in a topological category $\mathcal{E}$ over SET are introduced in [6].

Definition 3.1. Let $U : \mathcal{E} \to$ SET be topological, X and Y be objects in $\mathcal{E}$, and $f : X \to Y$ be a morphism in $\mathcal{E}$.

(1) f is said to be closed if and only if the image of each closed subobject of X is a closed subobject of Y.

(2) f is said to be strongly closed if and only if the image of each strongly closed subobject of X is a strongly closed subobject of Y.

(3) X is compact if and only if the projection $\pi_2 \colon X \times Y \to Y$ is closed for each object Y in $\mathcal{E}$.

(4) X is strongly compact if and only if the projection $\pi_2 \colon X \times Y \to Y$ is strongly closed for each object Y in $\mathcal{E}$.

For the category TOP of topological spaces, the notions of closed morphism and compactness reduce to the usual ones ([11] p. 97 and 103). Furthermore, by Theorem 2.2 and Definition 3.1, one can show that the notions of compactness and strong compactness are equivalent.

Theorem 3.2 ([9], Theorem 5.3)**.** (1) *All objects in* FCO (*resp.,* LFCO) *are compact.*

(2) (B, L) *in* FCO (*resp.,* LFCO) *is strongly compact if and only if every ultrafilter in B converges.*

Theorem 3.3. *Let $f : X \to Y$ be a morphism in* FCO (*resp.,* LFCO). *If X is* (*strongly*) *compact, then $f(X)$ is* (*strongly*) *compact.*

Proof. If X is compact, then, by Theorem 3.2, $f(X)$ is compact. It remains to show that if X is strongly compact, then $f(X)$ is strongly compact for FCO (resp., LFCO). Let α be an ultrafilter on $f(X)$. Note that $f^{-1}(\alpha)$ is a filter on X and consequently there exists an ultrafilter β on X with $\beta \supset f^{-1}(\alpha)$. Since X is strongly compact, by Theorem 3.2(2), there exists $x \in X$ such that $\beta \in L(x)$ and consequently, $f(\beta) \in S(f(x))$ and $f(\beta)$ is an ultrafilter on Y. It follows that $\alpha = f(f^{-1}(\alpha)) = f(\beta) \in S(f(x))$ (since $f : X \to Y$ is a morphism in FCO (resp., LFCO), α is an ultrafilter, and $f(f^{-1}(\alpha)) \supset \alpha$). Hence by Theorem 3.2(2), $f(X)$ is strongly compact. □

Definition 3.4 ([10, p. 5])**.** Let $\mathcal{E}$ be a topological category over SET and X be an object in $\mathcal{E}$.

(1) X is connected if and only if the only subsets of X both strongly open and strongly closed are X and $\emptyset$.

(2) X is strongly connected if and only if the only subsets of X both open and closed are X and $\emptyset$.

Note that for the category TOP of topological spaces, the notion of strong connectedness coincides with the usual notion of connectedness. If a topological space X is T_1, then, by Theorem 2.2 and Definition 3.4, the notions of connectedness and strong connectedness coincide.

Lemma 3.5 ([10, p. 5])**.** *Let* (B, L) *be in* FCO (*resp.,* LFCO)*.*

(B, L) *is strongly connected if and only if for any non-empty proper subset* M *of* B*, either the condition* (I) *or* (II) *holds*;

(I) *There exists a proper filter* α *in* $L(a)$ *such that* $\alpha \cup [M]$ *is proper for some* $a \in M^c$ *and* $[a] \in L(b)$ *for some* $b \in M$*.*

(II) *There exists a proper filter* α *in* $L(b)$ *such that* $\alpha \cup [M^c]$ *is proper for some* $b \in M$ *and* $[b] \in L(a)$ *for some* $a \in M^c$*.*

Lemma 3.6 ([10, p. 6])**.** *Let* (B, L) *be in* FCO (*resp.,* LFCO)*.*

(B, L) *is connected if and only if for any non-empty proper subset* M *of* B*, either the condition* (I) *or* (II) *holds.*

(I) *There exists a proper filter* α *in* $L(a)$ *such that* $\alpha \cup [M]$ *is proper for some* $a \in M^c$ *or* $[a] \in L(b)$ *for some* $b \in M$*.*

(II) *There exists a proper filter* α *in* $L(b)$ *such that* $\alpha \cup [M^c]$ *is proper for some* $b \in M$ *or* $[b] \in L(a)$ *for some* $a \in M^c$*.*

Recall that an objects X in a topological categories is connected (we call it **D-connected**, for simplicity) in the sense of [12, 13, 23, 24, 29, 30, 31] if and only if any morphism from X to a discrete object is constant.

Lemma 3.7 ([10, p. 10])**.** *Let* (B, L) *be in* FCO (*resp.,* LFCO)*.*

(B, L) *is D-connected if and only if for any non-empty proper subset* M *of* B*, either the condition* (I) *or* (II) *holds*;

(I) *There exists a proper filter* α *in* $L(a)$ *such that* $\alpha \cup [M]$ *is proper for some* $a \in M^c$*.*

(II) *There exists a proper filter* α *in* $L(b)$ *such that* $\alpha \cup [M^c]$ *is proper for some* $b \in M$*.*

Lemma 3.8. *Let* $f : (B, L) \to (A, S)$ *be a morphism in* FCO*,* (*resp.,* LFCO)*. If* (B, L) *is* (*strongly*) *connected or D-connected, then* $f(B)$ *is* (*strongly*) *connected or D-connected, respectively.*

Proof. Let (B, L), (A, S) be in FCO (resp., LFCO) and M any non empty proper subset of $f(B)$. Since $f^{-1}(M) \subset B$ and (B, L) is strongly connected, either conditions (I) or (II) in Lemma 3.5 holds. Suppose condition (I) in Lemma 3.5 holds. Then, there exists $\alpha \in L(a)$ such that $\alpha \cup [f^{-1}(M)]$ is proper for some $a \in (f^{-1}(M))^c$ and $[a] \in L(b)$ for some $b \in f^{-1}(M)$. Note that $f(a) \in M^c$ and $f(\alpha) \in S(f(a))$. By Lemma 2.5, $f(\alpha \cup [f^{-1}(M)]) \supset f(\alpha) \cup f([f^{-1}(M)]) \supset f(\alpha) \cup [M]$. Since $\alpha \cup [f^{-1}(M)]$ is proper, it follows that

$f(\alpha) \cup [M]$ is proper. Moreover, $[f(a)] \in S(f(b))$ and $f(b) \in M$. Similarly, if the condition (II) of Lemma 3.5 holds, $f(B)$ is strongly connected.

The proof for connectedness or D-connectedness is similar.

Let $\mathcal{E}$ be a complete category and C be a closure operator in the sense of Dikranjan and Giuli [17] of $\mathcal{E}$. An object X of $\mathcal{E}$ is called C-connected if the diagonal morphism $\delta_X = \langle 1_X, 1_X \rangle : X \to X \times X$ is C-dense. By $\nabla(C)$ we denote the full subcategory of C-connected objects (cf. [15], p. 158).

Note that if $\mathcal{E}$=TOP and $C = K$, the usual Kuratowski closure operator, then $\nabla(K)$ is the category of irreducible spaces (i.e., of spaces X for which $X = F \cup G$ with closed sets F, G is possible only for F=X or G=X) [15]. If $C = q$, the quasi-component closure operator which assigns to a subset M of X its quasi-component, i.e., the intersection of clopen sets in X containing M, then $\nabla(q)$ is the category of connected spaces [15]. □

If $\mathcal{E}$ = FCO (resp., LFCO) and C = cl (resp., scl) [8], the closure operators induced from the notions of closedness (resp., strong closedness) defined in Definition 2.1, then we have;

Lemma 3.9 ([10, p. 9])**.** *Let (B, L) be in* FCO *(resp.,* LFCO*).*

(1) *(B, L) is cl-connected if and only if for all $a, b \in B$ with $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$ and there exists $c \in B$ such that $[a]$ and $[b] \in L(c)$.*

(2) *(B, L) is scl-connected if and only if for all $a, b \in B$ with $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$ or there exists $c \in B$ such that $[a]$ and $[b] \in L(c)$.*

Lemma 3.10. *Let $f : (B, L) \to (A, S)$ be a morphism in* FCO*, (resp.,* LFCO*). If (B, L) is cl-connected (scl-connected), then $f(B)$ is cl-connected (scl-connected).*

Proof. Let (B, L), (A, S) be in FCO (resp., LFCO) and for any $a, b \in f(B)$ with $a \neq b$. There exist $x, y \in B$ with $x \neq y$ such that $f(x) = a$ and $f(y) = b$. Since (B, L) is cl-connected, then, by Lemma 3.9, there exists $c \in B$ such that $[x]$ and $[y] \in L(c)$. Note that $[f(x)]$, $[f(y)] \in L(f(c))$.

It remains to show that $S(a) \cap S(b) \neq \{[\emptyset]\}$. Note that $x, y \in B$ and (B, L) is cl-connected, then, there exists a proper filter $\alpha \in L(x) \cap L(y)$. It follows that $f(\alpha)$ is proper and $f(\alpha) \in S(f(x)) \cap S(f(y))$. Thus, $S(a) \cap S(b) \neq \{[\emptyset]\}$, which shows that $f(B)$ is cl-connected.

The proof for scl-connectedness is similar. □

4. Locally connected objects

In this section, the notions of locally connected and strongly locally connected objects in a set based topological category are introduced.

Let $\mathcal{E}$ be a topological category over SET, X be an object in $\mathcal{E}$, and $x \in U(X)$.

Definition 4.1. (1) The component $C(x)$ of x in X is the union of all connected subsets of X containing x.

(2) The strongly component $SC(x)$ of x in X is the union of all strongly connected subsets of X containing x.

(3) X is (strongly) totally disconnected if and only if $C(x) = \{x\}$ (resp., $SC(x) = \{x\}$).

Remark 4.2. (1) Let (B, L) be in FCO (resp., LFCO). If (B, L) is T_1, then, it follows easily from Theorem 2.8 and Lemma 3.5 that (B, L) is strongly connected if and only if B is a point or the empty set. By Definition 4.1, $SC(x) = \{x\}$ and consequently, (B, L) is strongly totally disconnected.

(2) Let (B, L) be in FCO (resp., LFCO). If (B, L) is either T_2' or T_3', then, by Theorem 2.8 and Theorem 2.10, (B, L) is (strongly) connected if and only if B is a point or the empty set, i.e., Card$B \leq 1$.

Definition 4.3. (1) X is locally connected if and only if the components of each open set in X are open sets.

(2) X is strongly locally connected if and only if the strongly components of each open set in X are strongly open sets.

Note that for the category TOP of topological spaces, by [11], the notion of strongly locally connected coincides with the usual one. Moreover, if a topological space X is T_1, then the notions of local connectedness and local strong connectedness coincide.

Theorem 4.4. *Let* (B, L) *be in* FCO (*resp.,* LFCO). *If* (B, L) *is* T_1, *then* (B, L) *is strongly locally connected.*

Proof. It follows from Theorem 2.8, Lemma 3.5, and Definition 4.3. □

Theorem 4.5. *Let* X *be in* FCO *or* LFCO *and* $f : X \to Y$ *be an epimorphism. If* X *is* T_1 *and* X *is either strongly connected or strongly locally connected, then so also is* Y.

Proof. Combine Theorem 2.8, Lemma 3.5, and Theorem 4.4. □

5. Preserving maps

Definition 5.1. Let $U : \mathcal{E} \to$ SET be topological, X and Y objects in $\mathcal{E}$, and $f : U(X) \to U(Y)$ be a map.

(1) f is a preserving map if and only if the image of every compact subobject of X is compact and the image of every strongly connected subobject of X is strongly connected.

(2) f is a strongly preserving map if and only if the image of every strongly compact subobject of X is strongly compact and the image of every connected subobject of X is connected.

Note that for $\mathcal{E}$=TOP, Definition 5.1(1) reduces to the usual one that is introduced in [20].

We now give characterizations of each of these concepts as well as interprete analogues and generalizations of theorems of Gerlits at al [20] in the categories of filter and local filter convergence spaces.

Let $\mathrm{Pr}(X, T_i)$ (resp., $\mathrm{SPr}(X, T_i)$) $(i = 1, 2, 3)$ denote the following statement: Every (resp., strongly) preserving map from an object X into any T_i object $(i = 1, 2, 3)$ is a morphism in $\mathcal{E}$.

Lemma 5.2. *Let $\mathcal{E}$ =FCO (resp., LFCO). All morphisms in $\mathcal{E}$ are (strongly) preserving, but the converse of implication is not true, in general.*

Proof. It follows from Theorem 3.3, Lemma 3.8, and Definition 5.1. □

Lemma 5.3. *Let $f : X \to Y$ be a morphism in $\mathcal{E}$ =FCO (resp., LFCO). For a T_1 object X, the following conditions are equivalent.*

(a) *If Y is T_1 and $f : X \to Y$ is a (strongly) connected preserving map, then f is a morphism (continuous).*

(b) *If Y is T_1 and $f : X \to Y$ is a (strongly) preserving map, then $\mathrm{Pr}(X, T_1)$ holds.*

Proof. Combine Theorem 2.8, Theorem 3.3, Lemma 3.8, and Definition 5.1. □

Theorem 5.4. *If an object (B, L) in FCO (resp., LFCO) is $T_2^{'}$, then any map from B to any set A is (strongly) preserving if and only if $f : (B, L) \to (A, S)$ is a morphism.*

Proof. It follows from Theorem 2.8, Theorem 3.3, Lemma 3.8, and Definition 5.1. □

Remark 5.5. Let (B, L) be in FCO (resp., LFCO).

(1) By Theorem 2.8 (resp., Theorem 2.10), if (B, L) is T_1 (resp., $T_3^{'}$), then every subspace of (B, L) is the coproduct of its components.

(2) By Theorem 2.8 (resp., Theorem 2.10), if (B, L) is $T_2^{'}$ (resp., $T_3^{'}$), then every subspace of (B, L) is discrete.

(3) By Theorem 4.4 (resp., Theorem 2.10), If $\mathrm{Pr}((B, L), T_1)$ holds for a T_1 (resp., $T_3^{'}$) space (B, L), then every subspace of (B, L) is locally connected.

(4) By Definition 3.4, Lemma 3.8 and Definition 5.1, the composition of (strongly) preserving maps is also (strongly) preserving.

Lemma 5.6. *Let $\mathcal{E}$ =FCO (resp., LFCO) and (B, L) be strongly connected in $\mathcal{E}$. $SPr((B, L), T_2^{'})$ holds if and only if any map B to any set A is constant.*

Proof. It follows from Theorem 2.8, Lemma 3.5, and Definition 5.1. □

Remark 5.7. Let $\mathcal{E}$ = FCO (resp., LFCO) and (B, L) be connected in $\mathcal{E}$. $\mathrm{Pr}((B, L), T_2^{'})$ holds if and only if any map B to any set A is constant.

Proof. It follows from Theorem 2.8, Lemma 3.6, and Lemma 5.6. □

Lemma 5.8. *Let $\mathcal{E}$ =FCO (resp., LFCO), (B, L) and (A, S) be in $\mathcal{E}$. If (B, L) is $T_2^{'}$, then $\mathrm{Pr}(X, T_i)$ (resp., $SPr(X, T_i)$) $(i = 1, 2, 3)$ holds.*

Proof. It follows from Definition 5.1 and Theorem 5.4. □

Lemma 5.9. *Let X and Y be in* FCO (*resp.,* LFCO)*. If $q : X \to Y$ is a quotient mapping of X onto Y, then, for any $i = 1, 2, 3$* $\mathrm{Pr}(X, T_i)$ *implies* $\mathrm{Pr}(Y, T_i)$.

Proof. Let $f : Y \to Z$ be a preserving map into the T_i space Z. The function $f \circ q : X \to Z$, as the composition of a continuous (and so preserving), and of a preserving function is also preserving. Since $\mathrm{Pr}(X, T_i)$ $(i = 1, 2, 3)$ holds, $f \circ q$ is a morphism (continuous). Hence, f is a morphism (continuous) because q is a quotient. □

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