

WEAK METRIC AND WEAK COMETRIC SCHEMES

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ABSTRACT. The notion of weak metric and weak cometric schemes are introduced as a generalization of metric and cometric schemes. They are given as the wreath product of a finite number of symmetric association schemes satisfying certain equivalent conditions which are analogous to the ones for metric or cometric schemes. We characterize those schemes and determine some of their parameters.

1. Introduction

Let \mathbb{F}_q be the finite field with q elements, and let $\mathbb{P} = ([n], \leq)$ be a poset on the underlying set $[n] = \{1, 2, \dots, n\}$ of coordinate positions of vectors in \mathbb{F}_q^n . Then the \mathbb{P} -weight $w_{\mathbb{P}}$ is the function on \mathbb{F}_q^n which is given by

$$w_{\mathbb{P}}(x) = |\{i \in [n] \mid i \leq j \text{ for some } j \in \text{Supp}(x)\}|.$$

Here $\text{Supp}(x) = \{j \in [n] \mid x_j \neq 0\}$ for $x = (x_1, \dots, x_n) \in \mathbb{F}_q^n$. Now, $d_{\mathbb{P}}(x, y) = w_{\mathbb{P}}(x - y)$ is a metric, called \mathbb{P} -metric. If \mathbb{P} is an antichain, \mathbb{P} -weight and \mathbb{P} -metric reduce respectively to Hamming weight and Hamming metric. The notion of \mathbb{P} -codes, namely subsets $\mathcal{C} \subseteq \mathbb{F}_q^n$ equipped with $w_{\mathbb{P}}$, were introduced in [4] by Brualdi et. al.

For each linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$, the \mathbb{P} -weight distribution of \mathcal{C} is $\{A_{\mathbb{P}, i}(\mathcal{C})\}_{i=0}^n$, where

$$A_{\mathbb{P}, i}(\mathcal{C}) = |\{x \in \mathcal{C} \mid w_{\mathbb{P}}(x) = i\}|.$$

We will denote by $\text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}})$ the group of all linear automorphisms $\tau : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ satisfying $w_{\mathbb{P}}(\tau x) = w_{\mathbb{P}}(x)$ for all $x \in \mathbb{F}_q^n$. Let $\mathbb{P}_0 = n_1 \mathbf{1} \oplus \dots \oplus n_t \mathbf{1}$ be the poset (\mathbb{P}_0 is called a weak order poset) given as the ordinal sum of the antichains $n_i \mathbf{1}$ on the set $\{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_{i-1} + n_i\}$ for $i = 1, \dots, t$, i.e., the underlying set is $[n]$ ($n = n_1 + \dots + n_t$) and the order relation is given by:

$$i < j \Leftrightarrow i \in n_l \mathbf{1}, j \in n_m \mathbf{1} \text{ for some } l < m.$$

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Then we have the following fundamental result.

Theorem A. *The following are equivalent.*

- (1) \mathbb{P} is a weak order poset on $[n]$.
- (2) $(\mathbb{F}_q^n, \{R_i\}_{i=0}^n)$, with $(x, y) \in R_i \Leftrightarrow d_{\mathbb{P}}(x, y) = i$ ($0 \leq i \leq n$) is a symmetric translation association scheme.
- (3) The \mathbb{P} -weight distribution $\{A_{\mathbb{P}, i}(\mathcal{C})\}_{i=0}^n$ of \mathcal{C} uniquely determines $\check{\mathbb{P}}$ -weight distribution $\{A_{\check{\mathbb{P}}, i}(\mathcal{C}^\perp)\}$ of \mathcal{C}^\perp for any linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$.
- (4) The group $\text{Aut}(\mathbb{F}_q^n, w_{\mathbb{P}})$ acts transitively on each \mathbb{P} -sphere $S_{\mathbb{P}}(r) = \{x \in \mathbb{F}_q^n \mid w_{\mathbb{P}}(x) = r\}$ for $0 \leq r \leq n$.

(1) \Leftrightarrow (2) is shown in [8]. (1) \Rightarrow (3) is proved in [5] and [7], and (3) \Rightarrow (1) in [7]. Finally, (1) \Rightarrow (4) is verified in [5] and (4) \Rightarrow (1) in [6]. Here we remark that the proofs for (1) \Rightarrow (3) were found in the form of MacWilliams-type identities by applying the discrete Poisson summation formula to suitable \mathbb{P} -weight enumerators (cf. [5-7]).

Let us now pay our attention to the equivalence (1) \Leftrightarrow (2) in Theorem A. We will denote the association scheme $(\mathbb{F}_q^n, \{R_i\}_{i=0}^n)$, with $(x, y) \in R_i \Leftrightarrow d_{\mathbb{P}_0}(x, y) = i$ ($\mathbb{P}_0 = n_1\mathbf{1} \oplus n_2\mathbf{1} \oplus \dots \oplus n_t\mathbf{1}$), by $H(n_1, \dots, n_t; q)$, which is what we call a weak Hamming scheme. Note that this becomes the usual Hamming scheme when $t = 1$. Recall the following theorem of Delsarte [3], which is usually called generalized MacWilliams identity.

Theorem B. *Let Y be an additive code of the translation association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^n)$. Then*

$$(a_j(Y^\circ))_{j=0}^n = \frac{1}{|Y|} (a_i(Y))_{i=0}^n (q_{ij}).$$

Here Y is just a subgroup of X , Y° is the additive code of the dual scheme $\mathfrak{X}^* = (X^*, \{R_i^*\}_{i=0}^n)$ of \mathfrak{X} , given by

$$\begin{aligned} Y^\circ &= \{\chi \in X^* \mid \chi(x) = 1 \text{ for all } x \in Y\}, \\ a_i(Y) &= |\{y \in Y \mid (0, y) \in R_i\}|, \\ a_i(Y^\circ) &= |\{\chi \in Y^\circ \mid (1, \chi) \in R_i^*\}|, \end{aligned}$$

and (q_{ij}) is the Q -matrix of the scheme \mathfrak{X} .

An immediate consequence of Theorem B is the classical MacWilliams identity which can be expressed in weight-enumerator-free form as: for any linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$,

$$(a_j(\mathcal{C}^\perp))_{j=0}^n = \frac{1}{|\mathcal{C}|} (a_i(\mathcal{C}))_{i=0}^n (p_j(i)),$$

where $p_j(x) = p_j(x; n, q)$ is the Krawtchouk polynomial defined by

$$p_j(x) = \sum_{l=0}^j (-1)^l (q-1)^{j-l} \binom{x}{l} \binom{n-x}{j-l} \quad (0 \leq j \leq n).$$

Likewise, we would have yet another way of obtaining “MacWilliams-type identity” if we could find the Q -matrix for the scheme $H(n_1, \dots, n_t; q)$, which is an example of weak metric schemes, i.e., a finite wreath product of metric schemes (cf. [1], [9]).

Indeed, we observe that, for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1$, $i_0 = 0$,

$$d_{\mathbb{F}_0}(x, y) = n_1 + \dots + n_{i-1} + i_0 \Leftrightarrow x_{i+1} = y_{i+1}, \dots, x_t = y_t, \quad d_H(x_i, y_i) = i_0,$$

where d_H is the Hamming metric. So identifying \mathbb{F}_q^n with $\mathbb{F}_q^{n_1} \times \dots \times \mathbb{F}_q^{n_t}$ by writing the elements $x \in \mathbb{F}_q^n$ as blocks of coordinates $x = (x_1, \dots, x_t) \in \mathbb{F}_q^{n_1} \times \dots \times \mathbb{F}_q^{n_t}$, we see that $H(n_1, \dots, n_t; q) = H(n_1, q) \wr \dots \wr H(n_t, q)$ is nothing but the wreath product of the Hamming schemes $H(n_1, q), \dots, H(n_t, q)$, which are metric schemes as is well-known. This motivates our study of weak metric and weak cometric schemes. In particular, this will contribute to better understanding of the important weak Hamming scheme $H(n_1, \dots, n_t; q)$ (cf. Theorem A above).

This paper is organized as follows. In Section 2, we will fix some notations that will be used throughout this paper. In Section 3, the notion of weak metric schemes is introduced as a finite wreath product of metric schemes. Several equivalent conditions for being weak metric schemes are presented in Theorem 2. These include the conditions on the shape of the “first intersection matrices for each level”, some modified polynomial relations on the adjacency matrices for each level and some modified polynomial relations on the p -numbers for each level. In Section 4, for schemes given as a finite wreath product of (not necessarily metric) symmetric association schemes, some of their parameters are determined. But, for Proposition 9(b), Lemma 10 and Theorem 11, we assume that the schemes are weak metric schemes. Note that, for weak metric schemes, Theorem 7(a) completely determines the first intersection matrices for each level, whereas Theorem 2 (b) gives only information about the shapes of those ones. Theorem 11 is an analogue of [2, Theorem 1.3, p. 197]. However, proving that requires considerably more work. In Section 5, the notion of weak cometric schemes, which is dual to weak metric schemes, is introduced and some equivalent conditions for being cometric schemes are presented. Finally, in Section 6, we will give an example illustrating modified polynomial relations in the case of a weak Hamming scheme.

2. Preliminaries and notations

Let n_1, \dots, n_t be positive integers with $n = n_1 + \dots + n_t$. For each $i = 1, 2, \dots, t$, let $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$ be a symmetric association scheme. Here we will always assume that the relations of $\mathfrak{X}^{(i)}$ are ordered as $R_0^{(i)}, R_1^{(i)}, \dots, R_{n_i}^{(i)}$. Then $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n)$ is the wreath product of $\mathfrak{X}^{(1)}, \mathfrak{X}^{(2)}, \dots, \mathfrak{X}^{(t)}$, so that, for $(x_1, \dots, x_t), (y_1, \dots, y_t) \in X$,

$$(x, y) \in R_{\underline{n_{i-1}+i_0}} \Leftrightarrow x_{i+1} = y_{i+1}, \dots, x_t = y_t \text{ and } (x_i, y_i) \in R_{i_0}^{(i)}$$

for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1, i_0 = 0$ (cf. [1], [9]). Here we will always assume that the relations of \mathfrak{X} are ordered as R_0, R_1, \dots, R_n .

For each $i = 1, \dots, t$, let $\Gamma_i = (X_i, R_1^{(i)})$ be the graph with the distance function ∂_i . Then, for $x = (x_1, \dots, x_t)$, $y = (y_1, \dots, y_t) \in X$, we define

$$(1) \quad d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \underline{n_{i-1}} + \partial_i(x_i, y_i), & \text{if } x_i \neq y_i \text{ and } x_{i+1} = y_{i+1}, \dots, x_t = y_t, \end{cases}$$

where $\underline{n_{i-1}} = n_1 + \dots + n_{i-1}$. Then d is a distance function. For this, we only need to check $d(x, y) \leq d(x, z) + d(z, y)$ for $x \neq y$. Assume $x_i \neq y_i$, $x_{i+1} = y_{i+1}, \dots, x_t = y_t$. Then $x_i \neq z_i$ or $z_i \neq y_i$, so that

$$\begin{aligned} d(x, z) + d(z, y) &\geq \underline{n_{i-1}} + \partial_i(x_i, z_i) + \partial_i(z_i, y_i) \\ &\geq \underline{n_{i-1}} + \partial_i(x_i, y_i) \\ &= d(x, y). \end{aligned}$$

Remark 1. $(x, y) \in R_{\underline{n_{i-1}}+1} \Leftrightarrow d(x, y) = \underline{n_{i-1}} + 1$ ($i = 1, \dots, t$).

For elementary facts about association schemes, one is referred to [2] and [3]. Throughout this paper, the following notations will be used.

- n_1, \dots, n_t positive integers with $n_1 + \dots + n_t = n$, $n_1 + \dots + n_i = \underline{n_i}$ ($1 \leq i \leq t$), $\underline{n_0} = 0$, $n_t + \dots + n_i = \overline{n_i}$ ($1 \leq i \leq t$), $\overline{n_{t+1}} = 0$.
- $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$ a symmetric association scheme with a fixed ordering $R_0^{(i)}, R_1^{(i)}, \dots, R_{n_i}^{(i)}$ of relations ($i = 1, \dots, t$).
- Some of the parameters of $\mathfrak{X}^{(i)}$ used are: valencies $v_0^{(i)} = 1, v_1^{(i)}, \dots, v_{n_i}^{(i)}$, multiplicities $m_0^{(i)} = 1, m_1^{(i)}, \dots, m_{n_i}^{(i)}$, intersection numbers $p_{jl}^{(i)k}$, Krein parameters $q_{jl}^{(i)k}$, the adjacency matrices $A_0^{(i)} = I, A_1^{(i)}, \dots, A_{n_i}^{(i)}$, the irreducible idempotents $E_0^{(i)} = |X_i|^{-1}J, E_1^{(i)}, \dots, E_{n_i}^{(i)}$, the first intersection matrix $L_1^{(i)} = (p_{1l}^{(i)k})_{0 \leq k, l \leq n_i}$, the first dual intersection matrix $M_1^{(i)} = (q_{1l}^{(i)k})_{0 \leq k, l \leq n_i}$.
- $\mathfrak{X} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n)$ the wreath product $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)}$ of $\mathfrak{X}^{(1)}, \dots, \mathfrak{X}^{(t)}$, with a fixed ordering R_0, R_1, \dots, R_n of relations, so that $(x, y) \in R_{\underline{n_{i-1}}+i_0} \Leftrightarrow x_{i+1} = y_{i+1}, \dots, x_t = y_t$ and $(x_i, y_i) \in R_{i_0}^{(i)}$ for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1, i_0 = 0$.
- Some of the parameters of \mathfrak{X} used are: p -numbers p_{jl} , q -numbers q_{jl} , intersection numbers p_{jl}^k , Krein parameters q_{jl}^k , valencies $v_0 = 1, v_1, \dots, v_n$, multiplicities $m_0 = 1, m_1, \dots, m_n$, the adjacency matrices A_0, A_1, \dots, A_n , the irreducible idempotents E_0, E_1, \dots, E_n , intersection matrices $L_j = (p_{jl}^k)_{0 \leq k, l \leq n}$, dual intersection matrices $M_j = (q_{jl}^k)_{0 \leq k, l \leq n}$.
- Further notations for the parameters of \mathfrak{X} are:
 $\theta_j^{(i)} = p_j, \underline{n_{i-1}}+1, v^{(i)} = v_{\underline{n_{i-1}}+1}$ ($v^{(i)} = \theta_0^{(i)}$),
 $\omega_j^{(i)} = q_j, \overline{n_{i+1}}+1, m^{(i)} = m_{\overline{n_{i+1}}+1}$ ($m^{(i)} = \omega_0^{(i)}$),

- $L^{(i)} = (p_{\underline{n_{i-1}+1}, l}^k)_{0 \leq k, l \leq n}$ the “first intersection matrix of level i ”,
- $M^{(i)} = (q_{\underline{n_{i+1}+1}, l}^k)_{0 \leq k, l \leq n}$ the “first dual intersection matrix of level i ”,
- $[L^{(i)}] = (p_{\underline{n_{i-1}+1}, l}^k)_{\underline{n_{i-1}} \leq k, l \leq \underline{n_i}}$ a submatrix of $L^{(i)}$,
- $[M^{(i)}] = (q_{\underline{n_{i+1}+1}, l}^k)_{\underline{n_{i+1}} \leq k, l \leq \underline{n_i}}$ a submatrix of $M^{(i)}$.
- $\Gamma_i = (X_i, R_1^{(i)})$ the graph with distance function ∂_i , d the function on X defined by (1).

3. Weak metric schemes

Let $\mathfrak{X} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)}$ be the symmetric association scheme which is given as the wreath product of the symmetric association schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$ (cf. Section 2). Then it will be called a *weak metric scheme* if \mathfrak{X} satisfies further the following equivalent conditions.

Theorem 2. *The following are equivalent.*

- (a) $\Gamma_i = (X_i, R_1^{(i)})$ is distance-regular for $i = 1, \dots, t$, and
- (2) $(x, y) \in R_j \Leftrightarrow d(x, y) = j$ for $j = 0, \dots, n$.
- (b) For $i = 1, \dots, t$, we have the following:
 - (i) The submatrix $[L^{(i)}] = (p_{\underline{n_{i-1}+1}, l}^k)_{\underline{n_{i-1}} \leq k, l \leq \underline{n_i}}$ of

$$L^{(i)} = (p_{\underline{n_{i-1}+1}, l}^k)_{0 \leq k, l \leq n}$$
 is a tridiagonal matrix with nonzero off-diagonal entries. Moreover, $L^{(i)}$ has the following entries:
 - (ii) $(p_{\underline{n_{i-1}+1}, l}^k)_{\underline{n_i}+1 \leq k, l \leq n} = v^{(i)}I$,
 - (iii) $p_{\underline{n_{i-1}+1}, \underline{n_{i-1}+1}}^k = v^{(i)}$ for $0 \leq k \leq \underline{n_{i-1}}$,
 - (iv) $p_{\underline{n_{i-1}+1}, l}^{\underline{n_{i-1}+1}} = v_l$ for $0 \leq l \leq \underline{n_{i-1}}$,
 - (v) All the other entries not appearing in (i)-(iv) are zeros.
 - (c) There are polynomials with real coefficients $\psi_0^{(i)}(x)$ of degree 0, $\psi_k^{(i)}(x)$ of degree k for $i = 1, \dots, t$, $1 \leq k \leq n_i$, with $\psi_k^{(i)}(0) = 0$ ($2 \leq i \leq t$), and real numbers $\alpha_0^{(1)} = \dots = \alpha_{n_1}^{(1)} = 0, \alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}, \dots, \alpha_1^{(t)}, \dots, \alpha_{n_t}^{(t)}$ such that

$$(3) \quad A_{\underline{n_{i-1}+i_0}} = \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_{i_0}^{(i)}(A_0 + \dots + A_{\underline{n_{i-1}}})$$
 for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1, i_0 = 0$.
 - (d) p -numbers satisfy

$$p_{j, \underline{n_{i-1}+i_0}} = \psi_{i_0}^{(i)}(\theta_j^{(i)}) + \alpha_{i_0}^{(i)}(p_{j0} + \dots + p_{j, \underline{n_{i-1}}}) \quad (0 \leq j \leq n),$$
 where $\psi_{i_0}^{(i)}$'s and $\alpha_{i_0}^{(i)}$'s are the same as in (c), and $\theta_j^{(i)} = p_{j, \underline{n_{i-1}+1}}$.
 - (e) $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$ is a metric scheme for $i = 1, \dots, t$.

Proof. We first note the following: for $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = 1$, $i_0 = 0$,

$$(4) \quad A_{\underline{n}_{i-1}+i_0} A_l = v_l A_{\underline{n}_{i-1}+i_0} \text{ for } 0 \leq l \leq \underline{n}_{i-1}.$$

For this, we need to see: for $1 \leq j < i$, $1 \leq k \leq n_j$, or $j = 1, k = 0$,

$$A_{\underline{n}_{i-1}+i_0} A_{\underline{n}_{j-1}+k} = v_{\underline{n}_{j-1}+k} A_{\underline{n}_{i-1}+i_0}.$$

Let $(x, y) \in R_m$. If $(x, z) \in R_{\underline{n}_{i-1}+i_0}$, $(z, y) \in R_{\underline{n}_{j-1}+k}$ for some $z \in X$, then $(x, y) \in R_{\underline{n}_{i-1}+i_0}$, and hence $m = \underline{n}_{i-1} + i_0$. On the other hand, if $(x, y) \in R_{\underline{n}_{i-1}+i_0}$, then

$$\begin{aligned} & \{z \in X \mid (x, z) \in R_{\underline{n}_{i-1}+i_0}, (z, y) \in R_{\underline{n}_{j-1}+k}\} \\ &= \{z \in X \mid (z, y) \in R_{\underline{n}_{j-1}+k}\}. \end{aligned}$$

This shows (4).

(a) \Rightarrow (b) (i) Let $k = \underline{n}_{i-1} + k_0$, $l = \underline{n}_{i-1} + l_0$, with $0 \leq k_0, l_0 \leq n_i$. Suppose $p_{\underline{n}_{i-1}+1}^k, l \neq 0$. Then $d(x, y) = k$, $d(x, z) = \underline{n}_{i-1} + 1$, $d(z, y) = l$ for some x, y, z . Then $\partial_i(x_i, y_i) = k_0$, $\partial_i(x_i, z_i) = 1$, $\partial_i(z_i, y_i) = l_0$. So $|k_0 - l_0| \leq 1$, and hence $|k - l| \leq 1$. Thus $p_{\underline{n}_{i-1}+1}^k, l = 0$, if $|k - l| \geq 2$, with $\underline{n}_{i-1} \leq k, l \leq n_i$. Also, $p_{\underline{n}_{i-1}+1}^k, l \neq 0$, if $|k - l| = 1$. For any $x_i, y_i \in X_i$, with $\partial_i(x_i, y_i) = k_0 (\geq 1)$, there is a path $x_i = z_0^{(i)}, z_1^{(i)}, \dots, z_{k_0}^{(i)} = y_i$. Then $\partial_i(x_i, z_1^{(i)}) = 1$, $\partial_i(z_1^{(i)}, y_i) = k_0 - 1$. Choose any points $x_j \in X_j$ for all $j \neq i$, and set

$$\begin{aligned} x &= (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_t), \\ y &= (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_t), \\ z &= (x_1, \dots, x_{i-1}, z_1^{(i)}, x_{i+1}, \dots, x_t). \end{aligned}$$

Then $d(x, y) = \underline{n}_{i-1} + k_0$, $d(x, z) = \underline{n}_{i-1} + 1$, $d(z, y) = \underline{n}_{i-1} + k_0 - 1$. So $p_{\underline{n}_{i-1}+1}^k, k-1 \neq 0$ for $\underline{n}_{i-1} + 1 \leq k \leq n_i$. Similarly, one shows that $p_{\underline{n}_{i-1}+1}^k, k+1 \neq 0$ for $\underline{n}_{i-1} \leq k \leq \underline{n}_i - 1$.

(ii) Let $\underline{n}_i + 1 \leq l \leq n$. For any x, y with $d(x, y) = l$,

$$\begin{aligned} p_{\underline{n}_{i-1}+1}^l, l &= |\{z \mid d(x, z) = \underline{n}_{i-1} + 1, d(z, y) = l\}| \\ &= |\{z \mid d(x, z) = \underline{n}_{i-1} + 1\}| \\ &= v^{(i)}. \end{aligned}$$

Indeed, if $l = \underline{n}_j + l_0$ ($1 \leq l_0 \leq n_{j+1}$) for some j with $i \leq j \leq t - 1$, then, for z with $d(x, z) = \underline{n}_{i-1} + 1$, $x_{j+1} = z_{j+1}$, $\partial_{j+1}(x_{j+1}, y_{j+1}) = l_0$, $z_{j+2} = y_{j+2}, \dots, z_t = y_t$, and hence $d(z, y) = l$ is automatic. In the same manner, one shows $p_{\underline{n}_{i-1}+1}^k, l = 0$ for all k, l with $\underline{n}_i + 1 \leq k, l \leq n$, $k \neq l$.

(iii), (iv) These can be shown similarly to the proof of (ii).

(v) As $p_{\underline{n}_{i-1}+1}^k, l = 0 \Leftrightarrow p_{\underline{n}_{i-1}+1}^l, k = 0$, we only need to see:

$$(5) \quad p_{\underline{n}_{i-1}+1}^k, l = 0 \text{ for } 0 \leq k \leq n \text{ (} k \neq \underline{n}_{i-1} + 1 \text{), } 0 \leq l \leq \underline{n}_{i-1},$$

and

$$p_{\underline{n_{i-1}+1}, l}^k = 0 \text{ for } \underline{n_{i-1}+1} \leq k \leq \underline{n_i}, \underline{n_i+1} \leq l \leq n.$$

Both of these are easy to see.

(b) \Rightarrow (a) First, we show (2) by induction on j . It is clearly true for $j = 0$. Assume that $k \geq 1$ and the assertion holds for all $j \leq k - 1$. Let $k = \underline{n_{i-1}} + k_0$ for some i, k_0 with $1 \leq i \leq t, 1 \leq k_0 \leq n_i$. Let $(x, y) \in R_k$. As $p_{\underline{n_{i-1}+1}, k-1}^k \neq 0, (x, z) \in R_{\underline{n_{i-1}+1}}, (z, y) \in R_{k-1}$ for some z . By induction hypothesis $d(z, y) = k - 1$, and $d(x, z) = \underline{n_{i-1}} + 1$, by Remark 1. Thus

$$\partial_i(z_i, y_i) = k_0 - 1, z_{i+1} = y_{i+1}, \dots, z_t = y_t;$$

$$\partial_i(x_i, z_i) = 1, x_{i+1} = z_{i+1}, \dots, x_t = z_t.$$

So $\partial_i(x_i, y_i) \leq k_0, x_{i+1} = y_{i+1}, \dots, x_t = y_t$, and hence $d(x, y) \leq k$. Also, we have $d(x, y) \geq k$, since otherwise we would have $(x, y) \in R_j$ for some $j \leq k - 1$, by induction hypothesis. Conversely, assume that $d(x, y) = k$. Here, in view of Remark 1, we may let $d(x, y) = k = \underline{n_{i-1}} + k_0$ for some i and k_0 with $2 \leq k_0 \leq n_i$. Then we must have $(x, y) \in R_j$ for some $j \geq k$. We now exclude the possibility that $(x, y) \in R_j$ for some $j \geq k + 1$. Assume, on the contrary, that $(x, y) \in R_j$ for some $j \geq k + 1$. Then there is a path $x_i = z_0^{(i)}, z_1^{(i)}, \dots, z_{k_0-1}^{(i)}, z_{k_0}^{(i)} = y_i$, and $x_{i+1} = y_{i+1}, \dots, x_t = y_t$. Put

$$z = (x_1, \dots, x_{i-1}, z_{k_0-1}^{(i)}, x_{i+1}, \dots, x_t).$$

Then $d(y, z) = \underline{n_{i-1}} + 1 \Rightarrow (y, z) \in R_{\underline{n_{i-1}}+1}$, by Remark 1, and $d(z, x) = k - 1 \Rightarrow (z, x) \in R_{k-1}$. So $p_{\underline{n_{i-1}+1}, k-1}^j \neq 0$. As $j \geq k + 1 \geq \underline{n_{i-1}} + 3, p_{\underline{n_{i-1}+1}, k-1}^j = 0$, by our assumption on the matrix $L^{(i)}$. This is a contradiction.

Next, we show that each Γ_i is distance-regular. Let $x_i, y_i, x'_i, y'_i \in X_i$, with $\partial_i(x_i, y_i) = \partial_i(x'_i, y'_i) = k_0$ ($0 \leq k_0 \leq n_i$). Then we must show:

$$(6) \quad \begin{aligned} & |\{u_i \in X_i \mid \partial_i(x_i, u_i) = l_0, \partial_i(u_i, y_i) = m_0\}| \\ &= |\{u_i \in X_i \mid \partial_i(x'_i, u_i) = l_0, \partial_i(u_i, y'_i) = m_0\}| \end{aligned}$$

for all l_0, m_0 with $0 \leq l_0, m_0 \leq n_i$. This is clearly true for $l_0 = 0$ or $m_0 = 0$. Thus we assume $l_0, m_0 \geq 1$. Choose $z_j \in X_j$ for all $j \neq i$, and set

$$\begin{aligned} \tilde{x} &= (z_1, \dots, z_{i-1}, x_i, z_{i+1}, \dots, z_t), \tilde{y} = (z_1, \dots, z_{i-1}, y_i, z_{i+1}, \dots, z_t), \\ \tilde{x}' &= (z_1, \dots, z_{i-1}, x'_i, z_{i+1}, \dots, z_t), \tilde{y}' = (z_1, \dots, z_{i-1}, y'_i, z_{i+1}, \dots, z_t). \end{aligned}$$

Then $d(\tilde{x}, \tilde{y}) = d(\tilde{x}', \tilde{y}')$. By (2), we have:

$$\begin{aligned} & |\{u \in X \mid d(\tilde{x}, u) = \underline{n_{i-1}} + l_0, d(u, \tilde{y}) = \underline{n_{i-1}} + m_0\}| \\ &= |\{u \in X \mid d(\tilde{x}', u) = \underline{n_{i-1}} + l_0, d(u, \tilde{y}') = \underline{n_{i-1}} + m_0\}|, \end{aligned}$$

which implies

$$|X_1| \cdots |X_{i-1}| \cdot |\{u_i \in X_i \mid \partial_i(x_i, u_i) = l_0, \partial_i(u_i, y_i) = m_0\}|$$

$$= |X_1| \cdots |X_{i-1}| \cdot |\{u_i \in X_i \mid \partial_i(x'_i, u_i) = l_0, \partial_i(u_i, y'_i) = m_0\}|.$$

So we get (6).

(b) \Rightarrow (c) Here we denote the entries of the submatrix $[L^{(i)}]$ of $L^{(i)}$ just as in (12) for $i = 1, \dots, t$. Define the polynomials $\psi_i^{(1)}(x)$ of degree i ($0 \leq i \leq n_1$) recursively by:

$$\begin{aligned} \psi_0^{(1)}(x) &= 1, \quad \psi_1^{(1)}(x) = x, \\ x\psi_i^{(1)}(x) &= b_{i-1}^{(1)}\psi_{i-1}^{(1)}(x) + a_i^{(1)}\psi_i^{(1)}(x) + c_{i+1}^{(1)}\psi_{i+1}^{(1)}(x) \quad (i \geq 1). \end{aligned}$$

Also, define

$$\alpha_0^{(1)} = \alpha_1^{(1)} = \cdots = \alpha_{n_1}^{(1)} = 0.$$

Then

$$A_{i_0} = \psi_{i_0}^{(1)}(A_1) + \alpha_{i_0}^{(1)}A_0 \text{ for } 0 \leq i_0 \leq n_1.$$

Let i ($2 \leq i \leq t$) be fixed. Define

$$\psi_1^{(i)}(x) = x, \quad \alpha_1^{(i)} = 0.$$

Then

$$A_{\underline{n_{i-1}+1}} = \psi_1^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_1^{(i)}(A_0 + \cdots + A_{\underline{n_{i-1}}}).$$

Also, if we define

$$\psi_2^{(i)}(x) = c_2^{(i)-1}(x^2 - a_1^{(i)}x), \quad \alpha_2^{(i)} = -c_2^{(i)-1}v^{(i)},$$

then

$$A_{\underline{n_{i-1}+2}} = \psi_2^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_2^{(i)}(A_0 + \cdots + A_{\underline{n_{i-1}}}).$$

Assume now that $\psi_1^{(i)}, \dots, \psi_{i_0}^{(i)}, \alpha_1^{(i)}, \dots, \alpha_{i_0}^{(i)}$ ($2 \leq i_0 < n_i$) are defined so that

$$A_{\underline{n_{i-1}+j_0}} = \psi_{j_0}^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_{j_0}^{(i)}(A_0 + \cdots + A_{\underline{n_{i-1}}}), \quad \psi_{j_0}^{(i)}(0) = 0$$

holds for all $1 \leq j_0 \leq i_0$. Define the polynomial $\psi_{i_0+1}^{(i)}(x)$ of degree $i_0 + 1$ by:

$$\begin{aligned} \psi_{i_0+1}^{(i)}(x) &= c_{i_0+1}^{(i)-1} \{x\psi_{i_0}^{(i)}(x) + \alpha_{i_0}^{(i)}(v_0 + \cdots + v_{\underline{n_{i-1}}})x \\ &\quad - b_{i_0-1}^{(i)}\psi_{i_0-1}^{(i)}(x) - \alpha_{i_0}^{(i)}\psi_{i_0}^{(i)}(x)\}, \end{aligned}$$

and a constant $\alpha_{i_0+1}^{(i)}$ by:

$$\alpha_{i_0+1}^{(i)} = -c_{i_0+1}^{(i)-1} (b_{i_0-1}^{(i)}\alpha_{i_0-1}^{(i)} + a_{i_0}^{(i)}\alpha_{i_0}^{(i)}).$$

Then

$$A_{\underline{n_{i-1}+i_0+1}} = \psi_{i_0+1}^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_{i_0+1}^{(i)}(A_0 + \cdots + A_{\underline{n_{i-1}}}).$$

Observe here that $\psi_{i_0+1}^{(i)}(0) = 0$.

(c) \Rightarrow (b) Here we must show the following: for $i = 1, \dots, t$,

$$(7) \quad A_{\underline{n_{i-1}+1}} A_l = v_l A_{\underline{n_{i-1}+1}} \quad (0 \leq l \leq \underline{n_{i-1}}),$$

$$(8) \quad A_{\underline{n_{i-1}+1}} A_{\underline{n_{i-1}+1}} = v^{(i)}(A_0 + \dots + A_{\underline{n_{i-1}}}) + a_1^{(i)} A_{\underline{n_{i-1}+1}} + c_2^{(i)} A_{\underline{n_{i-1}+2}}$$

with $c_2^{(i)} \neq 0$,

$$(9) \quad A_{\underline{n_{i-1}+1}} A_{\underline{n_{i-1}+i_0}} = b_{i_0-1}^{(i)} A_{\underline{n_{i-1}+i_0-1}} + a_{i_0}^{(i)} A_{\underline{n_{i-1}+i_0}} \\ + c_{i_0+1}^{(i)} A_{\underline{n_{i-1}+i_0+1}} \quad (2 \leq i_0 \leq n_i)$$

with $b_{i_0-1}^{(i)} \neq 0$ ($2 \leq i_0 \leq n_i$), $c_{i_0+1}^{(i)} \neq 0$ ($2 \leq i_0 \leq n_i - 1$), $c_{n_i+1}^{(i)} = 0$,

$$(10) \quad A_{\underline{n_{i-1}+1}} A_l = v^{(i)} A_l \quad (\underline{n_i} + 1 \leq l \leq n).$$

By (4), (7), and (10) are true. Next, we show (8) and (9). For $i = 1$, we only need to show (9) for $1 \leq l \leq n_1$. By our assumption $A_{i_0} = \psi_{i_0}^{(1)}(A_1)$ for $0 \leq i_0 \leq n_1$. Since $x\psi_l^{(1)}(x)$ is a linear combination of $\psi_{l+1}^{(1)}(x), \psi_l^{(1)}(x), \dots, \psi_0^{(1)}(x)$, $A_1 A_l$ is a linear combination of A_{l+1}, A_l, \dots, A_0 . Clearly, the coefficient of $A_{l+1} \neq 0$. As $A_1 A_l = \sum_{k=0}^n p_{1l}^k A_k$, $p_{1l}^k = 0$ for $k \geq l + 2$ and $p_{1l}^{l+1} \neq 0$. Since $p_{1l}^k = 0 \Leftrightarrow p_{1k}^l = 0$, $p_{1l}^k = 0$ if $|k - l| \geq 2$, and $p_{1l}^k \neq 0$ if $|k - l| = 1$. Let $i \geq 2$. First, we show (9). By (3) and (4), for $1 \leq i_0 \leq n_i$ we have:

$$(11) \quad A_{\underline{n_{i-1}+i_0}} A_{\underline{n_{i-1}+1}} = \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) A_{\underline{n_{i-1}+1}} + \alpha_{i_0}^{(i)}(v_0 + \dots + v_{\underline{n_{i-1}}}) A_{\underline{n_{i-1}+1}}.$$

As $\psi_k^{(i)}(0) = 0$ ($1 \leq k \leq n_i$),

$$\psi_{i_0}^{(i)}(x)x + \alpha_{i_0}^{(i)}(v_0 + \dots + v_{\underline{n_{i-1}}})x = \sum_{k_0=1}^{i_0+1} \beta_{k_0} \psi_{k_0}^{(i)}(x)$$

for some β_{k_0} with $\beta_{i_0+1} \neq 0$. Thus, for $2 \leq i_0 \leq n_i$,

$$A_{\underline{n_{i-1}+i_0}} A_{\underline{n_{i-1}+1}} = \sum_{k_0=1}^{i_0+1} \beta_{k_0} A_{\underline{n_{i-1}+k_0}} - \sum_{k_0=1}^{i_0+1} \beta_{k_0} \alpha_{k_0}^{(i)}(A_0 + \dots + A_{\underline{n_{i-1}}}),$$

by (3) and (11). Just as in the above argument, this implies that $p_{\underline{n_{i-1}+1}, \underline{n_{i-1}+l_0}}^{\underline{n_{i-1}+k_0}} = 0$ for $|k_0 - l_0| \geq 2$, and $p_{\underline{n_{i-1}+1}, \underline{n_{i-1}+l_0}}^{\underline{n_{i-1}+k_0}} \neq 0$ for $|k_0 - l_0| = 1$. Next, we show (8). By (3) with $i_0 = 2$,

$$A_{\underline{n_{i-1}+2}} = a A_{\underline{n_{i-1}+1}}^2 + b A_{\underline{n_{i-1}+1}} + \alpha_2^{(i)}(A_0 + \dots + A_{\underline{n_{i-1}}})$$

for some a, b with $a \neq 0$. So

$$A_{\underline{n_{i-1}+1}} A_{\underline{n_{i-1}+1}} = w(A_0 + \dots + A_{\underline{n_{i-1}}}) + a_1^{(i)} A_{\underline{n_{i-1}+1}} + c_2^{(i)} A_{\underline{n_{i-1}+2}}$$

with $c_2^{(i)} \neq 0$, where we put

$$w = -a^{-1}\alpha_2^{(i)}, \quad a_1^{(i)} = -a^{-1}b, \quad c_2^{(i)} = a^{-1}.$$

It remains to see that $w = v^{(i)}$. For $0 \leq k \leq \underline{n_{i-1}}$, let $(x, y) \in R_k$. Then $x_i = y_i, \dots, x_t = y_t$. Thus $p_{\underline{n_{i-1}+1}, \underline{n_{i-1}+1}}^k = v^{(i)}$, and hence $w = v^{(i)}$.

(c) \Leftrightarrow (d) This is straightforward.

(a) \Leftrightarrow (e) Note that (2) is equivalent to: for $i = 1, \dots, t$,

$$(x_i, y_i) \in R_j^{(i)} \Leftrightarrow \partial_i(x_i, y_i) = j \quad (0 \leq j \leq n_i) \quad (\text{cf. [2, Prop. 1.1, p. 189]}). \quad \square$$

Remark 3. (1) When \mathfrak{X} is a weak metric scheme, for $i = 1, \dots, t$ the submatrix $[L^{(i)}]$ of $L^{(i)}$ will be denoted by:

$$(12) \quad [L^{(i)}] = \begin{bmatrix} a_0^{(i)} & b_0^{(i)} & & & & \\ c_1^{(i)} & a_1^{(i)} & b_1^{(i)} & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_{n_{i-1}}^{(i)} & a_{n_{i-1}}^{(i)} & b_{n_{i-1}}^{(i)} & \\ & & & c_{n_i}^{(i)} & a_{n_i}^{(i)} & \\ & & & & & \end{bmatrix}.$$

Observe here that $a_0^{(i)} = 0, b_0^{(i)} = v^{(i)}, c_1^{(i)} = v_{\underline{n_{i-1}}}$.

(2) Put

$$\psi^{(1)}(x) = \psi_0^{(1)}(x) + \dots + \psi_{n_1}^{(1)}(x), \quad \psi^{(i)}(x) = \psi_1^{(i)}(x) + \dots + \psi_{n_i}^{(i)}(x) \quad (2 \leq i \leq t).$$

Then $\psi^{(i)}(x)$ is a polynomial of degree n_i for $1 \leq i \leq t$. Using (3), one can show by induction on i that, for $i = 2, \dots, t$,

$$(13) \quad A_0 + \dots + A_{\underline{n_{i-1}}} = \sum_{k=1}^{i-1} c_k^{(i)} \psi^{(k)}(A_{\underline{n_{k-1}+1}}),$$

where $c_k^{(i)} = \prod_{l=k+1}^{i-1} (1 + \alpha_1^{(l)} + \dots + \alpha_{n_l}^{(l)})$ with the understanding that $c_{i-1}^{(i)} = 1$.

(3) For $i = 2, \dots, t, 1 \leq i_0 \leq n_i$,

$$A_{\underline{n_{i-1}+i_0}} = \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_{i_0}^{(i)} \xi_{\underline{n_{i-1}+1}}^{(i)}(A_{\underline{n_{i-2}+1}}),$$

where $\xi_{\underline{n_{i-1}+1}}^{(i)}(x)$ is a polynomial with real coefficients of degree $n_{i-1} + 1$. Indeed, from (3) and (13), we have

$$(14) \quad A_{\underline{n_{i-1}+i_0}} = \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_{i_0}^{(i)} \sum_{k=1}^{i-1} c_k^{(i)} \psi^{(k)}(A_{\underline{n_{k-1}+1}}).$$

Multiplying both sides of (14) by $A_{\underline{n_{i-2}+1}}$ and using (4), we get:

$$\begin{aligned} & v^{(i-1)} A_{\underline{n_{i-1}+i_0}} \\ &= v^{(i-1)} \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) \end{aligned}$$

$$\begin{aligned}
 & + \alpha_{i_0}^{(i)} \sum_{k=1}^{i-2} c_k^{(i)} \psi^{(k)}(A_{\underline{n_{k-1}+1}}) A_{\underline{n_{i-2}+1}} + \alpha_{i_0}^{(i)} \psi^{(i-1)}(A_{\underline{n_{i-2}+1}}) A_{\underline{n_{i-2}+1}} \\
 = & v^{(i-1)} \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) \\
 & + \alpha_{i_0}^{(i)} \left\{ \sum_{k=1}^{i-2} c_k^{(i)} \psi^{(k)}(v^{(k)}) A_{\underline{n_{i-2}+1}} + \psi^{(i-1)}(A_{\underline{n_{i-2}+1}}) A_{\underline{n_{i-2}+1}} \right\}.
 \end{aligned}$$

This implies

$$A_{\underline{n_{i-1}+i_0}} = \psi_{i_0}^{(i)}(A_{\underline{n_{i-1}+1}}) + \alpha_{i_0}^{(i)} \xi_{n_{i-1}+1}^{(i)}(A_{\underline{n_{i-2}+1}}),$$

where we put

$$\xi_{n_{i-1}+1}^{(i)}(x) = v^{(i-1)^{-1}} \left\{ \sum_{k=1}^{i-2} c_k^{(i)} \psi^{(k)}(v^{(k)})x + \psi^{(i-1)}(x)x \right\}.$$

Proposition 4. Let $\mathfrak{X} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)}$ be a weak metric scheme with $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$. Let the entries of the submatrix $[L^{(i)}]$ of $L^{(i)}$ be as in (12). Then

(a)

$$(15) \quad a_l^{(1)} + b_l^{(1)} + c_l^{(1)} = v^{(1)} \text{ for } l \geq 2,$$

$$(16) \quad a_l^{(i)} + b_l^{(i)} + c_l^{(i)} = \begin{cases} v^{(i)}, & 0 \leq l \leq n_i \text{ and } l \neq 1, \\ v^{(i)} - (v_0 + v_1 + \dots + v_{\underline{n_{i-1}-1}}), & l = 1. \end{cases}$$

Here $c_0^{(i)} = 0, b_{n_i}^{(i)} = 0$ for $i = 1, \dots, t$.

- (b) $v_{\underline{n_{i-1}+i_0}} = v^{(i)} b_1^{(i)} b_2^{(i)} \dots b_{i_0-1}^{(i)} / c_2^{(i)} c_3^{(i)} \dots c_{i_0}^{(i)}$ for $i = 1, \dots, t, 2 \leq i_0 \leq n_i$.
- (c) $v^{(i)} = b_0^{(i)} \geq b_1^{(i)} \geq \dots \geq b_{n_i-1}^{(i)}$ for $i = 1, \dots, t$.
- (d) $v_{\underline{n_{i-1}}} = c_1^{(i)} \leq c_2^{(i)} \leq \dots \leq c_{n_i}^{(i)}$ for $i = 1, \dots, t$.

Proof. (a) The result follows from $\sum_{j=0}^n p_{\underline{n_{i-1}+1,j}}^k = v^{(i)}$, Theorem 2(b)(i), (iv), (v), and (12).

(b) Since $p_{ij}^k v_k = p_{ik}^j v_j, v_{\underline{n_{i-1}+i_0}} b_{i_0}^{(i)} = v_{\underline{n_{i-1}+i_0+1}} c_{i_0+1}^{(i)}$. So

$$v^{(i)} = \frac{c_2^{(i)} v_{\underline{n_{i-1}+2}}}{b_1^{(i)}} = \frac{c_2^{(i)} c_3^{(i)} v_{\underline{n_{i-1}+3}}}{b_1^{(i)} b_2^{(i)}} = \dots = \frac{c_2^{(i)} \dots c_{i_0}^{(i)} v_{\underline{n_{i-1}+i_0}}}{b_1^{(i)} b_2^{(i)} \dots b_{i_0-1}^{(i)}}.$$

(c) Let $x, y \in X$ with $d(x, y) = \underline{n_{i-1}} + i_0$ ($1 \leq i_0 \leq n_i - 1$). Let w be an element of X such that $d(x, w) = \underline{n_{i-1}} + 1, d(w, y) = \underline{n_{i-1}} + i_0 + 1$. As $\partial_i(x_i, y_i) = i_0$, we can choose $z_i \in X_i$, with $\partial_i(x_i, z_i) = i_0 - 1, \partial_i(z_i, y_i) = 1$. Then $\partial_i(z_i, w_i) \leq i_0$. Also, $\partial_i(z_i, w_i) \geq i_0$, since otherwise we would have $\partial_i(w_i, y_i) \leq i_0$. We put $z = (x_1, \dots, x_{i-2}, z_{i-1}, z_i, x_{i+1}, \dots, x_t)$, where $z_{i-1} \in X_{i-1}$ is an element with $\partial_{i-1}(x_{i-1}, z_{i-1}) = n_{i-1}$. Note here that such an z_{i-1} can be chosen, since, if necessary, after replacing $(i - 1)$ th component

x_{i-1} of x we may assume $\{a_{i-1} \in X_{i-1} \mid \partial_{i-1}(x_{i-1}, a_{i-1}) = n_{i-1}\} \neq \emptyset$. Then $d(x, z) = \underline{n_{i-1}} + i_0 - 1$. Observe that this holds even for $i_0 = 1$. Also, $d(w, z) = \underline{n_{i-1}} + i_0$. This shows that $b_{i_0}^{(i)} \leq b_{i_0-1}^{(i)}$ for $1 \leq i_0 \leq n_i - 1$.

(d) Let $x, z \in X$, with $d(x, z) = \underline{n_{i-1}} + i_0 + 1$ ($1 \leq i_0 \leq n_i - 1$). Choose $y_i \in X_i$ with $\partial_i(x_i, y_i) = i_0$, $\partial_i(y_i, z_i) = 1$. Put $y = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_t)$. Then $d(x, y) = \underline{n_{i-1}} + i_0$. Let w be an element of X such that $d(x, w) = \underline{n_{i-1}} + 1$, $d(w, y) = \underline{n_{i-1}} + i_0 - 1$. Just as in the proof of (c), we have $\partial_i(w_i, z_i) = i_0$, and hence $d(w, z) = \underline{n_{i-1}} + i_0$. This shows $c_{i_0}^{(i)} \leq c_{i_0+1}^{(i)}$ for $1 \leq i_0 \leq n_i - 1$. \square

4. Parameters of weak metric schemes

Let $\mathfrak{X} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)}$ be a symmetric association scheme which is given as the wreath product of the symmetric association schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$. Then we will determine some of the parameters of \mathfrak{X} , such as valencies, multiplicities, the adjacency matrices, the irreducible idempotents, P -matrix and Q -matrix. Further, we will determine the first intersection matrix for each level and the first dual intersection matrix for each level. Also, we will prove Theorem 11 which is an analogue of Theorem 1.3 in [2, p. 197].

In below, all the matrices in the i -th factor of tensor products are of size $|X_i|$ for $i = 1, \dots, t$, and accordingly I and J will denote respectively the identity and all-one matrices of various sizes.

Theorem 5. *Let $\mathfrak{X} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)}$ be a symmetric association scheme which is given as the wreath product of the symmetric association schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$. Then we have the following.*

- (a) $|X_1| \cdots |X_{i-1}| v_{i_0}^{(i)} = v_{\underline{n_{i-1}}+i_0}$ for $1 \leq i \leq t$, $1 \leq i_0 \leq n_i$, or $i = 1$, $i_0 = 0$.
- (b) $\lambda_i := \sum_{j=0}^{n_i} v_j = |X_1| \cdots |X_i|$ for $i = 1, \dots, t$.
- (c) $|X_t| \cdots |X_{i+1}| m_{i_0}^{(i)} = m_{\overline{n_{i+1}}+i_0}$ for $1 \leq i \leq t$, $1 \leq i_0 \leq n_i$, or $i = t$, $i_0 = 0$.
- (d) $\mu_i := \sum_{j=0}^{\overline{n_i}} m_j = |X_t| \cdots |X_i|$ for $i = 1, \dots, t$.
- (e)

$$\begin{aligned}
 A_0 &= A_0^{(1)} \otimes I \otimes \cdots \otimes I = I, \quad A_1 = A_1^{(1)} \otimes I \otimes \cdots \otimes I, \dots, \\
 A_{n_1} &= A_{n_1}^{(1)} \otimes I \otimes \cdots \otimes I, \quad A_{n_1+1} = J \otimes A_1^{(2)} \otimes I \otimes \cdots \otimes I, \dots, \\
 (17) \quad A_{n_1+n_2} &= J \otimes A_{n_2}^{(2)} \otimes I \otimes \cdots \otimes I, \dots, \\
 A_{\underline{n_{i-1}}+i_0} &= J \otimes \cdots \otimes J \otimes A_{i_0}^{(i)} \otimes I \otimes \cdots \otimes I \quad (1 \leq i_0 \leq n_i), \dots, \\
 A_{n_{t-1}+1} &= J \otimes \cdots \otimes J \otimes A_1^{(t)}, \dots, \quad A_n = J \otimes \cdots \otimes J \otimes A_{n_t}^{(t)},
 \end{aligned}$$

where in (17) there are $(i - 1)$ factors of J .

(f)

$$\begin{aligned}
 E_0 &= E_0^{(1)} \otimes \cdots \otimes E_0^{(t-1)} \otimes E_0^{(t)} = |X|^{-1}J, \\
 E_1 &= E_0^{(1)} \otimes \cdots \otimes E_0^{(t-1)} \otimes E_1^{(t)}, \\
 &\vdots \\
 E_{n_t} &= E_0^{(1)} \otimes \cdots \otimes E_0^{(t-1)} \otimes E_{n_t}^{(t)}, \\
 E_{n_t+1} &= E_0^{(1)} \otimes \cdots \otimes E_0^{(t-2)} \otimes E_1^{(t-1)} \otimes I, \\
 &\vdots \\
 E_{n_t+n_{t-1}} &= E_0^{(1)} \otimes \cdots \otimes E_0^{(t-2)} \otimes E_{n_{t-1}}^{(t-1)} \otimes I, \\
 &\vdots \\
 E_{\overline{n_{i+1}+i_0}} &= E_0^{(1)} \otimes \cdots \otimes E_0^{(i-1)} \otimes E_{i_0}^{(i)} \otimes I \otimes \cdots \otimes I \quad (1 \leq i_0 \leq n_i) \\
 &\vdots \\
 E_{\overline{n_2+1}} &= E_1^{(1)} \otimes I \otimes \cdots \otimes I, \dots, \quad E_n = E_{n_1}^{(1)} \otimes I \otimes \cdots \otimes I.
 \end{aligned}$$

For $i = 1, 2, \dots, t$, let $P^{(i)}$ and $Q^{(i)}$ be respectively the P -matrix and the Q -matrix of $\mathfrak{X}^{(i)}$. For these, we put

$$P^{(i)} = \left[\begin{array}{c|ccc} 1 & v_1^{(i)} & \cdots & v_{n_i}^{(i)} \\ \hline 1 & & & \\ \vdots & & & \\ 1 & & & \end{array} \right], \quad Q^{(i)} = \left[\begin{array}{c|ccc} 1 & m_1^{(i)} & \cdots & m_{n_i}^{(i)} \\ \hline 1 & & & \\ \vdots & & & \\ 1 & & & \end{array} \right].$$

(g) $P =$

1	$v_1 \cdots v_{n_1}$	$v_{n_1+1} \cdots v_{n_2}$	$v_{n_{t-2}+1} \cdots v_{n_{t-1}}$	$v_{n_{t-1}+1} \cdots v_n$
1	$v_1 \ v_{n_1}$	$v_{n_1+1} \ v_{n_2}$	$v_{n_{t-2}+1} \ v_{n_{t-1}}$	$\lambda_{t-1} \tilde{P}^{(t)}$
:	:	:	:	:	:
1	$v_1 \ v_{n_1}$	$v_{n_1+1} \ v_{n_2}$	$v_{n_{t-2}+1} \ v_{n_{t-1}}$	$\lambda_{t-1} \tilde{P}^{(t)}$
1	$v_1 \ v_{n_1}$	$v_{n_1+1} \ v_{n_2}$	$\lambda_{t-2} \tilde{P}^{(t-1)}$	0
:	:	:	:	:	:
1	$v_1 \ v_{n_1}$	$v_{n_1+1} \ v_{n_2}$	0	0
:	:	:	:	:	:
1	$v_1 \ v_{n_1}$	$\lambda_1 \tilde{P}^{(2)}$	0	0
:	:	:	:	:	:
1	$\lambda_0 \tilde{P}^{(1)}$	0	0	0
:	:	:	:	:	:
1	:	:	:	:

$$(h) Q = \begin{bmatrix} 1 & m_1 \cdots m_{n_t} & m_{n_t+1} \cdots m_{\overline{n_{t-1}}} & \cdots & m_{\overline{n_3+1}} \cdots m_{\overline{n_2}} & m_{\overline{n_2+1}} \cdots m_{\overline{n_1}} \\ 1 & m_1 m_{n_t} & m_{n_t+1} m_{\overline{n_{t-1}}} & \cdots & m_{\overline{n_3+1}} m_{\overline{n_2}} & \mu_2 \tilde{Q}^{(1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & m_1 m_{n_t} & m_{n_t+1} m_{\overline{n_{t-1}}} & \cdots & m_{\overline{n_3+1}} m_{\overline{n_2}} & \\ \vdots & \vdots & \vdots & \cdots & \mu_3 \tilde{Q}^{(2)} & 0 \\ 1 & m_1 m_{n_t} & m_{n_t+1} m_{\overline{n_{t-1}}} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & m_1 m_{n_t} & \mu_t \tilde{Q}^{(t-1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & m_1 m_{n_t} & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \mu_{t+1} \tilde{Q}^{(t)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \vdots & \vdots & \cdots & \vdots & \vdots \end{bmatrix}.$$

Proof. (a) Let $(x_1, x_2, \dots, x_t) \in X$. Then

$$\begin{aligned} v_{\overline{n_{i-1}+i_0}} &= |\{(y_1, \dots, y_i, x_{i+1}, \dots, x_t) \in X \mid (x_i, y_i) \in R_{i_0}^{(i)}\}| \\ &= |X_1| \cdots |X_{i-1}| v_{i_0}^{(i)}. \end{aligned}$$

(b) Induction on i . For $i = 1$, $\lambda_1 = \sum_{j=0}^{n_1} v_j = \sum_{j=0}^{n_1} v_j^{(1)} = |X_1|$, by (a). Assume that it holds for i ($i \geq 1$). Then, again by (a),

$$\begin{aligned} \lambda_{i+1} &= \lambda_i + \sum_{j=\overline{n_i+1}}^{\overline{n_{i+1}}} v_j \\ &= |X_1| \cdots |X_i| + |X_1| \cdots |X_i| \sum_{i_0=1}^{\overline{n_{i+1}}} v_{i_0}^{(i+1)} \\ &= |X_1| \cdots |X_i| + |X_1| \cdots |X_i| (|X_{i+1}| - 1) \\ &= |X_1| \cdots |X_{i+1}|. \end{aligned}$$

(e), (f) These can be shown directly. Or these will follow by generalizing the results in Section 4 of [9] stated for the wreath product of two association schemes. Observe, however, that in what will follow we reversed the order of factors in tensor product expressions of adjacency matrices and irreducible idempotents.

(c) From (f),

$$\begin{aligned} m_{\overline{n_{i+1}+i_0}} &= \text{tr} E_{\overline{n_{i+1}+i_0}} \\ &= \text{tr} E_0^{(1)} \cdots \text{tr} E_0^{(i-1)} \text{tr} E_{i_0}^{(i)} |X_{i+1}| \cdots |X_t| \\ &= |X_t| \cdots |X_{i+1}| m_{i_0}^{(i)}. \end{aligned}$$

(d) Reverse induction on i , starting from t .

(g) For the wreath product of two association schemes, this is also stated in Section 4 of [9]. The proof is left to the reader, as it can be shown just as in (h) below.

(h)

$$\begin{aligned}
 E_{\overline{n_{i+1}+i_0}} &= E_0^{(1)} \otimes \cdots \otimes E_0^{(i-1)} \otimes E_{i_0}^{(i)} \otimes I \otimes \cdots \otimes I \\
 &= |X_1|^{-1} J \otimes \cdots \otimes |X_{i-1}|^{-1} J \otimes |X_i|^{-1} \sum_{j=0}^{n_i} q_{j i_0}^{(i)} A_j^{(i)} \otimes I \otimes \cdots \otimes I \\
 &= |X|^{-1} \left\{ \mu_{i+1} \sum_{j=1}^{n_i} q_{j i_0}^{(i)} J \otimes \cdots \otimes J \otimes A_j^{(i)} \otimes I \otimes \cdots \otimes I \right. \\
 &\quad \left. + m_{\overline{n_{i+1}+i_0}} J \otimes \cdots \otimes J \otimes I \otimes I \otimes \cdots \otimes I \right\} \\
 &= |X|^{-1} \left\{ \mu_{i+1} \sum_{j=1}^{n_i} q_{j i_0}^{(i)} A_{\overline{n_{i-1}+j}} + m_{\overline{n_{i+1}+i_0}} (A_0 + A_1 + \cdots + A_{\overline{n_{i-1}}}) \right\},
 \end{aligned}$$

where the last equality is obtained by replacing various J by the corresponding sum of adjacency matrices for $\mathfrak{X}^{(j)}$ for $j = i - 1, i - 2, \dots, 1$. □

Remark 6. From the expression of P in Theorem 5 (g), we have:

$$\theta_0^{(i)}, \theta_1^{(i)}, \dots, \theta_n^{(i)} = \underbrace{\theta_0^{(i)}, \theta_0^{(i)}, \dots, \theta_0^{(i)}}_{\overline{n_{i+1}+1}}, \theta_{\overline{n_{i+1}+1}}^{(i)}, \dots, \theta_{\overline{n_i}}^{(i)}, \underbrace{0, \dots, 0}_{\overline{n_{i-1}}}.$$

Theorem 7. Let $\mathfrak{X} = (X = X_1 \times \cdots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \cdots \wr \mathfrak{X}^{(t)}$ be a symmetric association scheme which is given as the wreath product of the symmetric association schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$.

(a) The first intersection matrix $L^{(i)} = (p_{\overline{n_{i-1}+1}, l}^k)_{0 \leq k, l \leq n}$ of level i is

(18)

$$L^{(i)} = \left[\begin{array}{ccc|c|ccc}
 & & & \begin{array}{c} v^{(i)} \\ \vdots \\ v^{(i)} \end{array} & & & \\
 & 0 & & & & & 0 \\
 \hline
 & \overline{v_0 \ v_1 \ \cdots \ v_{\overline{n_{i-1}}}} & & \boxed{\lambda_{i-1} \tilde{L}_1^{(i)}} & & & \\
 & & & & & & \\
 & 0 & & & & \boxed{v^{(i)} I_{\overline{n-n_i}}} & \\
 \hline
 & & & & & & \\
 \hline
 \end{array} \right],$$

where the upper left diagonal block matrix in (18) is the zero matrix of size $\overline{n_{i-1} + 1}$ (with rows and columns indexed by $0, 1, \dots, \overline{n_{i-1}}$), and $\tilde{L}_1^{(i)}$ is the submatrix $(p_{1l}^{(i)k})_{1 \leq k, l \leq n_i}$ of the first intersection matrix $L_1^{(i)} = (p_{1l}^{(i)k})_{0 \leq k, l \leq n_i}$ of $\mathfrak{X}^{(i)}$.

$$= |X|^{-1} \left\{ \sum_{k=1}^{n_i} \mu_{i+1} q_{1l}^{(i)k} E_{\overline{n_{i+1}+k}} + m^{(i)} \delta_{1l} (E_0 + \dots + E_{\overline{n_{i+1}}}) \right\},$$

where the last equality is obtained by replacing various I by the corresponding sum of irreducible idempotents for $\mathfrak{X}^{(j)}$, for $j = i + 1, \dots, t$.

Let $j > i$, $1 \leq l \leq n_j$, or let $j = t$, $l = 0$. Then

$$\begin{aligned} & E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{j+1}+l}} \\ &= (E_0^{(1)} \otimes \dots \otimes E_0^{(i-1)} \otimes E_1^{(i)} \otimes I \otimes \dots \otimes I \otimes I \otimes I \otimes \dots \otimes I) \\ & \quad \circ (E_0^{(1)} \otimes \dots \otimes E_0^{(i-1)} \otimes E_0^{(i)} \otimes E_0^{(i+1)} \otimes \dots \otimes E_0^{(j-1)} \otimes E_l^{(j)} \otimes I \otimes \dots \otimes I) \\ &= |X_1|^{-1} E_0^{(1)} \otimes \dots \otimes |X_{i-1}|^{-1} E_0^{(i-1)} \otimes E_1^{(i)} \circ E_0^{(i)} \otimes |X_{i+1}|^{-1} I \otimes \dots \\ & \quad \otimes |X_{j-1}|^{-1} I \otimes I \circ E_l^{(j)} \otimes I \otimes \dots \otimes I \\ &= |X|^{-1} m_{\overline{n_{j+1}+l}} E_{\overline{n_{i+1}+1}}, \end{aligned}$$

where we note that

$$E_1^{(i)} \circ E_0^{(i)} = |X_i|^{-1} E_1^{(i)}, \quad I \circ E_l^{(j)} = |X_j|^{-1} m_l^{(j)} I.$$

Similarly to the just above case, one shows, for $j < i$, that $E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{j+1}+l}} = |X|^{-1} m^{(i)} E_{\overline{n_{j+1}+l}}$. \square

Remark 8. $\lambda_{i-1} L_1^{(i)}$ equals $[L^{(i)}]$ except for the (2,1) entry, namely the (2,1) entry of $\lambda_{i-1} L_1^{(i)}$ is $\lambda_{i-1} = \sum_{j=0}^{n_{i-1}} v_j$, whereas that of $[L^{(i)}]$ is $v_{\underline{n_{i-1}}}$ (cf. (19)). Also, $\mu_{i+1} M_1^{(i)}$ equals $[M^{(i)}]$ except for the (2,1) entry, namely the (2,1) entry of $\mu_{i+1} M_1^{(i)}$ is $\mu_{i+1} = \sum_{j=0}^{n_{i+1}} m_j$, whereas that of $[M^{(i)}]$ is $m_{\overline{n_{i+1}}}$ (cf. (21)).

Proposition 9. Let $\mathfrak{X} = (X = X_1 \times \dots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \dots \wr \mathfrak{X}^{(t)}$ be a symmetric association scheme which is given as the wreath product of the symmetric association schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$. Then we have:

- (a) $\theta_0^{(i)}, \dots, \theta_n^{(i)}$ are real numbers and eigenvalues of $L^{(i)}$ for $i = 1, 2, \dots, t$.
 Further, $|\theta_j^{(i)}| \leq v^{(i)}$ for $i = 1, 2, \dots, t$, $0 \leq j \leq n$.

If in addition \mathfrak{X} is a weak metric scheme, then we have:

- (b)

$$\begin{aligned} & \psi_0^{(1)}(x) = 1, \quad \psi_1^{(1)}(x) = x, \\ (22) \quad & c_{i_0}^{(1)} \psi_{i_0}^{(1)}(x) = (x - a_{i_0-1}^{(1)}) \psi_{i_0-1}^{(1)}(x) - b_{i_0-2}^{(1)} \psi_{i_0-2}^{(1)}(x) \quad (2 \leq i_0 \leq n_1) \\ & \alpha_0^{(1)} = \alpha_1^{(1)} = \dots = \alpha_{n_1}^{(1)} = 0 \text{ for } 2 \leq i \leq t, \end{aligned}$$

$$\begin{aligned} & \psi_1^{(i)}(x) = x, \quad \psi_2^{(i)}(x) = c_2^{(i)-1} (x^2 - a_1^{(i)} x), \\ (23) \quad & c_{i_0}^{(i)} \psi_{i_0}^{(i)}(x) = (x - a_{i_0-1}^{(i)}) \psi_{i_0-1}^{(i)}(x) - b_{i_0-2}^{(i)} \psi_{i_0-2}^{(i)}(x) \\ & \quad + \alpha_{i_0-1}^{(i)} (v_0 + \dots + v_{\underline{n_{i-1}}}) x \quad (3 \leq i_0 \leq n_i), \end{aligned}$$

then

$$\begin{aligned}
 P_{n_i+1}(x) &= \frac{1}{\tilde{c}_2^{(i)} \tilde{c}_3^{(i)} \cdots \tilde{c}_{n_i}^{(i)}} (x - p_{01}^{(i)})(x - p_{11}^{(i)}) \cdots (x - p_{n_i 1}^{(i)}) \\
 &= (x - p_{01}^{(i)}) \sum_{j=0}^{n_i} P_j(x).
 \end{aligned}$$

So we see that $\sum_{j=0}^{n_i} P_j(0) = \text{RHS of (27)}$, and (26) is satisfied with $\alpha_0^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{n_i}^{(i)}$ respectively replaced by $P_0(0), P_2(0), \dots, P_{n_i}(0)$. As $\mathfrak{X}^{(i)}$ is a metric scheme, the square matrix on the LHS of (26) is invertible, and hence $\alpha_0^{(i)} = P_0(0), \alpha_2^{(i)} = P_2(0), \dots, \alpha_{n_i}^{(i)} = P_{n_i}(0)$. In particular, $\sum_{j=0}^{n_i} P_j(0) = \alpha_0^{(i)} + \alpha_2^{(i)} + \cdots + \alpha_{n_i}^{(i)}$. \square

Theorem 11. *Let $\mathfrak{X} = (X = X_1 \times \cdots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \cdots \wr \mathfrak{X}^{(t)}$ be a weak metric scheme. Assume $\psi^{(i)}(x)$'s are as in (22) and (23). Let $\psi_{n_i+1}^{(1)}(x)$ be the polynomial of degree $n_i + 1$ defined by:*

$$\psi_{n_i+1}^{(1)}(x) = (x - a_{n_1}^{(1)})\psi_{n_1}^{(1)}(x) - b_{n_1-1}^{(1)}\psi_{n_1-1}^{(1)}(x).$$

Then

$$(28) \quad \psi_{n_i+1}^{(1)}(x) = \frac{1}{c_2^{(1)} \cdots c_{n_1}^{(1)}} (x - \theta_0^{(1)})(x - \theta_{\frac{n_2}{n_2+1}}^{(1)})(x - \theta_{\frac{n_2}{n_2+2}}^{(1)}) \cdots (x - \theta_{\frac{1}{n_1}}^{(1)})$$

$$(29) \quad = (x - \theta_0^{(1)}) \sum_{j=0}^{n_1} \psi_j^{(1)}(x).$$

For $2 \leq i \leq t$, let $\psi_{n_i+1}^{(i)}(x)$ be the polynomial of degree $n_i + 1$ defined by:

$$(30) \quad \psi_{n_i+1}^{(i)}(x) = (x - a_{n_i}^{(i)})\psi_{n_i}^{(i)}(x) - b_{n_i-1}^{(i)}\psi_{n_i-1}^{(i)}(x) + \alpha_{n_i}^{(i)}(v_0 + \cdots + v_{\underline{n_i-1}})x + \tau_i,$$

with

$$\tau_i = \frac{(-1)^{n_i+1} \theta_0^{(i)} \theta_{\frac{n_i}{n_i+1}+1}^{(i)} \cdots \theta_{\frac{n_i}{n_i}}^{(i)}}{c_2^{(i)} \cdots c_{n_i}^{(i)}}.$$

Then

$$(31) \quad \psi_{n_i+1}^{(i)}(x) = \frac{1}{c_2^{(i)} \cdots c_{n_i}^{(i)}} (x - \theta_0^{(i)})(x - \theta_{\frac{n_i}{n_i+1}+1}^{(i)}) \cdots (x - \theta_{\frac{n_i}{n_i}}^{(i)})$$

$$(32) \quad = (x - \theta_0^{(i)}) \sum_{j=0}^{n_i} \psi_j^{(i)}(x),$$

with

$$(33) \quad \psi_0^{(i)}(x) = \frac{(-1)^{n_i} \theta_{\frac{n_i}{n_i+1}+1}^{(i)} \cdots \theta_{\frac{n_i}{n_i}}^{(i)}}{c_2^{(i)} c_3^{(i)} \cdots c_{n_i}^{(i)}}.$$

Note here that

$$(34) \quad \tau_i = -\theta_0^{(i)}\psi_0^{(i)}(x) = -v^{(i)}\psi_0^{(i)}(x).$$

Proof. As $\psi_k^{(i)}(0) = 0$ for $2 \leq i \leq t$, $1 \leq k \leq n_i$, $\psi_0^{(i)}(x)$ is the constant polynomial in (33) if (31) and (32) are true.

Let $i = 1$. From (22), we see that

$$(35) \quad B \begin{bmatrix} \psi_0^{(1)}(x) \\ \psi_1^{(1)}(x) \\ \vdots \\ \psi_{n_1}^{(1)}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \psi_{n_1+1}^{(1)}(x) \end{bmatrix},$$

where $B = xI_{n_1+1} - {}^t[L^{(1)}]$. So, by multiplying the adjoint $\text{Adj}(B)$ on both sides, we get:

$$(36) \quad |B| \begin{bmatrix} \psi_0^{(1)}(x) \\ \psi_1^{(1)}(x) \\ \vdots \\ \psi_{n_1}^{(1)}(x) \end{bmatrix} = \text{Adj}(B) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \psi_{n_1+1}^{(1)}(x) \end{bmatrix}.$$

By looking at the first component, from (36) we have:

$$|B| = c_2^{(1)} \cdots c_{n_1}^{(1)}\psi_{n_1+1}^{(1)}(x).$$

So

$$(37) \quad \psi_{n_1+1}^{(1)}(x) = \frac{1}{c_2^{(1)} \cdots c_{n_1}^{(1)}} \text{char poly}([L^{(1)}]),$$

where $\text{char poly}([L^{(1)}])$ is an abbreviation for the characteristic polynomial of $[L^{(1)}]$. On the other hand, from $PL^{(1)}P^{-1} = \text{diag}(\theta_0^{(1)}, \theta_1^{(1)}, \dots, \theta_n^{(1)})$ and (18), we have:

$$\begin{aligned} & \text{char poly}([L^{(1)}])(x - \theta_0^{(1)})^{n-n_1} \\ &= \text{char poly}(L^{(1)}) \\ &= (x - \theta_0^{(1)}) \cdots (x - \theta_n^{(1)}). \end{aligned}$$

In view of Remark 6, this implies that

$$(38) \quad \text{char poly}([L^{(1)}]) = (x - \theta_0^{(1)})(x - \theta_{\frac{n_2+1}{n_1}}^{(1)}) \cdots (x - \theta_{\frac{1}{n_1}}^{(1)}).$$

Now, (28) follows from (37) and (38). Also, we have (29) by multiplying both sides of (35) by $[1 \cdots 1]$ on the left (cf. (15)).

where

$$D = \begin{bmatrix} xI_{\underline{n_{i-1}}} & * & 0 \\ 0 & xI_{n_{i+1}} - {}^t(\lambda_{i-1}L^{(i)}) & 0 \\ 0 & 0 & (x - v^{(i)})I_{\underline{n_{i+1}}} \end{bmatrix}.$$

D can be obtained from $xI_{n_{i+1}} - {}^tL^{(i)}$, by first adding the column $(\underline{n_{i-1}} + 1)$ to each of the columns one through $\underline{n_{i-1}}$ and then adding the columns one through $\underline{n_{i-1}}$ to the column $(\underline{n_{i-1}} + 1)$.

So, by Remark 6,

$$\text{char poly}(\lambda_{i-1}L_1^{(i)}) = (x - \theta_0^{(i)})(x - \theta_{\underline{n_{i+1}+1}}^{(i)}) \cdots (x - \theta_{\underline{n_i}}^{(i)}).$$

Now, after multiplying both sides of (39) by $\text{Adj}(C)$ on the left and looking at the second component, we get:

$$x \text{ char poly}(\lambda_{i-1}L_1^{(i)}) = |C'|,$$

where C' is the matrix obtained from C by replacing the second column by ${}^t[x\psi_0^{(i)}(x) \tau_i \ 0 \ \cdots \ 0 \ \psi_{\underline{n_{i+1}}}^{(i)}(x)]$. It is now easy to see that

$$|C'| = (c_2^{(i)} \cdots c_{n_i}^{(i)} \psi_{\underline{n_{i+1}}}^{(i)}(x))x.$$

Thus $\text{char poly}(\lambda_{i-1}L_1^{(i)}) = c_2^{(i)} \cdots c_{n_i}^{(i)} \psi_{\underline{n_{i+1}}}^{(i)}(x)$. This shows (31). □

5. Weak cometric schemes

Let $\mathfrak{X} = (X = X_1 \times \cdots \times X_t, \{R_j\}_{j=0}^n) = \mathfrak{X}^{(1)} \wr \cdots \wr \mathfrak{X}^{(t)}$ be the symmetric association scheme which is given as the wreath product of the symmetric association schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{n_i})$ (cf. Section 2). Then such a symmetric association scheme \mathfrak{X} will be called a weak cometric scheme if it satisfies the equivalent conditions in Theorem 13.

Lemma 12. *Let $W_1, \dots, W_m, W'_1, \dots, W'_n$ ($m, n \in \mathbb{Z}_{\geq 0}$), S, T be matrices with entries in a field. Assume that $W_1, \dots, W_m, W'_1, \dots, W'_n$ are nonzero. If $W_1 \otimes \cdots \otimes W_m \otimes S \otimes W'_1 \otimes \cdots \otimes W'_n = W_1 \otimes \cdots \otimes W_m \otimes T \otimes W'_1 \otimes \cdots \otimes W'_n$, then we have $S = T$.*

Proof. For this, we only need to show:

$$W \otimes S = W \otimes T \text{ and } W \neq 0, \text{ or } S \otimes W = T \otimes W \text{ and } W \neq 0 \Rightarrow S = T.$$

But this is trivial to see. □

Observe here that $e_0^{(i)} = 0$, $f_0^{(i)} = m^{(i)}$, $g_1^{(i)} = m_{\overline{n_{i+1}}}$.

Define the polynomials $\phi_i^{(t)}(x)$ of degree i ($0 \leq i \leq n_t$) recursively by:

$$\begin{aligned}\phi_0^{(t)}(x) &= |X|^{-1}, \quad \phi_1^{(t)}(x) = x, \\ x\phi_i^{(t)}(x) &= f_{i-1}^{(t)}\phi_{i-1}^{(t)}(x) + e_i^{(t)}\phi_i^{(t)}(x) + g_{i+1}^{(t)}\phi_{i+1}^{(t)}(x) \quad (i \geq 1).\end{aligned}$$

Also, define $\beta_0^{(t)} = \beta_1^{(t)} = \dots = \beta_{n_t}^{(t)} = 0$. Then

$$E_{i_0} = \phi_{i_0}^{(t)}(E_1) + \beta_{i_0}^{(t)}E_0 \text{ for } 0 \leq i_0 \leq n_t.$$

Here the multiplication is the Hadamard product and we understand that

$$\phi_0^{(t)}(E_1) = |X|^{-1}J = E_0.$$

Starting from $t-1$, we apply an inductive argument to show (40) for each $i = t-1, \dots, 1$, $1 \leq i_0 \leq n_i$.

Before proceeding further, we note, using (42), that (a) is equivalent to:

$$(43) \quad E_{\overline{n_{i+1}+1}} \circ E_l = |X|^{-1}m_l E_{\overline{n_{i+1}+1}} \quad (0 \leq l \leq \overline{n_{i+1}}),$$

$$(44)$$

$$E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{i+1}+1}} = |X|^{-1}\{m^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}) + e_1^{(i)}E_{\overline{n_{i+1}+1}} + g_2^{(i)}E_{\overline{n_{i+1}+2}}\}$$

with $g_2^{(i)} \neq 0$,

$$(45)$$

$$\begin{aligned}E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{i+1}+i_0}} \\ = |X|^{-1}\{f_{i_0-1}^{(i)}E_{\overline{n_{i+1}+i_0-1}} + e_{i_0}^{(i)}E_{\overline{n_{i+1}+i_0}} + g_{i_0+1}^{(i)}E_{\overline{n_{i+1}+i_0+1}}\} \quad (2 \leq i_0 \leq n_i)\end{aligned}$$

with $f_{i_0-1}^{(i)} \neq 0$ ($2 \leq i_0 \leq n_i$), $g_{i_0+1}^{(i)} \neq 0$ ($2 \leq i_0 \leq n_i - 1$), $g_{n_i+1}^{(i)} = 0$,

$$(46) \quad E_{\overline{n_{i+1}+1}} \circ E_l = |X|^{-1}m^{(i)}E_l \quad (\overline{n_i} + 1 \leq l \leq n).$$

Define $\phi_1^{(i)}(x) = x$, $\beta_1^{(i)} = 0$. Then

$$E_{\overline{n_{i+1}+1}} = \phi_1^{(i)}(E_{\overline{n_{i+1}+1}}) + \beta_1^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}).$$

Also, if we define

$$\phi_2^{(i)}(x) = g_2^{(i)-1}(|X|x^2 - e_1^{(i)}x), \quad \beta_2^{(i)} = -g_2^{(i)-1}m^{(i)},$$

then

$$E_{\overline{n_{i+1}+2}} = \phi_2^{(i)}(E_{\overline{n_{i+1}+1}}) + \beta_2^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}).$$

Assume that $\phi_1^{(i)}, \dots, \phi_{i_0}^{(i)}$, $\beta_1^{(i)}, \dots, \beta_{i_0}^{(i)}$ ($2 \leq i_0 < n_i$) are defined so that

$$E_{\overline{n_{i+1}+j_0}} = \phi_{j_0}^{(i)}(E_{\overline{n_{i+1}+1}}) + \beta_{j_0}^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}),$$

and $\phi_{j_0}^{(i)}(0) = 0$ holds for all $1 \leq j_0 \leq i_0$.

Define the polynomial $\phi_{i_0+1}^{(i)}(x)$ and $\beta_{i_0+1}^{(i)}$ respectively by:

$$\begin{aligned}\phi_{i_0+1}^{(i)}(x) &= g_{i_0+1}^{(i)-1}\{|X|x\phi_{i_0}^{(i)}(x) + \beta_{i_0}^{(i)}(m_0 + \dots + m_{\overline{n_{i+1}}})x \\ &\quad - f_{i_0-1}^{(i)}\phi_{i_0-1}^{(i)}(x) - e_{i_0}^{(i)}\phi_{i_0}^{(i)}(x)\},\end{aligned}$$

$$\beta_{i_0+1}^{(i)} = -g_{i_0+1}^{(i)-1} (f_{i_0-1}^{(i)} \beta_{i_0-1}^{(i)} + e_{i_0}^{(i)} \beta_{i_0}^{(i)}).$$

Then

$$E_{\overline{n_{i+1}+i_0+1}} = \phi_{i_0+1}^{(i)}(E_{\overline{n_{i+1}+1}}) + \beta_{i_0+1}^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}).$$

Note here that $\phi_{i_0+1}^{(i)}(0) = 0$. This shows (40).

(b) \Rightarrow (a) We have to show (43)-(46). By (41), (43), and (46) hold. Let $i = t$. Then we only need to show (45) for $1 \leq i_0 \leq n_t$. By our assumption, $E_{i_0} = \phi_{i_0}^{(t)}(E_1)$ for $0 \leq i_0 \leq n_t$. Since $x\phi_{i_0}^{(t)}(x)$ is a linear combination of $\phi_{i_0+1}^{(t)}(x), \phi_{i_0}^{(t)}(x), \dots, \phi_0^{(t)}(x)$, $E_1 \circ E_{i_0}$ is a linear combination of $E_{i_0+1}, E_{i_0}, \dots, E_0$. Clearly, the coefficient of E_{i_0+1} is not zero. This says $q_{1i_0}^j = 0$ for $j \geq i_0 + 2$, and $q_{1i_0}^j \neq 0$ for $j = i_0 + 1$. As $m_j q_{ki}^j = m_i q_{kj}^i$, $q_{1j}^i = 0 \Leftrightarrow q_{1i}^j = 0$. So $q_{1i_0}^j = 0$ if $|j - i_0| \geq 2$, and $q_{1i_0}^j \neq 0$ if $|j - i_0| = 1$.

Let $i \leq t - 1$. First, we show (45). By our assumption (40) and (41),

$$\begin{aligned} & E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{i+1}+i_0}} \\ &= \phi_{i_0}^{(i)}(E_{\overline{n_{i+1}+1}}) \circ E_{\overline{n_{i+1}+1}} + |X|^{-1} \beta_{i_0}^{(i)}(m_0 + \dots + m_{\overline{n_{i+1}}}) E_{\overline{n_{i+1}+1}}. \end{aligned}$$

So, if we put

$$x\phi_{i_0}^{(i)}(x) + |X|^{-1} \beta_{i_0}^{(i)}(m_0 + \dots + m_{\overline{n_{i+1}}})x = \sum_{j=0}^{i_0+1} \varepsilon_j \phi_j^{(i)}(x)$$

for some ε_j with $\varepsilon_{i_0+1} \neq 0$, then, by (40),

$$\begin{aligned} & E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{i+1}+i_0}} \\ &= - \sum_{j=0}^{i_0+1} \varepsilon_j \beta_j^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}) + \sum_{j=1}^{i_0+1} \varepsilon_j E_{\overline{n_{i+1}+j}}. \end{aligned}$$

By the same argument as above, $q_{\overline{n_{i+1}+1}, \overline{n_{i+1}+i_0}}^{\overline{n_{i+1}+j_0}} = 0$ if $|j_0 - i_0| \geq 2$, and $q_{\overline{n_{i+1}+1}, \overline{n_{i+1}+i_0}}^{\overline{n_{i+1}+j_0}} \neq 0$ if $|j_0 - i_0| = 1$. Next, we show (44). By (40) with $i_0 = 2$, we have:

$$E_{\overline{n_{i+1}+2}} = |X|eE_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{i+1}+1}} + fE_{\overline{n_{i+1}+1}} + \beta_2^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}})$$

for some e, f with $e \neq 0$. So, taking the proof of Theorem 7 into account, we get:

$$E_{\overline{n_{i+1}+1}} \circ E_{\overline{n_{i+1}+1}} = |X|^{-1} \{m^{(i)}(E_0 + \dots + E_{\overline{n_{i+1}}}) + e_1^{(i)} E_{\overline{n_{i+1}+1}} + g_2^{(i)} E_{\overline{n_{i+1}+2}}\}$$

with $g_2^{(i)} \neq 0$, where we put $e_1^{(i)} = -e^{-1}f$, $g_2^{(i)} = e^{-1}$.

(b) \Leftrightarrow (c) This is straightforward.

(b) \Leftrightarrow (d) For $i = 1, \dots, t$, $1 \leq i_0 \leq n_i$, or $i = t$, $i_0 = 0$, put

$$\Phi_{i_0}^{(i)}(x) = |X_1| \cdots |X_{i-1}| \phi_{i_0}^{(i)}(|X_1| \cdots |X_{i-1}|^{-1} x) + \beta_{i_0}^{(i)} |X_i|^{-1}.$$

Then we see that (40) is equivalent to:

$$E_0^{(1)} \otimes \dots \otimes E_0^{(i-1)} \otimes \Phi_{i_0}^{(i)}(E_1^{(i)}) \otimes I \otimes \dots \otimes I$$

$$= E_0^{(i)} \otimes \dots \otimes E_0^{(i-1)} \otimes E_{i_0}^{(i)} \otimes I \otimes \dots \otimes I.$$

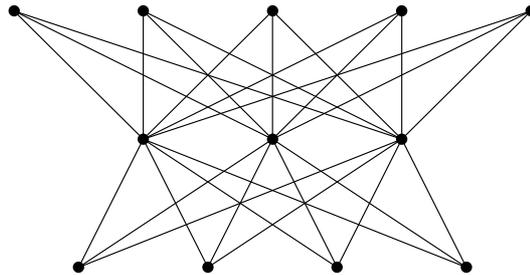
In view of Lemma 12, this in turn is equivalent to $E_{i_0}^{(i)} = \Phi_{i_0}^{(i)}(E_1^{(i)})$ (cf. [2, pp. 193–194]). \square

6. An example

Let $\mathfrak{X} = (X, \{R_j\}_{j=0}^n)$ be any symmetric association scheme. Let \mathcal{A} be the Bose-Mesner algebra of \mathfrak{X} generated by the adjacency matrices $A_0, A_1, \dots, A_n \in M_{|X|}(\mathbb{C})$, p_{jl}^k the intersection numbers of \mathfrak{X} , $L_j = (p_{jl}^k)$ ($j = 0, 1, \dots, n$) the intersection matrices of \mathfrak{X} , and let \mathcal{B} be the subalgebra of $M_{n+1}(\mathbb{C})$ generated by L_0, L_1, \dots, L_n . Then the map $\mathcal{A} \rightarrow \mathcal{B}$ ($A_i \mapsto L_i$) is an isomorphism as algebras over \mathbb{C} .

In view of this isomorphism and since $|X|$ is much bigger than $n + 1$, we will express the modified polynomial relations for $H(4, 3, 5; 2)$ below in terms of intersection matrices. For this example, observe that $|X| = 4096$, while $n + 1 = 13$.

Now, we consider the weak Hamming scheme $H(4, 3, 5; 2) = H(4, 2) \wr H(3, 2) \wr H(5, 2)$, which is associated with the poset $\mathbb{P}_0 = 4\mathbf{1} \oplus 3\mathbf{1} \oplus 5\mathbf{1}$.



The P -matrices of $H(4, 2), H(3, 2), H(5, 2)$ are respectively given by

$$\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 3 & 2 & -2 & -3 & -1 \\ 1 & 1 & -2 & 2 & 1 & 1 \\ 1 & -1 & -2 & 2 & 1 & -1 \\ 1 & -3 & 2 & 2 & -3 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix},$$

and the P -matrix of $H(4, 3, 5; 2)$ is

$$\begin{pmatrix} 1 & 4 & 6 & 4 & 1 & 48 & 48 & 16 & 640 & 1280 & 1280 & 640 & 128 \\ 1 & 4 & 6 & 4 & 1 & 48 & 48 & 16 & 384 & 256 & -256 & -384 & -128 \\ 1 & 4 & 6 & 4 & 1 & 48 & 48 & 16 & 128 & -256 & -256 & 128 & 128 \\ 1 & 4 & 6 & 4 & 1 & 48 & 48 & 16 & -128 & -256 & 256 & 128 & -128 \\ 1 & 4 & 6 & 4 & 1 & 48 & 48 & 16 & -384 & 256 & 256 & -384 & 128 \\ 1 & 4 & 6 & 4 & 1 & 48 & 48 & 16 & -640 & 1280 & -1280 & 640 & -128 \\ 1 & 4 & 6 & 4 & 1 & 16 & -16 & -16 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & -16 & -16 & 16 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & -48 & 48 & -16 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{aligned}
L_5, L_6 &= \frac{1}{32}L_5^2 - \frac{3}{2}(I + L_1 + \cdots + L_4), \quad L_7 = \frac{1}{1536}L_5^3 - \frac{7}{6}L_5, \\
L_8, L_9 &= \frac{1}{256}L_8^2 - \frac{5}{2}(I + L_1 + \cdots + L_7), \quad L_{10} = \frac{1}{98304}L_8^3 - \frac{13}{6}L_8, \\
L_{11} &= \frac{1}{50331648}L_8^4 - \frac{11}{1536}L_8^2 + \frac{15}{8}(I + L_1 + \cdots + L_7), \\
L_{12} &= \frac{1}{32212254720}L_8^5 - \frac{1}{65536}L_8^3 + \frac{149}{120}L_8.
\end{aligned}$$

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