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DOMINATION IN GRAPHS OF MINIMUM DEGREE FOUR

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Abstract. A dominating set for a graph G is a set D of vertices of G such that every vertex of G not in D is adjacent to a vertex of D. Reed [11] considered the domination problem for graphs with minimum degree at least three. He showed that any graph G of minimum degree at least three contains a dominating set D of size at most $\frac{3}{8}|V(G)|$ by introducing a covering by vertex disjoint paths. In this paper, by using this technique, we show that every graph on n vertices of minimum degree at least four contains a dominating set D of size at most $\frac{4}{11}|V(G)|$.

1. Introduction

Throughout this paper, by a graph G we always mean a finite, undirected, and simple graph with vertex set $V(G)$ and edge set $E(G)$. For $x, y \in V(G)$, xy denotes the edge with ends x and y. If $xy \in E(G)$, we say that y is a neighbor of x or y is joined to x, and denote by $N(x)$ the set of neighbors of x. $d(x) = |N(x)|$ is called the degree of x. A subgraph H is said to be induced by U if $V(H) = U$ and $xy \in E(H)$ if and only if $xy \in E(G)$, $x, y \in U$. A set D of vertices of a graph G is called a dominating set if every vertex of $V(G) - D$ is adjacent to at least one element of D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. It has been proved [5] that the decision problem corresponding to the domination number for arbitrary graphs is NP-complete. Thus, the exploration of lower and upper bounds for the domination number as sharp as possible is of great significance. Many results on upper bounds on the domination number in terms of some basic parameters such as the numbers of vertices and edges, the minimum and maximum degree and so on, have been obtained. The terminologies not presented here can be found in [6].

Let $\delta = \delta(G)$ denote the minimum degree of graph G. An early result of Ore (see [9]) states that $\gamma(G) \le \frac{n}{2}$ if G is a graph of order n with the minimum degree at least one. This result was improved to $\gamma(G) \le \frac{2n}{5}$ by McCuaig and Shepherd in [8] for the connected graph G which has minimum degree at least

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two and is not one of seven exceptional graphs. Reed in [11] considered the case for the graphs with minimum degree at least three, and obtained that $\gamma(G) \leq \frac{3n}{8}$. In this direction, an obvious conjecture (see [6]) seems to be that for any graph G with $\delta(G) \geq k$, $\gamma(G) \leq \frac{k}{3k-1}n$. However, Caro and Roditty (see [2], [3] and also [1]) proved that for any graph G with minimum degree δ, $\gamma(G) \leq n[1 - \delta(\frac{1}{\delta+1})^{1+\frac{1}{\delta}}]$. For $\delta(G) \geq 7$, it is easy to verify that $n[1 - \delta(\frac{1}{\delta+1})^{1+\frac{1}{\delta}}] < \frac{\delta}{3\delta-1}n$ by using calculus. Thus, the question remains open only for graphs G with $4 \leq \delta(G) \leq 6$. The purpose of this paper is to give a positive answer for the graph G with minimum degree $\delta(G) \geq 4$.

Main Theorem. *Let G be a graph of order n with minimum degree at least four. Then*

$$\gamma(G) \leq \frac{4}{11}n.$$

The proof of Main Theorem is completed by choosing a dominating set D of G based on the so-called vertex disjoint paths cover, which was introduced by Reed [11]. By cases analysis, we prove three basic facts, from these, $|D| \leq \frac{4}{11}n$ is obtained. For convenience, we use $|G|$ for the number of vertices of the graph G.

A cover of vertex disjoint paths of G, or simply a *vdp*-cover, is a set of vertex disjoint paths $P_1, \ldots, P_k$ such that $V(G) = V(P_1) \cup \cdots \cup V(P_k)$. A path P is called a 0-, 1- or 2-path if $|P|$ is congruent to 0, 1 or 2 mod 3, respectively. For a *vdp*-cover S of G, let S_i $(i = 0, 1, 2)$ be the set of i-paths in S. If $P = P'xP''$, where P' is an i-path and P'' is a j-path (and x is on neither of those paths), then we say x is an (i, j)-vertex of P. Let $P \in S$ and x be an endvertex of P. We say that x is an out-endvertex if it has a neighbor which is not on P. If P is a 2-path, we say that x is a $(2, 2)$-endvertex if it is not an out-endvertex and is adjacent to some $(2, 2)$-vertex of P.

2. Choose a dominating set

In the below, we always assume that G is a graph of order n with $\delta(G) \geq 4$. For convenience, we assume that G is connected. We first choose a *vdp*-cover S of G such that

(1) $2|S_1| + |S_2|$ is minimized,
(2) Subject to (1), $|S_2|$ is minimized,
(3) Subject to (2), $\sum_{P_i \in S_0} |P_i|$ is minimized,
(4) Subject to (3), $\sum_{P_i \in S_1} |P_i|$ is minimized.

Let x be an out-endvertex of $P_i \in S_1 \cup S_2$, y a neighbor of x on some path P_j distinct from P_i. Let $P_j = P_j' y P_j''$. Then, we have the following assertion (for the proof, see [11], Observation 1-3).

Assertion 1. *P_j is not a 1-path. If P_j is a 0-path, then both P_j' and P_j'' are 1-paths; if P_j is a 2-path, then both P_j' and P_j'' are 2-paths.*

Having chosen the minimal *vdp*-cover $S = \{P_1, \ldots, P_k\}$, we rearrange the paths of S to obtain a new *vdp*-cover $S' = \{P'_1, \ldots, P'_k\}$ such that P'_i is a Hamilton path on $V(P_i)$, and so that the number of out-endvertices is maximized, and subject to this, the number of $(2,2)$-endvertices is maximized. Clearly, S' is still minimal with respect to the above four conditions. For convenience, we still denote the new *vdp*-cover of G by S.

Now, for each 1-path P in S which has an out-endvertex we choose some vertex $y \notin P$ which is adjacent to the endvertex of P. We say that y is the acceptor for P. For each 2-path P in S which has two out-endvertices, for each of these endvertices we choose a vertex of $G - P$ which is adjacent to it and designate it as the acceptor corresponding to that endvertex. For each 2-path P in S which has precisely one out-endvertex x and $|P| \leq 8$, we choose some vertex $y \notin P$ which is adjacent to it and designate it as the acceptor for P. We call a path in S accepting if it contains an acceptor. We next specify a set $A \subseteq S$ of 2-paths. Initially, let A be the set of accepting 2-paths. While there is any out-endvertex x of a path in A for which we have not chosen an acceptor, we choose a neighbor of this endvertex in $G - P$ and designate it as an acceptor for x. If this new acceptor is on a previously non-accepting 2-path P', then we add P' to A. We continue this process until there is an acceptor for every out-endvertex of the paths in A. In addition, for any $(2,2)$-endvertex x of any path P in A, we choose a $(2,2)$-vertex y of P which is adjacent to x and designate it as an inacceptor for x.

For any accepting 2-path P, we partition $P = P_1P_2P_3$ such that P_1 and P_3 are both 1-paths which contain neither acceptors nor inacceptors, and maximal with this property. We say that P_1 and P_3 are tips of P and P_2 is its central path. By the maximality of P_1, P_3 and Assertion 1, if $x \in P_2$ is adjacent in P_2 to an endvertex of P_2, then it is an acceptor or inacceptor.

Before the description of choosing the dominating set, we present the following fact.

Assertion 2. *Let $P \in S$ be a 2-path. If P has precisely one out-endvertex x and $|P| \leq 8$, then $V(P)$ has a subset of $\lfloor \frac{|P|}{3} \rfloor$ vertices which dominate $V(P) - x$.*

We will prove Assertion 2 in next section. Now, we choose a dominating set D of G in the following manner:

Step 1: For each 0-path P, we put every $(1,1)$-vertex of P in D.

Step 2: For each accepting 2-path P, we put into D every $(2,2)$-vertex of P which is in the central path of P.

Step 3: For each 1-path P with at least one out-endvertex, we choose $\lfloor \frac{|P|}{3} \rfloor$ vertices of P which dominate all of the vertices of P except for the endvertex of P which is adjacent to the acceptor of P. We put these vertices in D. For each non-accepting 2-path P with two out-endvertices we choose $\lfloor \frac{|P|}{3} \rfloor$ vertices of P which dominate its interior vertices. We put these vertices in D. For each non-accepting 2-path P which has precisely one out-endvertex x and $|P| \leq 8$, we choose $\lfloor \frac{|P|}{3} \rfloor$ vertices of P which dominate all of the vertices of P except for the endvertex x (By Assertion 2, we can do that). We put these vertices in D.

Step 4: For each 1-path P with no out-endvertex, we choose a subset of $V(P)$ which dominate $V(P)$ to put in D. If possible, we choose a set of $\lfloor \frac{|P|}{3} \rfloor$ vertices; otherwise we choose a set of $\lceil \frac{|P|}{3} \rceil$ vertices. For each non-accepting 2-path P which has no out-endvertex, or has precisely one out-endvertex and $|P| \geq 11$, we choose a subset of $V(P)$ which dominate $V(P)$ to put in D. If possible, we choose a set of $\lfloor \frac{|P|}{3} \rfloor$ vertices, otherwise we choose a set of $\lceil \frac{|P|}{3} \rceil$ vertices.

Step 5: For each tip P_1 of an accepting 2-path P, if the common endvertex x of P_1 and P is adjacent to a vertex chosen in Step 1 or 2, we choose $\lfloor \frac{|P_1|}{3} \rfloor$ of vertices of P_1 which dominate the remaining vertices of P_1 and put them in D. If x is not adjacent to a vertex chosen in Step 1 or 2, we choose a set which dominates P_1 to put in D. If possible, we choose $\lfloor \frac{|P_1|}{3} \rfloor$ vertices, otherwise we choose $\lceil \frac{|P_1|}{3} \rceil$ vertices.

It is easy to see that D is a dominating set of G (or see [11], Observation 5-8). To calculate the size of D, we define the following sets.

(i) O_1: the set of 1-paths P which either have an out-endvertex or contain a dominating set of size $\lfloor \frac{|P|}{3} \rfloor$.

(ii) O_2: the set of non-accepting 2-paths P which have two out-endvertices or contain a set of size $\lfloor \frac{|P|}{3} \rfloor$ that dominates all of the vertices of P, and all non-accetping 2-paths which have precisely one out-endvertex and $|P| \leq 8$.

(iii) I_1: the set of 1-paths not in O_1.

(iv) I_2: the set of non-accepting 2-paths not in O_2.

(v) E: the set of such tips P_1 of an accepting 2-path P, which is in E if and only if the corresponding endvertex of P is neither an out-endvertex nor a $(2,2)$-endvertex and we cannot dominate P_1 using $\lfloor \frac{|P_1|}{3} \rfloor$ vertices.

(vi) W: the set of $(2,2)$-endvertices of accepting 2-paths for which we have chosen an inacceptor.

Then

$$|D| = \sum_{P\in O_1} \frac{|P|-1}{3} + \sum_{P\in O_2} \frac{|P|-2}{3} + \sum_{P\in I_1} \frac{|P|+2}{3} + \sum_{P\in I_2} \frac{|P|+1}{3} + \sum_{P\in S_0} \frac{|P|}{3} + \sum_{P\in A} \frac{|P|-2}{3} + |E|.$$

Equivalently,

$$|D| = \frac{n}{3} - \frac{1}{3}|O_1| - \frac{2}{3}|O_2| + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| - \frac{2}{3}|A| + |E|.$$

Note that each accepting 2-path corresponds to an endvertex of some path in $O_1 \cup O_2$ or to an endvertex of an accepting 2-path of A which is not in

$E \cup W$. Thus, we have $|A| \leq |O_1| + 2|O_2| + 2|A| - |E| - |W|$, and so $|E| \leq |O_1| + 2|O_2| + |A| - |W|$. Also, $|E| \leq 2|A| - |W|$. Thus,

$$|D| \leq \frac{n}{3} + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{|E|}{2} - \frac{|W|}{2}.$$

For each element T of E, there is an accepting 2-path P_T such that T is a tip of P_T. Now we define $E' \subseteq E$ by saying that each $T \in E$ is in E' if the endpoint of P_T not in T is not an element of W.

Clearly, $|E'| \geq |E| - |W|$, and so

$$(*) \qquad |D| \leq \frac{n}{3} + \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{1}{2}|E'|$$

In next section, we will prove some facts (Lemma 3, Lemma 4, Lemma 5), with $(*)$ that imply $|D| \leq \frac{4}{11}n$.

3. 1-paths with short length

Essentially, in this section we will prove that every 1-path P of short length with some additional conditions can be dominated by $\lfloor \frac{|P|}{3} \rfloor$ vertices. We will assume the same conditions and use the notations as in last section. We first state three simple observations.

(Q_1) Let $P = x_1x_2 \cdots x_{3k+1}$ $(k \geq 1)$ be a 1-path. If x_1 is adjacent to a vertex x_{3i} for some $1 \leq i \leq k$, then P can be dominated by k vertices.

(Q_2) Let C be a cycle of $3k+1$ $(k \geq 1)$ vertices, $B = b_1b_2b_3$ be a path such that $V(C) \cap V(B) = \emptyset$. If b_2 has a neighbor in C, then $V(C) \cup V(B)$ can be dominated by $k+1$ vertices.

(Q_3) Let $P = x_1x_2 \cdots x_{3k-1}$ $(k \geq 1)$ be a path, and $x \notin P$. If x is adjacent to some vertex of $\cup_{i=1}^{k}\{x_{3i-2}, x_{3i-1}\}$, then $V(P) \cup \{x\}$ can be dominated by k vertices.

Next we show two technical results.

Lemma 1. *Let $C = x_1x_2 \cdots x_{3k+1}x_1$ $(1 \leq k \leq 4)$ be a cycle of G, H the subgraph induced by $V(C)$. If $N(x_i) \subseteq V(C)$ for any $x_i \in V(C)$ such that there is a Hamilton path from x_i to x_{3k+1} in H, then H has a dominating set of k vertices.*

Proof. Assume to the contrary that H has no any dominating set of k vertices. We only prove for $k = 4$. For $k \leq 3$ we can deduce a contradiction by the same reasoning.

Let $k = 4$ and $C = x_1x_2 \cdots x_{13}x_1$. Then, both x_1, x_{12} are the endvertices of some Hamilton paths to x_{13} in H, thus $N(x_1) \subseteq V(C)$ and $N(x_{12}) \subseteq V(C)$. First we check the possible neighbors of x_1. By (Q_1), x_1 is not adjacent to any of x_3, x_6, x_9, x_{12}. If x_1 is adjacent to x_{10}, as x_{12} has a neighbor in the cycle $C' = x_1x_2 \cdots x_{10}x_1$, by (Q_2), then H has a dominating set of four vertices, a contradiction. On the other hand, if x_{12} is adjacent to both x_8 and x_9, then x_{10} is an endvertex of a Hamilton path of H to x_{13}, by applying (Q_2)

to $x_9x_{10}x_{11}$ and the cycle $x_1x_2\cdots x_8x_{12}x_{13}x_1$, then H has a dominating set of four vertices, a contradiction. Thus, if x_1 is adjacent to x_7, applying (Q_2) to $x_{11}x_{12}x_{13}$ and the cycle $x_1x_2\cdots x_7x_1$, x_{12} is not adjacent to any vertex of the cycle $x_1x_2\ldots x_7x_1$. So we can deduce that x_{12} is adjacent to both x_8 and x_9, a contradiction.

Hence, x_1 has extra neighbors only in $\{x_4, x_5, x_8, x_{11}\}$. Symmetrically, x_{12} has extra neighbors only in $\{x_9, x_8, x_5, x_2\}$. Now, if x_1 is adjacent to x_4, by (Q_2), then x_{12} is not adjacent to x_2. As x_{12} is not adjacent to both x_8 and x_9, x_{12} is adjacent to x_5. Note that, x_3 is an endvertex of a Hamilton path of H to x_{13}. As x_1 dominates x_2, x_4, x_{13}, H has a dominating set of four vertices for any choice of the neighbor of x_3 in the 8-cycle $C'' = x_5x_6\cdots x_{12}x_5$, a contradiction. So, x_1 is not adjacent to x_4, and symmetrically, x_{12} is not adjacent to x_9. If x_1 is adjacent to x_{11}, then x_{10} is an endvertex of a Hamilton path of H to x_{13}. Thus, x_{12} is still not adjacent to x_8, for otherwise, we apply (Q_2) to $x_9x_{10}x_{11}$ and the cycle $x_1x_2\cdots x_8x_{12}x_{13}x_1$ to obtain a contradiction. So, x_{12} is adjacent to both x_2, x_5. Then, x_3 is an endvertex of a Hamilton path of H to x_{13}. By applying (Q_2) to $x_2x_3x_4$ and the cycle $x_1x_{11}x_{10}\cdots x_5x_{12}x_{13}x_1$, also a contradiction. Hence, x_1 is adjacent only to both x_5 and x_8, and symmetrically, x_{12} is also adjacent to only both x_8, x_5. Then, x_7 is an endvertex of a Hamilton path of H to x_{13}. By (Q_1) and (Q_2), x_7 is adjacent to neither x_9 nor any vertex of the cycle $x_1\cdots x_5x_{12}x_{13}x_1$. Thus, x_7 is adjacent to x_{10}. Then, $\{x_2, x_5, x_{10}, x_{12}\}$ dominates H, a contradiction. This proves Lemma 1. □

Lemma 2. *Let $C = x_1x_2\cdots x_{3k+2}x_1$ $(1 \le k \le 4)$ be a cycle of G, H the subgraph induced by $V(C)$. If $N(x_i) \subseteq V(C)$ for any $x_i \in V(C)$ such that there is a Hamilton path from x_i to x_{3k+2} in H, then $V(C) - \{x_{3k+2}\}$ can be dominated by k vertices.*

Proof. Assume to the contrary that $V(C) - \{x_{3k+2}\}$ can not be dominated by k vertices. We still prove only for $k = 4$ and omit for $k \le 3$.

Let $k = 4$ and $C = x_1x_2\cdots x_{14}x_1$. Then, both x_1, x_{13} are the endvertices of some Hamilton paths to x_{14} in H, thus $N(x_1) \subseteq V(C)$ and $N(x_{13}) \subseteq V(C)$. Note that, x_1 and x_{13} are symmetrical. By (Q_1), x_1 is not adjacent to any of x_3, x_6, x_9, x_{12}. If x_1 is adjacent to both x_4, x_5, then both x_2, x_3 are the endvertices of some hamiltonian paths of H to x_{14}. As x_1 dominates x_2, x_4, x_5, by (Q_3), x_3 has extra neighbors only in $\{x_5, x_8, x_{11}, x_{14}\}$. On the other hand, as x_4 dominates x_1, x_3, x_5, by (Q_3), x_2 has at least one neighbor in $\{x_5, x_8, x_{11}, x_{14}\}$. Hence, $\{x_5, x_8, x_{11}, x_{14}\}$ dominates $V(C) - \{x_{3k+2}\}$, a contradiction. So, x_1 is not adjacent to both x_4, x_5, and symmetrically, x_{13} is not adjacent to both x_9, x_{10}. In the sequel, we distinguish five cases according to the possible neighbors of x_1.

1. x_1 is adjacent to x_{13}. Then, both x_2 and x_{12} are the endvertices of Hamilton paths to x_{14} in H. As x_1 has one more neighbor in $\{x_4, x_5, x_7, x_8, x_{10}, x_{11}\}$, we consider the following subcases.

(1.1) x_1 is adjacent to x_4. Then, x_3 is an endvertex of Hamilton path to x_{14} in H. As x_1 dominates x_2, x_4 and x_{13}, by (Q_3), x_3 has extra neighbors only in $\{x_7, x_{10}, x_{13}, x_{14}\}$. If x_3 is adjacent to x_{10}, as $N(x_{12}) \subseteq V(C)$, we apply (Q_2) to $x_{11}x_{12}x_{13}$ and the cycle $x_1x_4x_5 \cdots x_{10}x_3x_2x_1$, then there is a dominating set of four vertices of $V(C) - \{x_{14}\}$, a contradiction. If x_3 is adjacent to x_{13}, then x_{13} dominates x_1, x_3 and x_{12}. As $N(x_2) \subseteq V(C)$, still by (Q_3), then x_2 has extra neighbors only in $\{x_6, x_9, x_{12}, x_{14}\}$, and thus x_2 is adjacent to at least one vertex of $\{x_6, x_9, x_{12}\}$. Then, $\{x_4, x_6, x_9, x_{12}\}$ dominates $V(C) - \{x_{14}\}$, a contradiction. Hence, x_3 is adjacent to x_{14}. On the other hand, as x_4 dominates x_1, x_3 and x_5, by (Q_3), x_2 has extra neighbors only in $D' := \{x_5, x_8, x_{11}, x_{14}\}$. Then, D' dominates $V(C) - \{x_{14}\}$, a contradiction.

(1.2) x_1 is adjacent to x_{10}. As $N(x_{12}) \subseteq V(C)$, by applying (Q_2) to $x_{11}x_{12}x_{13}$ and the cycle $x_1x_2 \cdots x_{10}x_1$, we have that four vertices dominate $V(C) - \{x_{14}\}$, a contradiction.

(1.3) x_1 is adjacent to x_5. By (1.1) and (1.2), assume that both x_1, x_{13} are not adjacent to x_4 and x_{10}. Now x_4 is an endvertex of Hamilton path to x_{14} in H, we have $N(x_4) \subseteq V(C)$. By (Q_1) and above result, x_4 has extra neighbors only in $\{x_7, x_8, x_{10}, x_{11}, x_{14}\}$. If x_4 is adjacent to both x_7 and x_8, then x_6 is an endvertex of a Hamilton path of H to x_{14}, and thus, by applying (Q_2) to $x_5x_6x_7$ and the cycle $x_1 \cdots x_4x_8 \cdots x_{13}x_1$, we have a contradiction. If x_4 is adjacent to x_{10}, then we have a 10-cycle without three vertices $\{x_{11}, x_{12}, x_{13}\}$, thus, by (Q_2), we also have a contradiction. If x_4 is adjacent to x_{14}, then $x_3x_2x_1x_{13} \cdots x_4x_{14}$ is a Hamilton path of H, and thus $N(x_3) \subseteq V(C)$. By applying (Q_2) to $x_2x_3x_4$ and the cycle $x_1x_5 \cdots x_{13}x_1$, we have a contradiction. Summarizing, x_4 must be adjacent to x_{11}. Then, x_{11} dominates x_{10}, x_{12}, x_4. By applying (Q_3) to x_{13} and the path $x_3x_2x_1x_5 \cdots x_9$, we have that x_{13} must be adjacent to x_7. As $N(x_2) \subseteq V(C)$, by (Q_2), x_2 has no neighbor in the cycle $x_7x_8 \cdots x_{13}x_7$. Again by (Q_1), x_2 has two extra neighbors only in $\{x_5, x_6, x_{14}\}$. If x_2 is adjacent to x_{14}, then x_3 is also an endvertex of a Hamilton path of H to x_{14}, by using the same reasoning as above, we have a contradiction. Thus, x_2 is adjacent to both x_5, x_6. By applying (Q_2) to $x_3x_4x_5$ and the cycle $x_1x_2x_6 \cdots x_{13}x_1$, we have a contradiction.

(1.4) x_1 is adjacent to x_7. Then, by (Q_2) and (Q_1), x_{12} has extra neighbors only in $\{x_8, x_9, x_{14}\}$. If x_{12} is adjacent to both x_8 and x_9, then x_{10} is also an endvertex of a Hamilton path of H to x_{14}, and thus $N(x_{10}) \subseteq V(C)$. By applying (Q_2) to $x_9x_{10}x_{11}$ and the cycle $x_1 \cdots x_8x_{12}x_{13}x_1$, we have a contradiction. So, x_{12} must be adjacent to x_{14} and one of x_8, x_9. In this case $x_{12} \cdots x_1x_{13}x_{14}x_{12}$ is a Hamilton cycle and that x_{12}, x_{13} are adjacent. Note that x_{12} is adjacent to x_8 or x_9, this is the same situation as (1.1) or (1.3), a contradiction.

(1.5) x_1 is adjacent to x_8 or x_{11}. By symmetry and (1.1)-(1.4), x_{13} has one more neighbor only in $\{x_3, x_6\}$. Then $\{x_3, x_6, x_8, x_{11}\}$ dominates $V(C) - \{x_{14}\}$, a contradiction.

2. x_1 is adjacent to x_{11}. By 1, x_1, x_{13} are not adjacent. As x_{11} dominates $\{x_1, x_{10}, x_{12}\}$, by (Q_3), x_{13} has extra neighbors only in $\{x_4, x_7, x_{10}\}$.

If x_{13} is adjacent to x_{10}, then x_{12} is an endvertex of a Hamilton path of H to x_{14}. As x_{10} dominates $\{x_9, x_{11}, x_{13}\}$, still by (Q_3), x_{12} has extra neighbors only in $\{x_3, x_6, x_9, x_{14}\}$. If x_{12} is adjacent to x_{14}, then it is the same situation as 1, a contradiction. If x_{12} is adjacent to x_9, by noting that x_{13} has one more neighbor in $\{x_4, x_7\}$, then $\{x_1, x_4, x_7, x_9\}$ dominates $V(C) - \{x_{14}\}$, a contradiction. If x_{12} is adjacent to both x_3, x_6, then there is a 10-cycle $x_3 \cdots x_{10}x_{13}x_{12}x_3$ which excludes $x_2x_1x_{11}$. As x_1 has one more neighbor in this cycle, by (Q_2), we also have a contradiction.

Otherwise, x_{13} is adjacent to both x_4 and x_7. Then, x_5 is an endvertex of a Hamilton path of H to x_{14}, so $N(x_5) \subseteq V(C)$. By (Q_2), x_5 has no neighbor in the cycle $x_7x_8 \cdots x_{13}x_7$. By (Q_1), x_5 has extra neighbors only in $\{x_1, x_3, x_{14}\}$. If x_5 is adjacent to x_1, then x_2 is an endvertex of a Hamilton path of H to x_{14}, by applying (Q_2) to $x_1x_2x_3$ and the cycle $x_4x_5 \cdots x_{13}x_4$, we have a contradiction; otherwise, x_5 is adjacent to x_3, then $\{x_1, x_5, x_9, x_{13}\}$ dominates $V(C) - \{x_{14}\}$, also a contradiction.

3. x_1 is adjacent to x_{10}. By **1** and **2**, x_1 has one more neighbor only in $\{x_4, x_5, x_7, x_8\}$. In this case, x_9 is an endvertex of a Hamilton path of H to x_{14}. Clearly, by applying (Q_2), we have the following claim.

(F) x_{12} has no neighbor in the cycle $x_1x_2 \cdots x_{10}x_1$.

Then, x_9, x_{13} are not adjacent, for otherwise x_{12} is an endvertex of a Hamilton path of H to x_{14}, and thus $N(x_{12}) \subseteq V(C)$, as $\delta(G) \geq 4$, contradicting (F). Hence, by noting that x_{13}, x_1 are symmetrical, x_{13} has extra neighbors only in $\{x_{10}, x_7, x_6, x_4\}$. Clearly, by (Q_1), x_9 is not adjacent to x_{12}. And also x_9 is not adjacent to x_{11}, for otherwise x_{12} is an endvertex of a Hamilton path of H to x_{14}, contradicting (F). So, by (Q_1), x_9 has extra neighbor only in $\{x_2, x_3, x_5, x_6, x_{14}\}$. In the following we distinguish four subcases.

(3.1) x_1 is adjacent to x_4. Then, x_3 is an endvertex of a Hamilton path of H to x_{14}. As x_1 dominates $\{x_2, x_4, x_{10}\}$, by (Q_3) and (Q_1) and above result, x_3 has extra neighbors only in $\{x_7, x_{10}, x_{11}, x_{14}\}$. If x_3 is adjacent to x_7, we apply (Q_2) to $x_8x_9x_{10}$ and the cycle $x_1x_2x_3x_7 \cdots x_4x_1$ to obtain a contradiction. If x_3 is adjacent to both x_{11}, x_{14}, then it is the same situation as **2**, a contradiction. If x_3 is adjacent to both x_{10}, x_{11}, then x_2 is an endvertex of a Hamilton path of H to x_{14}. Thus, we can similarly deduce that x_2 has extra neighbors only in $\{x_5, x_8, x_{11}, x_{14}\}$, and hence $\{x_5, x_8, x_{11}, x_{14}\}$ dominates $V(C) - \{x_{14}\}$, a contradiction. So, x_3 must be adjacent to both x_{10}, x_{14}. Next we check the neighbors of x_9.

By (Q_2), x_9 has no neighbors in the cycle $x_1x_2x_3x_4x_1$. Then, x_9 has extra neighbors only in $\{x_5, x_6, x_{14}\}$. First assume that x_9 is adjacent to x_5. Then, x_{13} is not adjacent to x_7, for otherwise $\{x_2, x_5, x_7, x_{11}\}$ dominates $V(C) - \{x_{14}\}$, a contradiction. Moreover, if x_{13} is adjacent to x_6, then

$x_{12}x_{11}x_{10}x_1\cdots x_5x_9\cdots x_6\ x_{13}x_{14}$ is a Hamilton path of H, contradict (F). Hence, in this case, x_{13} is adjacent to both x_4, x_{10}. Note that

$$x_6\cdots x_9x_5\cdots x_1x_{10}\cdots x_{14}$$

is a Hamilton path of H, we have $N(x_6)\subseteq V(C)$. We can similarly deduce that x_6 has extra neighbors only in $\{x_9, x_{10}, x_{14}\}$. If x_6 is adjacent to x_9, then $x_7x_8x_9x_6\cdots x_1x_{10}\cdots x_{14}$ is a Hamilton path of H; if x_6 is adjacent to x_{10}, then $x_7x_8x_9x_5x_6x_{10}\cdots x_{13}x_4\ x_3\cdots x_{14}$ is a Hamilton path of H. Thus, $N(x_7)\subseteq V(C)$. Now we apply (Q_2) to the cycle $x_1\cdots x_5x_9x_{10}x_1$ and the cycle $x_{10}\cdots x_{13}\ x_{10}$ to deduce that x_7 has only one extra neighbor x_{14}, a contradiction.

Hence, x_9, x_5 are not adjacent, and thus x_9 is adjacent to both x_6, x_{14}. Now if x_{13} is adjacent to x_7, then $x_{12}x_{11}x_{10}x_1\cdots x_6x_9x_8x_7x_{13}x_{14}$ is a Hamilton path of H, contradict (F). So, x_{13} has extra neighbors only in $\{x_4, x_6, x_{10}\}$. If x_{13} is adjacent to both $\{x_4, x_{10}\}$, by symmetry, we can similarly deduce that x_5 is adjacent to both x_8, x_{14}, and thus $\{x_3, x_7, x_{11}, x_{14}\}$ dominates $V(C)-\{x_{14}\}$, a contradiction. If x_{13} is adjacent to both $\{x_4, x_6\}$, by noting that x_{13} dominates $\{x_4, x_6, x_{12}\}$ and that both x_9, x_{12} are not adjacent to x_5, we can deduce that x_5 has only one extra neighbor x_{14}, a contradiction. If x_{13} is adjacent to both $\{x_6, x_{10}\}$, then $x_5\cdots x_1x_{10}\cdots x_{13}x_6\cdots x_9x_{14}$ is a Hamilton path of H. By (Q_1), x_5 is not adjacent to x_8. If x_5 is adjacent to x_7 or x_{14}, then $\{x_3, x_7, x_{11}, x_{14}\}$ dominates $V(C)-\{x_{14}\}$, a contradiction. As x_5 is not adjacent to x_9, then x_5 has a neighbor in the cycle $C_1' := x_1x_2x_3x_{10}x_1$ or the cycle $C_2' := x_{10}x_{11}x_{12}x_{13}x_{10}$. By applying (Q_2) to $x_4x_5x_6$ and C_1', or to $x_4x_5x_6$ and C_2', we have a contradiction.

By (3.1) and the symmetry of x_1 and x_{13}, x_{13} is also not adjacent to both x_4 and x_{10}.

(3.2) x_1 is adjacent to x_5. In this case, if either x_{13} is adjacent to x_4, or x_{13} is adjacent to both x_6, x_7, then x_{12} is an endvertex of a Hamilton path of H to x_{14}, contradict (F). So, we only need check for that x_{13} is adjacent to both x_6, x_{10}, or to both x_7, x_{10}.

First, let x_{13} be adjacent to both x_6, x_{10}. Then, x_{11} is an endvertex of a Hamilton path of H to x_{14}. As x_{13} dominates $\{x_6, x_{10}, x_{12}\}$, if x_{11} has one neighbor in the cycle $x_1\cdots x_5x_1$, then we can easily find four vertices to dominate $V(C)-\{x_{14}\}$. On the other hand, if x_{11} is adjacent to x_7, then $x_{12}x_{11}x_7\cdots x_{10}x_{13}x_6\cdots x_1x_{14}$ is a Hamilton path of H, contradict (F). Clearly, x_{11} is not adjacent to x_8, x_9, x_{13}, and thus x_{11} is adjacent to both x_6, x_{14}. Then, $x_2\cdots x_5x_1x_{10}\cdots x_6x_{11}\cdots x_{14}$ is a Hamilton path of H. By (Q_2), x_2 has no neighbor in the cycle $C_3' := x_{10}\cdots x_{13}x_{10}$. If x_2 is adjacent to any vertex in $\{x_5, x_6, x_9\}$, then x_3 is an endvertex of a Hamilton path of H to x_{14}. Thus, by applying (Q_2) to $x_2x_3x_4$ the cycle $x_1x_5\cdots x_{10}x_1$ and C_3', we have a contradiction. Otherwise, by (Q_1), x_2 is adjacent to x_8. Then $\{x_4, x_8, x_{10}, x_{13}\}$ dominates $V(C)-\{x_{14}\}$, a contradiction.

Secondly, let x_{13} be adjacent to both x_7, x_{10}. Then, x_8 is an endvertex of a Hamilton path of H to x_{14}. By applying (Q_2) to $x_7x_8x_9$ and the cycle C_3', we have that x_8 has no neighbor in C_3'. As x_{10} dominates x_1, x_9, x_{11}, we apply (Q_3) to x_8 and the path $x_{12}x_{13}x_7 \cdots x_2$, then x_8 has extra neighbors only in $\{x_{14}, x_4, x_1\}$. Then, x_8 is adjacent to x_1 or x_4, and thus, x_{12} is an endvertex of a Hamilton path of H to x_{14}, contradict (F).

(3.3) By above, x_1 is adjacent to x_7 or x_8. If x_1 is adjacent to x_7, then we apply (Q_2) to $x_8x_9x_{10}$ and the cycle $x_1 \cdots x_7x_1$ to deduce a contradiction. If x_1 is adjacent to x_8, then, by (F) and (Q_3), we can deduce that x_{13} is adjacent to both x_4, x_{10}, as above claim, also a contradiction.

4. x_1 is adjacent to x_8. From **1-3** and the symmetry, x_{13} has extra neighbors only in $\{x_6, x_7, x_9, x_{10}\}$. If x_{13} is adjacent to x_6, then $\{x_3, x_6, x_8, x_{11}\}$ dominates $V(C) - \{x_{14}\}$, a contradiction. Otherwise, as x_{13} is not adjacent to both x_9, x_{10}, then x_{13} is adjacent to x_7, then x_2 is an endvertex of a Hamilton path of H to x_{14}. By applying $x_1x_2x_3$ and the cycle $x_7 \cdots x_{13}x_7$, we know that x_7 has no neighbor in this cycle. If x_2 is adjacent to both x_5, x_6 Then, x_4 is an endvertex of a Hamilton path H to x_{14}, by (Q_2) (for $x_3x_4x_5$ and the cycle $x_1x_2x_6x_7x_{13} \cdots x_8x_1$), we have a contradiction. Otherwise, x_2 must be adjacent to x_{14}, this is the same situation as **1**, a contradiction.

5. From **1-4** and the fact x_1 is not adjacent to both x_4, x_5, we have that x_1 is adjacent to x_7. Symmetrically, x_{13} is also adjacent to x_7. On the other hand, x_1 has one more neighbor in $\{x_4, x_5\}$. First let x_1 be adjacent to x_4. As x_6 is an endvertex of a Hamilton path of H to x_{14}, we can similarly use (Q_1) and (Q_2) to deduce that, x_6 has extra neighbors only in $\{x_8, x_{10}, x_{11}, x_{14}\}$. As the former cases shown, x_6 is not adjacent to x_{14}. Thus, x_6 is adjacent to either x_8, or both x_{10}, x_{11}. If x_6 is adjacent to x_8, then x_{12} is an endvertex of a Hamilton path of H to x_{14}, and thus, by (Q_2) and (Q_1) and **1** shown, x_{12} is adjacent to both x_8, x_9. Then, x_{10} is an endvertex of a Hamilton path of H to x_{14}. Again by Q_2 for $x_9x_{10}x_{11}$ and the cycle $x_1 \cdots x_8x_{12}x_{13}x_1$, we have a contradiction. Otherwise, x_6 is adjacent to both x_{10}, x_{11}. Then, x_9 is an endvertex of a Hamilton path of H to x_{14}. By (Q_2) for $x_8x_9x_{10}$ and the cycle $x_1 \cdots x_6x_{11}x_{12}x_{13}x_7x_1$, we also have a contradiction.

Hence, x_1 is adjacent to x_5, and by symmetry, x_{13} is adjacent to x_9. As x_6 is an endvertex of a Hamilton path of H to x_{14}, by (Q_1), x_6 has extra neighbors only in $\{x_2, x_3, x_8, x_{10}, x_{11}, x_{14}\}$. If x_6 is adjacent to x_3 or x_{11}, then $\{x_1, x_3, x_9, x_{12}\}$ or $\{x_1, x_4, x_9, x_{11}\}$ dominates $V(C) - \{x_{14}\}$, a contradiction. If x_6 is adjacent to x_2, as x_4 is an endvertex of a Hamilton path of H to x_{14}, by (Q_2) for $x_3x_4x_5$ and the cycles $x_1x_2x_6x_7x_1$ and $x_7 \cdots x_{13}x_7$, we have a contradiction. Hence, x_6 has extra neighbors only in $\{x_8, x_{10}, x_{14}\}$, and thus, by the former cases shown, x_6 is adjacent to both x_8, x_{10}. Then, x_{12} is an endvertex of a Hamilton path of H to x_{14}, then, by (Q_2) for $x_{11}x_{12}x_{13}$ and the cycle $x_1 \cdots x_6x_{10} \cdots x_7x_1$, we also have a contradiction. This proves Lemma 2. □

A lasso L is defined as a graph by identifying one vertex in a cycle C with an endvertex of a path P. The other endvertex of the path P is called the end of L, the common vertex of C and P is called the connecting vertex (sometimes, with a little abuse, we also regard a cycle as a lasso). Now we use Lemma 1 and Lemma 2 to deduce the results we need.

Lemma 3. *Let $P \in S$ be a 2-path with at most one out-endvertex. If $|P| \leq 8$, then $V(P)$ has a subset of $\lfloor \frac{|P|}{3} \rfloor$ vertices which dominate all vertices of $V(P)$ except for the possible out-endvertex.*

Proof. Clearly, Lemma 3 implies Assertion 2. If $|P| = 2$, then P has two out-endvertices. If $|P| = 5$ and P has at most one out-endvertex, it is also easy to verify that P can be dominated by one vertex. Now let $P = x_1x_2 \cdots x_8$. Let H be the subgraph induced by $V(P)$. If $|G| = 8$, then $G = H$. As each vertex of G has degree at least four, we can verify directly that H can be dominated by two vertices. So, let $|G| > 8$. We first claim that P has one out-endvertex. Otherwise, both x_1 and x_8 have at least four neighbors in $V(P)$, from that it is easy to see that H has a Hamilton cycle. As G is connected, there exists at least one edge joining $V(P)$ and $V(G) - V(P)$. By our choices to maximize the number of the out-endvertices of S. This is a contradiction. Hence, P has precisely one out-endvertex.

Let x_8 be the out-endvertex of P. We choose a lasso L with x_8 as the end of L, such that the cycle of L has maximum length. For convenience, we denote the vertices of L along a Hamilton path of L from the end as $x_8, x_7, \ldots, x_1$. Let v be the connecting vertex. Clearly, x_1 is adjacent to v. Let $C' = x_1x_2 \cdots vx_1$. By (Q_1), $v = x_{3k+1}$ or $v = x_{3k+2}$ $(1 \leq k \leq 2)$. By the choice of L, C' satisfies the condition of Lemma 1 or Lemma 2. By Lemma 1 or Lemma 2, we can deduce that $V(P) - \{x_8\}$ can be dominated by two vertices. □

Lemma 4. *Let $P \in S$ be a 1-path with no out-endvertex. If $|P| \leq 19$, then P can be dominated by $\lfloor \frac{|P|}{3} \rfloor$ vertices.*

Proof. We first prove that, if $|G| = |P| = 19$ and G has a Hamilton cycle, then G has a dominating set of 8 vertices. Let $C = b_1b_2 \cdots b_{19}b_1$ be a Hamilton cycle of G. Assume the conclusion is not true. By (Q_1), b_1 is not adjacent to b_{3k} $(1 \leq k \leq 8)$. Note that b_1 is adjacent to b_5, for otherwise, by applying (Q_2) for $b_2b_3b_4$ and the cycle $b_1b_5 \cdots b_{19}b_1$, we have a contradiction. Similarly, b_3 is also not adjacent to any of b_5, b_7. If b_1 is adjacent to b_8, then, by (Q_2) for $b_2b_3b_4$ and the cycle $b_1b_8 \cdots b_{19}b_1$, b_3 has no neighbor in this cycle, and thus, b_3 must be b_5 or b_7, a contradiction. So, b_1 is also not adjacent to b_8. By symmetry, b_1 is not adjacent to any of x_{16}, x_{13}. Clearly, for each $2 \leq i \leq 19$, b_i has the similar property as b_1. Now, if b_1 is adjacent to b_4, as b_1 dominates b_2, b_4, b_{19}, by (Q_3), b_3 has extra neighbors only in $\{b_{13}, b_{16}, b_{19}\}$; as b_4 dominates b_1, b_3, b_5, by (Q_3), b_2 has extra neighbors only in $\{b_5, b_8, b_{11}\}$. Since $\delta(G) \geq 4$, $\{b_5, b_8, b_{11}, b_{13}, b_{16}, b_{19}\}$ dominates $V(G)$, a contradiction. Hence, b_1 is also not adjacent to b_4, symmetrically, not adjacent to b_{17}. And hence, b_1 is not adjacent to b_{11}, for otherwise, we look at b_6, as b_6 has similar properties as b_1, then b_6 must have neighbors in the cycle $b_1 b_{11} \cdots b_{19} b_1$, and thus, by applying (Q_2) for $b_5 b_6 b_7$ and this cycle, we have a contradiction. Symmetrically, b_1 is not adjacent to b_{10}. So, b_1 is adjacent to both x_7, x_{14}. Similarly, b_3 is adjacent to both b_9, b_{16}. Then, $\{b_1, b_5, b_9, b_{12}, b_{15}, b_{18}\}$ dominates $V(G)$, a contradiction. So, in this case, G has a dominating set of 8 vertices.

Next we assume that either $|G| > 19$ or G has no hamiltonian cycle. Let $|P| = 3m + 1$ $(1 \le m \le 6)$. Let H be the subgraph induced by $V(P)$. Now we claim that H has no Hamilton cycle. For otherwise, if $|G| > |P|$, as G is connected, then P has at least one out-endvertex, a contradiction; if $|G| = |P| = 19$, then $G = H$, contradicting that G has no Hamilton cycle. Now we choose a lasso L in H such that the number of vertices on the cycle of the lasso is maximum. For convenience, we label the vertices of L along a Hamilton path on L from the end of L as $x_{3m+1}, x_{3m}, \ldots, x_1$. Let v be the connecting vertex. By the labelling, x_1 is adjacent to v. By (Q_1) and our assumption, we may assume that $v = x_{3k+1}$ or x_{3k+2} $(1 \le k < m)$. If $k \le 4$, then, by the choice of L, the cycle $C' = x_1 x_2 \cdots v x_1$ satisfies the condition of Lemma 1 or Lemma 2, and thus we can obtain the desired result by Lemma 2 or Lemma 3. So, we next let $k \ge 5$. Hence, $m = 6, k = 5$ and $|P| = 19$. We still prove by contradiction.

Case 1. $v = x_{3k+1} = x_{16}$. Denote the cycle $C' := x_1 x_2 \cdots x_{16} x_1$. By the choice of L, $N(x_1) \subseteq V(C')$ and $N(x_{15}) \subseteq V(C')$. We can similarly as in the proof of Lemma 1 deduce the following.

(F) x_1 is not adjacent to both x_4, x_5; x_{15} is not adjacent to both x_9, x_{10}.

By (Q_2), x_1 is not adjacent to x_{13}. If x_1 is adjacent to x_{10}, then, by (Q_2), x_{15} has no neighbor in the cycle $x_1 \cdots x_{10} x_1$, and thus, by (Q_1), x_{15} is adjacent to both x_9, x_{10}, a contradiction. So, x_1 is also not adjacent to x_{10}. Symmetrically, x_{15} is not adjacent to any of x_4, x_7. Now, we check the possible neighbors of x_{19}.

Clearly, $N(x_{19}) \subseteq V(P)$. Moreover, by the choice of L and (Q_1), x_{19} has extra neighbors only in $\{x_4, x_7, x_9, x_{12}, x_{16}\} \cup \{x_6, x_{10}\}$. If x_{19} is adjacent to x_6, we look at the Hamilton path$x_1 \cdots x_6 x_{19} \cdots x_7$ of H, by the choice of L and (Q_1), we can deduce that x_1 is adjacent to both x_4, x_5, contradict (F). Note that x_6 and x_{10} are symmetrical, we have x_{19} is not adjacent to any of x_6, x_{10}. If x_{19} is adjacent to both x_4, x_9, then we can similarly deduce that x_1 must be adjacent to both x_4, x_8, and thus $x_2 x_3 \cdots x_1 x_8 x_{16} x_{15} \cdots x_9 x_{19} x_{18} x_{17}$ is a hamiltonian path of H. By applying (Q_2) for $x_1 x_2 x_3$ and the cycle $x_4 \cdots x_{19} x_4$, we have a contradiction. Hence, x_{19} is adjacent to at most one of x_4, x_9, and symmetrically, is adjacent to at most one of x_7, x_{12}. So, x_{19}, x_{16} are adjacent. Thus, x_{17} is also an endvertex of a hamiltonian path of H. Then, x_{17}, x_{19} have the same properties.

Note that x_{19} must be adjacent to one of x_4, x_9, and one of x_7, x_{12}. First let x_{19} be adjacent to x_4. As x_{17} is also adjacent to one of x_7, x_{12}, if x_{17} is adjacent to x_4, then $\{x_2, x_6, x_9, x_{12}, x_{15}, x_{19}\}$ dominates P; if x_{17} is adjacent to x_7, then $x_7 \cdots x_{16}x_1 \cdots x_4x_{19}x_{18}x_{17}x_7$ is a cycle of H which exclude x_5, x_6, contradict the choice of L. Then, by symmetry, x_{19} is adjacent to both x_7, x_9. Also, x_{17} is adjacent to these two vertices. Hence, $x_1 \cdots x_7x_{19}x_{18}x_{17}x_9 \cdots x_{16}x_1$ is a cycle of H which exclude x_8, contradict the choice of L.

Case 2. $v = x_{3k+2} = x_{17}$. By (Q_1) and the choice of L, x_{19} is not adjacent to any of x_{17}, x_1, x_{16}. As x_{17} dominates x_{18}, x_1, x_{16}, by (Q_3), x_{19} has extra neighbors only in $\{x_4, x_7, x_{10}, x_{13}\}$. Since x_{19} is adjacent to three of them, by symmetry, we assume that x_{19} is adjacent to both x_4, x_7. Then, x_5 is an endvertex of a Hamilton path of H, and thus $N(x_5) \subseteq V(P)$. As $x_7 \cdots x_{19}x_7$ is a 13-cycle, by looking at $x_4x_5x_6$ and this cycle, we deduce that x_5 has no neighbor in this cycle. So, x_5 has extra neighbors only in $\{x_1, x_2, x_3\}$. If x_5 is adjacent to x_3, then $\{x_5, x_1, x_{15}, x_{12}, x_9, x_{19}\}$ dominates P; otherwise x_5 is adjacent to both x_1, x_2, then $\{x_4, x_5, x_8, x_{11}, x_{14}, x_{17}\}$ dominates P, a contradiction. This proves Lemma 4. □

Lemma 5. *Let $T \in E'$ be a tip of a 2-path P in A. If $|T| \leq 13$, then T can be dominated by $\lfloor \frac{|T|}{3} \rfloor$ vertices.*

Proof. Let $T = a_0 \cdots a_{3m+1} \in E'$ $(m \leq 4)$ and $C = c_0 \cdots c_l$ be the central path of P. Assume that c_0 is adjacent to a_{3m+1} on the path P. By definition, c_1 is an acceptor or inacceptor. As $T \in E'$, neither endpoint of P is $(2, 2)$-endpoint, and thus c_1 is an acceptor. We first present a claim (for the proof, see [11, p. 285, Fact 11]).

Claim. a_0 is only adjacent to the vertices of $T \cup \{c_0\}$.

By the choice of S and the claim, if $a'_0 \cdots a'_{3m+1}$ is a Hamilton path on $V(T)$ such that a'_{3m+1} is adjacent to c_0, then a'_0 is also only adjacent to the vertices of $T \cup \{c_0\}$.

Let H be the subgraph induced by $V(T) \cup \{c_0\}$. We choose a lasso L with c_0 as the end of L, such that the cycle of L has maximum length. For convenience, we denote the vertices of L along a hamiltonian path of L from c_0 as $x_{3m+2}, x_{3m+1}, \ldots, x_1$. Let v be the connecting vertex. Clearly, x_1 is adjacent to v. Let $C' = x_1x_2 \cdots vx_1$. By (Q_1), $v = x_{3k+1}$ or $v = x_{3k+2}$ $(1 \leq k \leq m \leq 4)$. By the choice of L, C' satisfies the conditions of Lemma 1 or Lemma 2. By Lemma 1 or Lemma 2, we can deduce the desired result. □

Proof of Main Theorem. By Lemma 4, $|P| \geq 22$ for any path P in I_1. By Lemma 3, $|P| \geq 11$ for any path P in I_2. Hence,

$$\sum_{P \in I_1} |P| \geq 22|I_1|; \ \sum_{P \in I_2} |P| \geq 11|I_2|.$$

By Lemma 5,

$$\sum_{P\in A} |P| \geq 17|E'|.$$

From that we have,

$$n \geq \sum_{P\in I_1} |P| + \sum_{P\in I_2} |P| + \sum_{P\in A} |P| \geq 22|I_1| + 11|I_2| + 17|E'|,$$

yielding, $\frac{n}{33} \geq \frac{2}{3}|I_1| + \frac{1}{3}|I_2| + \frac{1}{2}|E'|$. Combining with $(*)$, we have $|D| \leq \frac{4}{11}n$. This proves Main Theorem. □

Remark. We have verified by the same method, that the conjecture for the remaining two cases $k = 5, 6$ is also true.

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