J. Korean Math. Soc. **46** (2009), No. 4, pp. 747–758
DOI 10.4134/JKMS.2009.46.4.747

GROWTH AND FIXED POINTS OF MEROMORPHIC SOLUTIONS OF HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the growth and fixed points of meromorphic solutions of higher order linear differential equations with meromorphic coefficients and their derivatives. Because of the restriction of differential equations, we obtain that the properties of fixed points of meromorphic solutions of higher order linear differential equations with meromorphic coefficients and their derivatives are more interesting than that of general transcendental meromorphic functions. Our results extend the previous results due to M. Frei, M. Ozawa, G. Gundersen, and J. K. Langley and Z. Chen and K. Shon.

1. Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions (see [13, 21]). The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$. In addition, we will use notations $\sigma(f)$ to denote the order of growth of a meromorphic function $f(z)$, $\lambda(f)$ to denote the exponents of convergence of the zero-sequence of a meromorphic function $f(z)$, $\overline{\lambda}(f)$ to denote the exponents of convergence of the sequence of distinct zeros of $f(z)$.

In order to give some estimates of fixed points, we recall the following definitions (see [3, 16]).

Definition 1.1. Let $z_1, z_2, \ldots, (|z_j| = r_j, 0 \leq r_1 \leq r_2 \leq \cdots)$ be the sequence of distinct fixed points of transcendental meromorphic function f. Then $\overline{\tau}(f)$, the exponent of convergence of the sequence of distinct fixed points of f, is

Received September 26, 2007.
2000 *Mathematics Subject Classification.* Primary 34M10, 30D35.
Key words and phrases. linear differential equation, meromorphic function, fixed point.
This work was supported by NSF of China(No.10771121), NSFC-RFBR and SRFDP (No.20060422049).

defined by

$$\overline{\tau}(f) = \inf\{\tau > 0 \mid \sum_{j=1}^{\infty} |z_j|^{-\tau} < +\infty\}.$$

It is evident that $\overline{\tau}(f) = \overline{\lim\limits_{r\to\infty}} \frac{\log \overline{N}(r, \frac{1}{f-z})}{\log r}$ and $\overline{\tau}(f) = \overline{\lambda}(f-z)$.

For the second order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1}$$

where $B(z)$ is an entire function of finite order, it is well known that each solution f of (1) is an entire function. If f_1 and f_2 are any two linearly independent solutions of (1), then at least one of f_1, f_2 must have infinite order ([14]). Hence, "most" solutions of (1) will have infinite order.

Thus a natural question is: what condition on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1) will have infinite order? Frei, Ozawa, Amemiya and Langley, and Gundersen studied the question. For the case that $B(z)$ is a transcendental entire function, Gundersen [10] proved that if $\rho(B) \neq 1$, then for every solution $f \not\equiv 0$ of (1) has infinite order.

For the above question, there are many results for second order linear differential equations (see for example [1, 2, 7, 8, 12, 17]). In 2002, Chen considered the problem and obtained the following result in [2].

Theorem A. *Let a, b be nonzero complex numbers and $a \neq b$, $Q(z) \not\equiv 0$ be a nonconstant polynomial or $Q(z) = h(z)e^{bz}$, where $h(z)$ is a nonzero polynomial. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + e^{bz} f' + Q(z) f = 0$$

has infinite order.

In 2005, Chen [5] investigated the more general equation with meromorphic coefficients, and obtained the following result.

Theorem B. *Let $A_j(z)(\not\equiv 0)$ $(j = 0, 1)$ be meromorphic functions with $\sigma(A_j) < 1$, a, b be nonzero complex numbers and* $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$. *Then every solution $f \not\equiv 0$ of the equation*

$$f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f = 0 \tag{2}$$

has infinite order.

In this paper, we continue the research in the direction and obtain the following result which greatly extends the previous results of M. Frei, M. Ozawa, G. Gundersen, and J. K. Langley and Z. Chen and K. Shon.

Theorem 1.1. *Suppose that $A_j \not\equiv 0$ $(j = 0, 1, \ldots, k-1)$ be meromorphic functions with $\sigma(A_j) < 1$ $(j = 0, 1, \ldots, k-1)$. Let $a_0, a_1, \ldots, a_{k-1}$ be nonzero complex constants such that for* (i) $\arg a_j = \arg a_0$ *and* $a_j = c_j a_0$ $(0 < c_j < 1)$

or (ii) $\arg a_j \neq \arg a_0$ $(j = 0, 1, \ldots, k-1)$. *Then for* $k \geq 2$, *every transcendental meromorphic solution* $f(\not\equiv 0)$ *of the equation*

$$f^{(k)} + A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \cdots + A_1 e^{a_1 z}f' + A_0 e^{a_0 z}f = 0. \tag{3}$$

have infinite order.

Remark 1.2. In (i), if $c_j = c\,(0 < c < 1)$, then (i) becomes $a_j = ca_0 \mod 2\pi$, $j = 1, 2, \ldots, k-1$. Obviously, Theorem 1.1 generalizes Theorem B to the high order differential equation and ([6]), Theorem 1.5 from the entire coefficients to meromorphic ones.

Since the beginning of the last four decades, a substantial number of research articles have been written to describe the fixed points of general transcendental meromorphic functions (see [23]). However, there are few studies on the fixed points of solutions of the general differential equation. In [3], Z. X. Chen first studied the problems on the fixed points of solutions of second order linear differential equations with entire coefficients. Since then, Wang and Yi [20, 19], Laine and J. Rieppo [16], Chen and Shon [5] studied the problems on the fixed points of solutions of second order linear differential equations with meromorphic coefficients and their derivatives. The other main purpose of this paper is to extend some results in [5] to the case of higher order linear differential equations with meromorphic coefficients.

Theorem C. *Let* $A_j(z), a, b, c$ *satisfy the additional hypotheses of Theorem 1.1. If* $f \not\equiv 0$ *is any meromorphic solution of the equation* (2), *then* f, f', f'' *all have infinitely fixed points and satisfy*

$$\bar{\tau}(f) = \bar{\tau}(f') = \bar{\tau}(f'') = \infty.$$

Remark 1.3. In the proof of Theorem C, the authors gave an important lemma, see [5], Lemma 7, to prove the conclusion. However it seems too complicated to deal with the high differential equations. In this paper, we use the Lemma 2.1 in Section 2 to solve the difficulty easily.

Theorem 1.4. *Let* $A_j(z), a_j, c_j$ *satisfy the additional hypotheses of Theorem 1.1. If* $f \not\equiv 0$ *is any meromorphic solution of the equation* (3), *then* f, f', f'' *all have infinitely fixed points and satisfy*

$$\bar{\tau}(f) = \bar{\tau}(f') = \bar{\tau}(f'') = \infty.$$

2. Lemmas

The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$. The logarithmic measure of a set $E \subset [1, +\infty)$ is defined by

$$lm(E) = \int_1^{+\infty} \chi_E(t)/t\,dt,$$

where $\chi_E(t)$ is the characteristic function of E. The upper and lower densities of E are

$$\overline{\text{dens}}E = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r}, \quad \underline{\text{dens}}E = \liminf_{r\to+\infty} \frac{m(E\cap[0,r])}{r}.$$

The following lemma, due to Gross [9], is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well.

Lemma 2.1 ([9, 22])**.** *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.
(ii) *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.*
(iii) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$ $(r \to \infty, r \notin E)$.*

Then $f_j(z) \equiv 0\,(j = 1, 2, \dots, n)$.

Lemma 2.2 ([11])**.** *Let f be a transcendental meromorphic function of finite order σ. Let $\varepsilon > 0$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold*:

(a) *There exists a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, such that for all z satisfying $|z| \notin E_1 \bigcup [0, 1]$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{4}$$

(b) *There exists a set $E_2 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_2$, then there is a constant $R = R(\theta) > 0$ such that* (4) *holds for all z satisfying* $\arg z = \theta$ *and $R \leq |z|$.*

Lemma 2.3. *Let $f(z) = g(z)/d(z)$, where $g(z)$ is transcendental entire, and let $d(z)$ be the canonical product* (*or polynomial*) *formed with the non-zero poles of $f(z)$. Then we have*

$$f^{(n)} = \frac{1}{d}\left[g^{(i)} + B_{i,i-1}g^{(k-1)} + \cdots + B_{i,1}g' + B_{i,0}g\right],$$

where $B_{i,j}$ are defined as a sum of a finite number of terms of the type

$$\sum_{(j_1\cdots j_i)} C_{jj_1\cdots j_i}\left(\frac{d'}{d}\right)^{j_1}\cdots\left(\frac{d^{(i)}}{d}\right)^{j_i},$$

$C_{jj_1\cdots j_i}$ are constants, and $j + j_1 + 2j_2 + \cdots + ij_i = n$.

Using mathematical induction, we can easily prove the lemma.

Lemma 2.4 ([2])**.** *Let $g(z)$ be a meormorphic function with $\sigma(g) = \beta < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi)\backslash E$, then there is a constant $R = R(\psi) > 1$ such that, for all z satisfying $\arg z = \psi$ and $|z| = r > R$, we have*

$$\exp\{-r^{\beta+\varepsilon}\} \le |g(z)| \le \exp\{r^{\beta+\varepsilon}\}.$$

Lemma 2.5 ([18])**.** *Consider $g(z) = A(z)e^{az}$, where $A(z)(\not\equiv 0)$ is a meromorphic function with $\sigma(A) = \alpha < 1$, a is a complex constant, $a = |a|e^{i\varphi}(\varphi \in [0, 2\pi))$. Set $E_0 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$, then E_0 is a finite set. Then for any given ε $(0 < \varepsilon < 1 - \alpha)$, there is a set $E_1 \in [0, 2\pi)$ that has linear measure zero, if $z = re^{i\theta}$, $\theta\backslash(E_0 \bigcup E_1)$, then we have when r is sufficiently large*:

(i) *If $\cos(\varphi + \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)r\delta(az, \theta))\} \le |g(z)| \le \exp\{(1 + \varepsilon)r\delta(az, \theta))\};$$

(ii) *If $\cos(\varphi + \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)r\delta(az, \theta))\} \le |g(z)| \le \exp\{(1 - \varepsilon)r\delta(az, \theta))\};$$

where $\delta(az, \theta) = |a| \cos(\varphi + \theta)$.

Lemma 2.6 ([4])**.** *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ are finite order meromorphic function. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = F,$$

then f satisfies $\lambda(f) = \overline{\lambda}(f) = \sigma(f) = \infty$.

3. Proof of Theorem 1.1

First of all we prove that the equation (3) can't have a meromorphic solution $f \not\equiv 0$ with $\sigma(f) < 1$. Assume a meromorphic function $f \not\equiv 0$ with $\sigma(f) = \sigma_1 < 1$ satisfies the equation (3). Then $\sigma(f^{(j)}) = \sigma_1 < 1$ $(j = 1, 2, \ldots, k-1)$. By Lemma 2.4, for any given ε_1 $(0 < 3\varepsilon_1 < \min\{1 - \sigma_1, \frac{1-c}{2}\})$, $c = \max_{1\le j\le k-1}\{c_j\}$, there is a set $E_1 \in [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi)\backslash E_1$, then there is a constant $R > 1$, such that for all $\arg z = \theta$ and $|z| = r > R$, we have

$$|f^{(j)}| \le \exp\{r^{\sigma_1+\varepsilon_1}\}. \tag{5}$$

If $\arg a_j \ne \arg a_0$ $(j = 1, 2, \ldots, k-1)$, then from Lemma 2.5 and $\sigma(A_jf^{(j)}) < 1$ $(j = 0, 1, \ldots, k-1)$, we know that for the above ε_1, there is a ray $\arg z = \theta_0 \in [0, 2\pi)\backslash(E_1 \bigcup E_2 \bigcup E_0)$, where $E_2 \in [0, 2\pi)$ that has linear measure zero,

$$E_0 = \{\theta \in [0, 2\pi) : \delta(a_jz, \theta) = 0 (j \ne 0) \text{ or } \delta(a_0z, \theta) = 0\},$$

where $\delta(a_jz, \theta) = |a_j| \cos(\arg a_j + \theta_0)$ $(j \ne 0)$, $\delta(a_0z, \theta) = |a_0| \cos(\arg a_0 + \theta_0)$, such that $\text{Re}\{a_jz\} = \delta(a_jz, \theta_0)r < 0$, $\text{Re}\{a_0z\} = \delta(a_0z, \theta_0)r > 0$. For a sufficiently large r, combining with (5) we have

$$|A_0(re^{i\theta_0})e^{a_0re^{i\theta_0}}f(re^{i\theta_0})| \ge \exp\{(1 - \varepsilon_1)\delta(a_jz, \theta_0)r\}, \tag{6}$$

(7)

$$\begin{aligned}&\left|f^{(k)}(re^{i\theta_0})+A_{k-1}(re^{i\theta_0})e^{a_{k-1}re^{i\theta_0}}f^{(k-1)}(re^{i\theta_0})+\cdots+A_1(r_1e^{i\theta_0})e^{a_1re^{i\theta_0}}f'(re^{i\theta_0})\right|\\ \le&\ \exp\{r^{\sigma_1+\varepsilon_1}\}+\sum_{j=1}^{k-1}\exp\{(1-\varepsilon_1)\delta(a_jz,\theta_0)r\}\\ \le&\ \exp\{r^{\sigma_1+\varepsilon_1}\}+1,\end{aligned}$$

By (3), (6), and (7), we have

$$\exp\{(1-\varepsilon_1)\delta(a_jz,\theta_0)r\}\le\exp\{r^{\sigma_1+\varepsilon_1}\}+1.$$

This is absurd by $\sigma_1+\varepsilon_1<1$.

If $\arg a_j=\arg a_0$, and $a_j=c_ja_0$ $(0<c_j<1)$, then $\delta(a_jz,\theta)=c_j\delta(a_0z,\theta)$ for $z=re^{i\theta}$. Using the same reasoning as above, we know that there is a ray $\arg z=\theta_0\in[0,2\pi)\backslash(E_1\bigcup E_2\bigcup E_0)$ satisfying $\delta(a_jz,\theta_0)=c_j\delta(a_0z,\theta_0)>0$, and for the above ε_1 and a sufficiently large r, we have

(8)

$$\begin{aligned}\exp\{(1-\varepsilon_1)\delta(a_jz,\theta_0)r\}&\le|A_0(re^{i\theta_0})e^{a_0re^{i\theta_0}}f(r_0e^{i\theta_0})|\\ &\le|f^{(k)}+A_{k-1}(re^{i\theta_0})e^{a_{k-1}re^{i\theta_0}}f^{(k-1)}(re^{i\theta_0})+\cdots\\ &\quad+A_1(re^{i\theta_0})e^{a_1re^{i\theta_0}}f'(re^{i\theta_0})|\\ &\le\exp\{r^{\sigma_1+\varepsilon_1}\}+\exp\{(1-\varepsilon_1)c_j\delta(a_jz,\theta_0)r\}\\ &\le\exp\{r^{\sigma_1+\varepsilon_1}\}\exp\{(1-\varepsilon_1)c_j\delta(a_jz,\theta_0)r\}.\end{aligned}$$

By (8), we can get

$$\exp\left\{\frac{1-c}{2}\delta(a_jz,\theta_0)r\right\}\le\exp\{r^{\sigma_1+\varepsilon_1}\}.$$

This is a contradiction. Hence $\sigma(f)\ge1$. □

Now assume f is a meromorphic function of the equation (3) with $1\le\sigma(f)=\sigma<\infty$. From the equation (3), we know that the poles of $f(z)$ can occur only at the poles of A_j $(j=0,1,\dots,k-1)$. Let $f=g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\sigma(d)=\beta\le\alpha=\max\{\sigma(A_j):j=0,1,\dots,k-1\}<1$, g be an entire function and $1\le\sigma(g)=\sigma(f)=\sigma<\infty$. Substituting $f=g/d$ into (3), by Lemma 2.3 we can get

(9)

$$\begin{aligned}&g^{(k)}+g^{(k-1)}\big[A_{k-1}e^{a_{k-1}z}+B_{k,k-1}\big]+\cdots+g'\big[A_1e^{a_1z}+\sum_{i=2}^{k-1}A_ie^{a_iz}B_{i,1}+B_{k,1}\big]\\ &+g\big[A_0e^{a_0z}+\sum_{i=1}^{k-1}A_ie^{a_iz}B_{i,1}+B_{k,0}\big]=0.\end{aligned}$$

By Lemma 2.2, for any given ε $(0<3\varepsilon<\min\{1-\alpha,\frac{1-c}{6}\},c=\max\{c_j,1\le j\le k-1\})$, there exists a set $E\in[0,2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_0$, we have

$$\frac{g^{(j)}(z)}{g(z)} \leq |z|^{k(\sigma-1+\varepsilon)}, \quad (j = 1, 2, \dots, k) \tag{10}$$

and

$$\frac{d^{(j)}(z)}{d(z)} \leq |z|^{k(\beta-1+\varepsilon)}, \quad (j = 1, 2, \dots, k). \tag{11}$$

Setting $z = re^{i\theta}$, then

$$\operatorname{Re}\{a_j z\} = \delta(a_j z, \theta) r, \quad \operatorname{Re}\{a_0 z\} = \delta(a_0 z, \theta) r. \tag{12}$$

Now suppose that $\arg a_j \neq \arg a_0$ $(j = 1, 2, \dots, k-1)$. In view of Lemma 2.5 and (12), it is easy to see for the above ε there is a ray $\arg z = \theta$ such that $\theta \in [0, 2\pi) \backslash (E_1 \bigcup E_2 \bigcup E_0)$ (where E_2 and E_0 are defined as in Lemma 2.5, $E_1 \bigcup E_2 \bigcup E_0$ is of linear measure zero) satisfying $\delta(a_j z, \theta) < 0$, $c_j \delta(a_0 z, \theta) > 0$, and for a sufficiently large r, we have

$$|A_0(re^{i\theta}) e^{a_0 re^{i\theta}} f(re^{i\theta})| \geq \exp\{(1-\varepsilon)\delta(a_0 z, \theta) r\}, \tag{13}$$

$$|A_j(re^{i\theta}) e^{a_j re^{i\theta}}| \leq \exp\{(1-\varepsilon)\delta(a_j z, \theta) r\} \, (j = 1, \dots, k-1). \tag{14}$$

By (11), (13), and (14), we have

$$\left|A_{k-1} e^{a_{k-1} z} + B_{k,k-1}\right| \leq \exp\{(1-\varepsilon)\delta(a_j z, \theta) r\} + M r^{k(\beta-1+\varepsilon)}, \dots, \tag{15}$$

$$\left|A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}\right| \leq \exp\{(1-\varepsilon)\delta(a_j z, \theta) r\} + M r^{k(\beta-1+\varepsilon)}, \tag{16}$$

and

$$\left|A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0}\right| \geq \exp\{(1-\varepsilon)\delta(a_0 z, \theta) r\}(1 - o(1)), \tag{17}$$

where $M > 0$ is a constant, it can be different in different occurrences.

By (9), (10), and (15)-(17), we have

$$\begin{aligned}
&\exp\{(1-\varepsilon)\delta(a_0 z, \theta) r\}(1 - o(1)) \\
\leq &\left|A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0}\right| \\
\leq &\left|\frac{g^{(k)}(z)}{g(z)}\right| + \left|\frac{g^{(k-1)}(z)}{g(z)} \left(A_{k-1} e^{a_{k-1} z} + B_{k,k-1}\right)\right| + \cdots \\
&+ \left|\frac{g'(z)}{g(z)} \left(A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}\right)\right| \\
\leq &\, r^{k(\sigma-1+\varepsilon)} + r^{(k-1)(\sigma-1+\varepsilon)} \left[\exp\{(1-\varepsilon)\delta(a_j z, \theta) r_j\} + M r^{k(\beta-1+\varepsilon)}\right] + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ r^{(\sigma-1+\varepsilon)}\left[\exp\{(1-\varepsilon)\delta(a_j z,\theta)r_j\} + Mr^{k(\beta-1+\varepsilon)}\right]\\
&\le r^M.
\end{aligned}
$$

This is absurd which implies $\sigma(g)=\infty$, i.e., $\sigma(f)=\infty$. □

Now suppose that $\arg a_j = \arg a_0$, and $a_j = c_j a_0$ $(0 < c_j < 1)$; then $\delta(a_j z,\theta) = c_j\delta(a_0 z,\theta)$, $\mathrm{Re}\{a_j z\} = c_j \mathrm{Re}\{a_0 z\}$. Using the same argument as above, we know that (10), (11) hold. Moreover, there is a ray $\arg z = \theta$ satisfying $\delta(a_j z,\theta) = c_j\delta(a_0 z,\theta) > 0$, then for a sufficiently large r, we have (13) and

$$
|A_j(re^{i\theta})e^{a_j re^{i\theta}}| \le \exp\{(1+\varepsilon)c_j\delta r(a_0 z,\theta)\}\ (j=1,\dots,k-1). \tag{18}
$$

By (11), (13), and (18), we have

$$
\left|A_{k-1}e^{a_{k-1}z} + B_{k,k-1}\right| \le \exp\{(1+\varepsilon)c_j\delta(a_0 z,\theta)r\},\dots, \tag{19}
$$

$$
\left|A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}\right| \le \exp\{(1+\varepsilon)c_j\delta(a_0 z,\theta)r\}, \tag{20}
$$

and

$$
\left|A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0}\right| \ge \exp\{(1-\varepsilon)\delta(a_0 z,\theta)r\}(1-o(1)). \tag{21}
$$

By (9), (10), and (19)-(21), we have

$$
\begin{aligned}
&\exp\{(1-\varepsilon)\delta(a_0 z,\theta)r\}(1-o(1))\\
&\le \left|A_0 e^{a_0 z} + \sum_{i=1}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,0}\right|\\
&\le \left|\frac{g^{(k)}(z)}{g(z)}\right| + \left|\frac{g^{(k-1)}(z)}{g(z)}\left(A_{k-1}e^{a_{k-1}z} + B_{k,k-1}\right)\right| + \cdots\\
&\quad + \left|\frac{g'(z)}{g(z)}\left(A_1 e^{a_1 z} + \sum_{i=2}^{k-1} A_i e^{a_i z} B_{i,1} + B_{k,1}\right)\right|\\
&\le r^{k(\sigma-1+\varepsilon)} + r^{(k-1)(\sigma-1+\varepsilon)}\exp\{(1+\varepsilon)c_j\delta(a_0 z,\theta)r\}(1+o(1)) + \cdots\\
&\quad + r^{(\sigma-1+\varepsilon)}\exp\{(1+\varepsilon)c_j\delta(a_0 z,\theta)r\}(1+o(1))\\
&\le Mr^{k(\sigma-1+\varepsilon)}\exp\{(1+\varepsilon)c_j\delta(a_0 z,\theta)r\}(1+o(1)).
\end{aligned}
$$

From this and $3\varepsilon < \frac{1-c}{6}$, we get

$$
\exp\left\{\frac{1-c}{2} r\delta(a_0 z,\theta)\right\} \le Mr^{k(\sigma-1+\varepsilon)}.
$$

It is a contradiction. The proof of Theorem 1.1 is completed. □

4. Proof of Theorem 1.4

Assume $f(\not\equiv 0)$ is a meromorphic function of (3); then $\sigma(f)=\infty$ by Theorem 1.1. Set $g_0(z)=f(z)-z$, then z is a fixed point of $f(z)$ if and only if $g_0(z)=0$. $g_0(z)$ is a meromorphic function and $\sigma(g_0)=\sigma(f)=\infty$. Substituting $f=g_0+z$ into (3), we have

$$g_0^{(k)}+A_{k-1}e^{a_{k-1}z}g_0^{(k-1)}+\cdots+A_1e^{a_1z}g_0'+A_0e^{a_0z}g_0=-A_1e^{a_1z}-zA_0e^{a_0z}. \tag{22}$$

We can rewrite (22) as the following form:

$$g_0^{(k)}+h_{0,k-1}g_0^{(k-1)}+\cdots+h_{0,1}g_0'+h_{0,0}g_0=-h_{0,1}-zh_{0,0}.$$

Obviously, $h_0=-[h_{1,0}+zh_{0,0}]=-A_1e^{a_1z}-zA_0e^{a_0z}\not\equiv 0$. Here we just consider the meromorphic solutions of infinite order satisfying $g_0=f-z$, by Lemma 2.6 we know that $\bar{\lambda}(g_0)=\bar{\tau}(f)=\infty$ holds.

Now we consider the fixed points of $f'(z)$.

Let $g_1(z)=f'-z$. Then z is a fixed point of $f'(z)$ if and only if $g_1(z)=0$. $g_1(z)$ is a meromorphic function and $\sigma(g_1)=\sigma(f')=\sigma(f)=\infty$. Differentiating both sides of the equation (3), we have

$$\begin{aligned}&f^{(k+1)}+A_{k-1}e^{a_{k-1}z}f^{(k)}+[(A_{k-1}e^{a_{k-1}z})'+A_{k-2}e^{a_{k-2}z}]f^{(k-1)}\\&+\cdots+[(A_3e^{a_3z})'+A_2e^{a_2z}]f'''+[(A_2e^{a_2z})'+A_1e^{a_1z}]f''\\&+[(A_1e^{a_1z})'+A_0e^{a_0}z]f'+(A_0e^{a_0z})'f=0.\end{aligned} \tag{23}$$

By (3), we have

$$f=-\frac{1}{A_0e^{a_0z}}\left[f^{(k)}+A_{k-1}e^{a_{k-1}z}f^{(k-1)}+\cdots+A_2e^{a_2z}f''+A_1e^{a_1z}f'\right]. \tag{24}$$

Substituting (24) into (23), we have

(25)

$$\begin{aligned}&f^{(k+1)}+[A_{k-1}e^{a_{k-1}z}-\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}]f^{(k)}+[(A_{k-1}e^{a_{k-1}z})'+A_{k-2}e^{a_{k-2}z}-\\&\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_{k-1}e^{a_{k-1}z}]f^{(k-1)}+\cdots+[(A_3e^{a_3z})'+A_2e^{a_2z}-\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_3e^{a_3z}]f'''\\&+[(A_2e^{a_2z})'+A_1e^{a_1z}-\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_2e^{a_2z}]f''\\&+[(A_1e^{a_1z})'+A_0e^{a_0z}-\frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_1e^{a_1z}]f'=0.\end{aligned}$$

We can denote the equation by the following form:

$$f^{(k+1)}+h_{1,k-1}f^{(k)}+h_{1,k-2}f^{(k-1)}+\cdots+h_{1,2}f'''+h_{1,1}f''+h_{1,0}f'=0, \tag{26}$$

where $h_{1,j}$ $(j=0,1,\ldots,k-1)$ is the meromorphic functions defined by the equation (25). Substituting $f'=g_1+z$, $f''=g_1'+1$, $f^{(j+1)}=g_1^{(j)}$ $(2\le j\le k)$ into (26), we get

$$g_1^{(k)}+h_{1,k-1}g_1^{(k-1)}+\cdots+h_{1,1}g'+h_{1,0}g_1=h_1, \tag{27}$$

where

$$h_1 = -(h_{1,1} + zh_{1,0}) = -\Big[\big(A_2' + a_2A_2 - \frac{A_0'}{A_0}A_2 + a_0A_2\big)e^{a_2z} + \big(A_1 + zA_1' + za_1A_1 - zA_1\frac{A_0'}{A_0} - za_0A_1\big)e^{a_1z} + zA_0e^{a_0z}\Big].$$

We claim $h_1 \not\equiv 0$. Since a_2, a_1, a_0 are different each other, if $h_1 \equiv 0$ by Lemma 2.1, we conclude by Lemma 2.1 that $A_0 \equiv 0$, a contradiction. Therefore, $h_1 \not\equiv 0$. Applying Lemma 2.6 to (27) above, we obtain $\bar{\lambda}(g_1) = \bar{\lambda}(f' - z) = \bar{\tau}(f') = \sigma(g_1) = \sigma(f) = \infty$.

Now we prove that $\bar{\tau}(f'') = \bar{\lambda}(f'' - z) = \infty$. Set $g_2(z) = f'' - z$. Using the same argument as above, we need to prove only that $\bar{\lambda}(g_2) = \infty$.

We differentiate both sides of (26), and obtain

(28)
$$f^{(k+2)} + h_{1,k-1}f^{(k+1)} + [h_{1,k-1}' + h_{1,k-2}]f^{(k)} + \cdots + [h_{1,1}' + h_{1,0}]f'' + h_{1,0'}f' = 0.$$

By (26) and (28), we have

(29)
$$f^{(k+2)} + \left[h_{1,k-1} - \frac{h_{1,0}'}{h_{1,0}}\right] f^{(k+1)} + \left[h_{1,k-1}' + h_{1,k-2} - \frac{h_{1,0}'}{h_{1,0}}h_{1,k-1}\right] f^{(k)} + \cdots + \left[h_{1,2}' + h_{1,1} - \frac{h_{1,0}'}{h_{1,0}}h_{1,2}\right] f''' + \left[h_{1,1}' + h_{1,0} - \frac{h_{1,0}'}{h_{1,0}}h_{1,1}\right] f'' = 0.$$

We can write (28) to the following form

(30)
$$f^{(k+2)} + h_{2,k-1}f^{(k+1)} + h_{2,k-2}f^{(k)} + \cdots + h_{2,1}f''' + h_{2,0}f'' = 0,$$

where $h_{2,j}$ are meromorphic functions with $\sigma(h_{2,j}) < 1$ $(j = 0, 1, \ldots, k-1)$, and

(31)
$$h_{2,1} = h_{1,2}' + h_{1,1} - \frac{h_{1,0}'}{h_{1,0}}h_{1,2},$$
$$h_{2,0} = h_{1,1}' + h_{1,0} - \frac{h_{1,0}'}{h_{1,0}}h_{1,1},$$

where

(32)
$$h_{1,2} = (A_3e^{a_1z})' + A_2e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_3e^{a_3z},$$
$$h_{1,1} = (A_2e^{a_1z})' + A_1e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_2e^{a_2z},$$
$$h_{1,0} = (A_1e^{a_1z})' + A_0e^{a_0z} - \frac{(A_0e^{a_0z})'}{A_0e^{a_0z}}A_1e^{a_1z}.$$

Substituting $f'' = g_2 + z$, $f''' = g_2' + 1$, $f^{(j+2)} = g_2^{(j)}$ $(2 \le j \le k)$ into (30), we get

$$g_2^{(k)} + h_{2,k-1}g_2^{(k-1)} + \cdots + h_{2,1}g_2' + h_{2,0}g_2 = -(h_{2,1} + zh_{2,0}). \tag{33}$$

We claim $h_{2,1} + zh_{2,0} \not\equiv 0$. By (31), (32) we know $h_{2,1} + zh_{2,0}$ can write into the following form

$$h_2 = -[h_{2,1} + zh_{2,0}] = \frac{-1}{h_{1,0}} \left[zA_0^2 e^{2a_0 z} + \sum_{\gamma \in \Lambda_2} D_\gamma e^{\gamma z} \right],$$

where D_γ are meromorphic functions with the order less than 1 which are different in different places. The index set Λ_2 denotes the sums of a_i, a_j $(0 \le i, j \le 3)$, except for $2a_0$. Obviously, the differences of every sum are not the constant which satisfies the condition (ii) and (iii) in Lemma 2.1. Similarly with the above, if $h_{2,1} + zh_{2,0} \equiv 0$, by Lemma 2.1, there must be $A_0 \equiv 0$, it is a contradiction. Then applying Lemma 2.6 to (33), we have $\bar{\lambda}(g_2) = \bar{\lambda}(f'' - z) = \bar{\tau}(f'') = \infty$.

This proves the theorem. □

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