J. Korean Math. Soc. **46** (2009), No. 4, pp. 713–731
DOI 10.4134/JKMS.2009.46.4.713

ANALYSIS OF A DELAY PREY-PREDATOR MODEL WITH DISEASE IN THE PREY SPECIES ONLY

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Abstract. In this paper, a three-dimensional eco-epidemiological model with delay is considered. The stability of the two equilibria, the existence of Hopf bifurcation and the permanence are investigated. It is found that Hopf bifurcation occurs when the delay τ passes though a sequence of critical values. The estimation of the length of delay to preserve stability has also been calculated. Numerical simulation with a hypothetical set of data has been done to support the analytical findings.

1. Introduction

The mathematical modelling of epidemics has become a very important subject of research after the seminal model of Kermac-McKendric (1927) on SIRS (susceptible-infected-removed-susceptible) systems, in which the evolution of a disease which gets transmitted upon contact is described. Important studies in the following decades have been carried out, with the aim of controlling the effects of diseases and of developing suitable vaccination strategies [12, 18, 25]. After the seminal models of Vito Volterra and Alfred James Lotka in the mid 1920s for predator-prey interactions, mutualist and competitive mechanisms have been studied extensively in the recent years by researchers [15, 16, 17].

In the natural world, however, species do not exist alone, it is of more biological significance to study the persistence-extinction threshold of each population in systems of two or more interacting species subjected to parasitism. Mathematical biology, namely predator-prey systems and models for transmissible diseases are major fields of study in their own right. But little attention has been paid so far to merge these two important areas of research (see [5, 6, 13, 26]). In order to study the influence of disease on an environment where two or more interacting species are present. In this paper, we shall put emphasis on such an eco-epidemiological system consisting of three species, namely, the

Received September 15, 2007.
2000 *Mathematics Subject Classification.* 92D25, 92D30, 34K18, 34K20.
Key words and phrases. predator-prey model, eco-epidemiology, delay, Hopf bifurcation.
This work is supported by the National Natural Science Foundation of China (No. 10771179), the Henan Innovation Project for University Prominent Research Talents (No.2005KYCX017) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

sound prey (which is susceptible), the infected prey (which becomes infective by some viruses) and the predator population.

We have two populations:

1. The prey, whose total population density is denoted by N.
2. The predator, whose population density is denoted by y.

We make the following assumptions:

(A_1) In the absence of infection and predation, the prey population density grows logistically with carrying capacity K $(K > 0)$ and an intrinsic birth rate constant r $(r > 0)$,

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right). \tag{1.1}$$

(A_2) In the presence of disease, the total prey population N are divided into two distinct classes, namely, susceptible populations, S, and infected populations, I. Therefore, at any time t, the total density of prey population is

$$N(t) = S(t) + I(t). \tag{1.2}$$

(A_3) We assume that only susceptible prey S are capable of reproducing with logistic law (Eq.(1.1)); i.e., the infected prey I are removed by death (say its death rate is a positive constant μ), or by predation before having the possibility of reproducing. However, the infective population I still contributes with S to population growth toward the carrying capacity.

(A_4) We assume that the force of infection at time t is given by $\beta S(t)I(t-\tau)$, where β is the average number of contacts per infective per day and $\tau > 0$ is a fixed time during which the infectious agents develop in the vector and it is only after that time that the infected vector can infect a susceptible prey [3, 19, 21]. Hence, the SI model of the infected prey is:

$$\begin{cases} \dot{S} = rS\left(1 - \dfrac{S}{K}\right) - \beta SI(t-\tau), \\ \dot{I} = \beta SI(t-\tau) - \mu I. \end{cases} \tag{1.3}$$

(A_5) It is assumed that predator can distinguish between infected and health prey. We assume that the predator eats only the infected prey with Leslie-Gower ratio-dependent schemes [1, 2, 14, 20, 22]. That is to say, the predator consumes the prey according to the ratio-dependent functional response and the predator grows logistically with intrinsic growth rate δ and carrying capacity proportional to the prey populations size I.

From the above assumptions we have the following model:

$$\begin{cases} \dfrac{dS}{dt} = rS\left(1 - \dfrac{S}{K}\right) - \beta SI(t-\tau), \\ \dfrac{dI}{dt} = \beta SI(t-\tau) - \dfrac{cyI}{my + I} - \mu I, \\ \dfrac{dy}{dt} = \delta y\left(1 - \dfrac{hy}{I}\right). \end{cases} \tag{1.4}$$

The initial conditions for system (1.4) take the form

$$\begin{aligned} &S(\theta) = \varphi_1(\theta),\ \ I(\theta) = \varphi_2(\theta),\ y(\theta) = \varphi_3(\theta),\\ &\varphi_1(\theta) \geq 0,\ \varphi_2(\theta) \geq 0,\ \varphi_3(\theta) \geq 0,\ \theta \in [-\tau, 0],\\ &\varphi_1(0) > 0,\ \varphi_2(0) > 0,\ \varphi_3(0) > 0, \end{aligned} \tag{1.5}$$

where $(\varphi_1(\theta),\ \varphi_2(\theta),\ \varphi_3(\theta)) \in C([-\tau, 0],\ R^3_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_{+0}, where $R^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

It is well known by the fundamental theorem of functional differential equations that system (1.4) has a unique solution $(S(t), I(t), y(t))$ satisfying initial conditions (1.5).

The paper is organized as follows. In Section 2, we present the positivity and the boundedness of solutions. We find conditions for local stability and bifurcation results in Section 3. In Section 4, the time delay is estimated for witch local stability is preserved. The permanence of system is given in Section 5. Some numerical simulations are performed for a hypothetical of parameter values in the last section.

2. Positivity and boundedness of solutions

It is important to show positivity and boundedness for the system (1.4) as they represent populations. Positivity implies that the populations survive and boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources. The model system can be put into the matrix form $\dot{X} = G(X)$, where $X = (S, I, y)^T \in R^3$ and $G(X)$ is given by

$$G(X) = \begin{pmatrix} G_1(X) \\ G_2(X) \\ G_3(X) \end{pmatrix} = \begin{pmatrix} rS(1 - \frac{S}{K}) - \beta SI(t-\tau) \\ \beta SI(t-\tau) - \dfrac{cyI}{my + I} - \mu I \\ \delta y(1 - \dfrac{hy}{I}) \end{pmatrix}.$$

Let $R^3_+ = [0, +\infty)^3$ be the nonnegative octant in R^3, the $G : R^{3+1}_+ \to R^3$ is locally Lipschitz and satisfy the condition

$$G_i(X)|_{X_i(t)=0},\ X \in R^3_+ \geq 0,$$

where $X_1 = S$, $X_2 = I$, $X_3 = y$.

Due to Lemma in [27] and Theorem A4 in [24] any solutions of (1.4) with positive initial conditions exist uniquely and each component of X remains the interval $[0, b)$ for some $b > 0$. Furthermore, if $b < +\infty$, then $\limsup[S(t) + I(t) + y(t)] = +\infty$.

Next, we present the boundedness of solutions. Since

$$\frac{dS}{dt} \leq rS\left(1 - \frac{S}{K}\right),$$

by a standard comparison theorem, we have $\limsup_{t\to+\infty} S(t) \le M_1$, where $M_1 = \max\{S(0), K\}$. Define the function

$$W(t) = S(t) + I(t).$$

The time derivative along a solution of (1.4) is

$$\frac{dW}{dt} = rS\left(1 - \frac{S}{K}\right) - \frac{cyI}{my + I} - \mu I \le M_1(r+1) - qW(t),$$

where $q = \min\{1, \mu\}$. Thus, $\frac{dW}{dt} + qW \le M_1(r+1)$. Applying a theorem in differential inequalities, we obtain

$$W(t) \le \frac{M_1(r+1)}{q} + \left[W(S(0), I(0)) - \frac{M_1(r+1)}{q}\right] e^{-qt}.$$

Therefore, there exists $M_2 > 0$ and some $T_1 > 0$ such that $I(t) \le M_2$, $t \ge T_1$.

Lastly, we consider the boundedness of $y(t)$. From the third equation of system (1.4), we get

$$\frac{dy}{dt} \le \delta y \left(1 - \frac{hy}{M_2}\right).$$

By a standard comparison theorem, we have $\limsup_{t\to+\infty} y(t) \le M_3$, where $M_3 = \max\{y(0), \frac{M_2}{h}\}$. So, all solutions of system (1.4) with initial condition enter the region $B = \{(S(t), I(t), y(t)) : 0 \le S(t) \le M_1, 0 \le I(t) \le M_2, 0 \le y(t) \le M_3\}$.

3. Stability analysis and Hopf bifurcation

In this section, we focus on investigating the stability of the equilibria and Hopf bifurcation of the positive equilibrium of the system (1.4). System (1.4) has the boundary equilibrium $E_1(\frac{\mu}{\beta}, \frac{r}{\beta}(1-\frac{\mu}{\beta K}), 0) \triangleq (S_1, I_1, y_1)$ and the positive equilibrium $E_2(S_2, I_2, y_2)$, where $S_2 = \frac{\mu h + c + \mu m}{\beta(m+h)}$, $I_2 = \frac{r(\beta m K + \beta h K - \mu h - c - \mu m)}{\beta^2 K(m+h)}$, $y_2 = \frac{r(\beta m K + \beta h K - \mu h - c - \mu m)}{\beta^2 h K(m+h)}$. Clearly, if $1 - \frac{\mu}{\beta K} > 0$, then E_1 exists and remains positive. And E_2 exists and remains positive if $\beta > \frac{1}{K}(\mu + \frac{c}{m+h}) \triangleq \beta_0$.

Let $E^*(S^*, I^*, y^*)$ be any arbitrary equilibrium. Then the characteristic equation about E^* is given by

(3.1)
$$\begin{vmatrix} r - \dfrac{2rS^*}{K} - \beta I^* - \lambda & -\beta S^* e^{-\lambda\tau} & 0 \\ \beta I^* & \beta S^* e^{-\lambda\tau} - \dfrac{cmy^{*2}}{(my^* + I^*)^2} - \mu - \lambda & -\dfrac{cI^{*2}}{(my^* + I^*)^2} \\ 0 & \dfrac{\delta h y^{*2}}{I^{*2}} & \delta - \dfrac{2\delta h y^*}{I^*} - \lambda \end{vmatrix} = 0.$$

For equilibrium E_1, (3.1) reduces to

$$\begin{vmatrix} -\frac{rS_1}{K}-\beta I_1-\lambda & -\beta S_1 e^{-\lambda\tau} & 0 \\ \beta I_1 & \beta S_1 e^{-\lambda\tau}-\mu-\lambda & -c \\ 0 & 0 & \delta-\lambda \end{vmatrix} = 0. \tag{3.2}$$

It is easy to see that the equilibrium E_1 is a saddle.

For equilibrium E_2, (3.1) reduces to

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 + (B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau} = 0, \tag{3.3}$$

where

$$\begin{aligned}
A_1 &= \delta + \mu + \frac{cmy_2^2}{(my_2+I_2)^2} + \frac{rS_2}{K},\\
A_2 &= \delta\left(\mu + \frac{cmy_2^2}{(my_2+I_2)^2}\right) + \frac{rS_2}{K}\left(\delta + \mu + \frac{cmy_2^2}{(my_2+I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2+I_2)^2},\\
A_3 &= \frac{rS_2}{K}\left[\delta\left(\mu + \frac{cmy_2^2}{(my_2+I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2+I_2)^2}\right],\\
B_1 &= -\beta S_2,\\
B_2 &= -\delta\beta S_2 - \frac{r\beta S_2^2}{K} + \beta^2 S_2 I_2,\\
B_3 &= -\frac{r\delta\beta S_2^2}{K} + \delta\beta^2 S_2 I_2.
\end{aligned}$$

For $\tau = 0$, the transcendental equation (3.3) reduces to (3.4):

$$\lambda^3 + (A_1+B_1)\lambda^2 + (A_2+B_2)\lambda + A_3 + B_3 = 0. \tag{3.4}$$

We can easily get

$$\begin{aligned}
A_1 + B_1 &= \frac{rS_2}{K} + \delta - \delta^* = \frac{r\beta_0}{K} + \delta - \delta^* > 0,\\
A_2 + B_2 &= \frac{rS_2}{K}(\delta - \delta^*) + \beta^2 S_2 I_2 = r\left(\mu + \frac{c}{m+h}\right)\left(1 + \frac{\delta-\delta^*}{K\beta} - \frac{\beta_0}{\beta}\right),\\
A_3 + B_3 &= \delta\beta^2 S_2 I_2 = r\delta\left(\mu + \frac{c}{m+h}\right)\left(1 - \frac{\beta_0}{\beta}\right) > 0,\\
A_3 - B_3 &= r\delta S_2\left[3\frac{\mu(m+h)+c}{K(m+h)} - \beta\right] = r\delta S_2[3\beta_0 - \beta],
\end{aligned}$$

where $\delta^* = \frac{ch}{(m+h)^2}$.

By Routh-Hurwitz Criterion, we know that all the roots of equation (3.4) have negative real parts, i.e., the positive equilibrium E_2 is locally asymptotically stable provided that the conditions

(H_1): $(A_1+B_1)(A_2+B_2) - (A_3+B_3) > 0$,
(H_2): $\delta - \delta^* > 0$, and
(H_3): $\beta > \beta_0$ hold.

We now turn to an investigation of the type of stability for system (1.4) at the positive equilibrium E_2. We shall firstly introduce two lemmas.

Lemma 3.1 ([23]). *For the polynomial equation* $z^3 + a_1 z^2 + a_2 z + a_3 = 0$,

(1) *If* $a_3 < 0$, *the equation has at least one positive root*;

(2) *If* $a_3 \geq 0$ *and* $\triangle = a_1^2 - 3a_2 \leq 0$, *the equation has no positive roots*;

(3) *If* $a_3 \geq 0$ *and* $\triangle = a_1^2 - 3a_2 > 0$, *the equation has positive roots if and only if* $z_1^* = \frac{-a_1+\sqrt{\triangle}}{3}$ *and* $h(z_1^*) \leq 0$, *where* $h(z) = z^3 + a_1 z^2 + a_2 z + a_3$.

Lemma 3.2. (i) *The positive equilibrium* E_2 *of system* (1.4) *is absolutely stable if and only if the equilibrium* E_2 *of the corresponding ordinary differential equation* (ODE) *system is asymptotically stable and the characteristic equation* (3.3) *has no purely imaginary roots for any* $\tau > 0$;

(ii) *The positive equilibrium* E_2 *of system* (1.4) *is conditionally stable if and only if all roots of the characteristic equation* (3.3) *have negative real parts at* $\tau = 0$ *and there exist some positive values* τ *such that the characteristic equation* (3.3) *has a pair of purely imaginary roots* $\pm i\omega_0$.

Theorem 3.1. *For system* (1.4), *if the conditions* (H_1), (H_2) *and*

$$(H_4): \quad \beta > 3\beta_0$$

hold, the positive equilibrium E_2 *is conditionally stable.*

Proof. Assume that for some $\tau > 0$, $i\omega$ ($\omega > 0$) is a root of characteristic equation (3.3). Now substituting $\lambda = i\omega$ ($\omega > 0$) in (3.3) and separating the real and imaginary parts, we obtain the system of transcendental equations

$$A_1\omega^2 - A_3 = (B_3 - B_1\omega^2)\cos(\omega\tau) + B_2\omega\sin(\omega\tau), \tag{3.5}$$

$$\omega^3 - A_2\omega = B_2\omega\cos(\omega\tau) - (B_3 - B_1\omega^2)\sin(\omega\tau). \tag{3.6}$$

Squaring and adding (3.5) and (3.6) we get

$$(B_3 - B_1\omega^2)^2 + B_2^2\omega^2 = (A_1\omega^2 - A_3)^2 + (\omega^3 - A_2\omega)^2. \tag{3.7}$$

We finally have

$$\omega^6 + P_1\omega^4 + P_2\omega^2 + P_3 = 0,$$

where

$$\begin{aligned} P_1 &= A_1^2 - 2A_2 - B_1^2, \\ P_2 &= A_2^2 - B_2^2 - 2A_1A_3 + 2B_1B_3, \\ P_3 &= A_3^2 - B_3^2. \end{aligned}$$

We know $P_3 < 0$ provided that the condition (H_4) holds. By Lemma 3.1, there is at least a positive ω_0 satisfying equation (3.7), i.e., the characteristic equation (3.3) has a pair of purely imaginary roots of the form $\pm i\omega_0$. From equations (3.5) and (3.6), we can get the corresponding $\tau_k > 0$ such that the characteristic equation (3.3) has a pair of purely imaginary roots

$$\tau_k = \frac{1}{\omega_0} \arccos \left[\frac{(A_1\omega_0^2 - A_3)(B_3 - B_1\omega_0^2) + (\omega_0^3 - A_2\omega_0)B_2\omega_0}{(B_3 - B_1\omega_0^2)^2 + (B_2\omega_0)^2} \right] + \frac{2n\pi}{\omega_0}, \ (n = 0, 1, 2, 3, \ldots).$$

We know that under the conditions of (H_1), (H_2), (H_4), all the roots of characteristic equation (3.3) have negative real parts when $\tau = 0$. By Lemma 3.2 the positive equilibrium E_2 of system (1.4) is conditionally stable. This completes the proof. □

Theorem 3.2. *Under the condition* (H_4) *and*

$$(H_5): \ \delta^2 + (\mu + \frac{cm}{(m+h)^2})^2 + (\frac{rS_2}{K})^2 - \beta^2 S_2^2 - \frac{2\delta ch}{(m+h)^2} > 0,$$

system (1.4) *undergoes Hopf bifurcation at the positive equilibrium* E_2 *when* $\tau = \tau_k$.

Proof. Let $\lambda(\tau) = u(\tau) + i\omega(\tau)$ be a root of the characteristic equation (3.3). Separating the real and imaginary parts of transcendental equation (3.4), we then have

$$\begin{cases} H_1(u, \omega, \tau) = 0, \\ H_2(u, \omega, \tau) = 0, \end{cases} \tag{3.8}$$

where

$$\begin{aligned} H_1(u, \omega, \tau) =& u^3 - 3u\omega^2 + A_1u^2 - A_1\omega^2 + A_2u + A_3 + (B_1u^2 - B_1\omega^2 + B_2u \\ &+ B_3)e^{-u\tau}\cos(\omega\tau) + (2B_1u\omega + B_2\omega)e^{-u\tau}\sin(\omega\tau), \\ H_2(u, \omega, \tau) =& -\omega^3 + 3u^2\omega + 2A_1u\omega + A_2\omega - (B_1u^2 - B_1\omega^2 + B_2u \\ &+ B_3)e^{-u\tau}\sin(\omega\tau) + (2B_1u\omega + B_2\omega)e^{-u\tau}\cos(\omega\tau). \end{aligned}$$

By Theorem 3.1 we have $H_1(0, \omega, \tau) = H_2(0, \omega, \tau) = 0$. To check that the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial H_1}{\partial u} & \frac{\partial H_1}{\partial \omega} \\ \frac{\partial H_2}{\partial u} & \frac{\partial H_2}{\partial \omega} \end{pmatrix}$$

satisfies $|J|_{(0,\omega_0,\tau_k)} > 0$. By means of the implicit function theorem, we deduce that equation (3.8) define u, ω as functions of τ in a neighborhood of $(0, \omega_0, \tau_k)$ such that $u(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$. We now investigate how the real part of the roots of characteristic equation (3.3) varies as τ varies in a small neighborhood of τ_k. Next, we turn to show

$$\frac{d(\mathrm{Re}\lambda)}{d\tau}|_{\tau=\tau_k} > 0.$$

This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_k$. Differentiating the transcendental equation (3.3) with respect τ, we get

$$[(3\lambda^2 + 2A_1\lambda + A_2) + e^{-\lambda\tau}(2B_1\lambda + B_2) - \tau e^{-\lambda\tau}(B_1\lambda^2 + B_2\lambda + B_3)]\frac{d\lambda}{d\tau}$$
$$= (B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau}\lambda.$$

Thus,

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + 2A_1\lambda + A_2}{\lambda e^{-\lambda\tau}(B_1\lambda^2 + B_2\lambda + B_3)} + \frac{2B_1\lambda + B_3}{\lambda(B_1\lambda^2 + B_2\lambda + B_3)} - \frac{\tau}{\lambda}\\ &= \frac{3\lambda^2 + 2A_1\lambda + A_2}{-\lambda(\lambda^3 + A_1\lambda^2 + A_2\lambda + B_3)} + \frac{2B_1\lambda + B_3}{\lambda(B_1\lambda^2 + B_2\lambda + B_3)} - \frac{\tau}{\lambda}\\ &= \frac{2\lambda^3 + A_1\lambda^2 - A_2}{-\lambda^2(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)} + \frac{B_1\lambda^2 - B_3}{\lambda^2(B_1\lambda^2 + B_2\lambda + B_3)} - \frac{\tau}{\lambda}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Theta =&\text{sign}\left[\text{Re}\left(\frac{2\lambda^3 + A_1\lambda^2 - A_2}{-\lambda^2(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)} + \frac{B_1\lambda^2 - B_3}{\lambda^2(B_1\lambda^2 + B_2\lambda + B_3)} - \frac{\tau}{\lambda}\right)\right]_{\lambda=i\omega_0}\\ =&\frac{1}{\omega_0^2}\text{sign}\left[\text{Re}\left(\frac{(A_3 + A_1\omega_0^2) + i2\omega_0^3}{(A_1\omega_0^2 - A_3) + i(\omega_0^3 - A_3\omega_0)}\right) + \frac{B_1\omega_0^2 + B_3}{(B_3 - B_1\omega_0^2) + iB_3\omega_0}\right]\\ =&\frac{1}{\omega_0^2}\text{sign}\left[\frac{(A_3 + A_1\omega_0^2)(A_1\omega_0^2 - A_3) + 2\omega_0^3(\omega_0^3 - A_2\omega_0)}{(A_1\omega_0^2 - A_3)^2 + (\omega_0^3 - A_2\omega_0)^2}\right.\\ &\left.+\frac{(B_1\omega_0^2 + B_3)(B_3 - B_1\omega_0^2)}{(B_3 - B_1\omega_0^2)^2 + (B_2\omega_0^2)^2}\right]\\ =&\frac{1}{\omega_0^2}\text{sign}\left[\frac{2\omega_0^6 + (A_1^2 - 2A_2 - B_1^2)\omega_0^4 + (B_3^2 - A_3^2)}{(B_3 - B_1\omega_0^2)^2 + (B_2\omega_0^2)^2}\right].\end{aligned}$$

As $A_1^2 - 2A_2 - B_1^2$ and $B_3^2 - A_3^2$ are both positive by virtue of (H_5) and (H_4) respectively, we have

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_0,\tau=\tau_k} > 0.$$

Therefore, the transversality condition holds and hence Hopf bifurcation occurs at $\omega = \omega_0, \tau = \tau_k$. □

Remark 3.1. It must be pointed out that Theorem 3.1 cannot determine the stability of bifurcation periodic orbits, that is, the periodic solutions may exist either for $\tau > \tau_0$ or for $\tau < \tau_0$, near τ_0. Further, we can investigate the stability of bifurcating periodic orbits by analyzing higher-order terms. The calculation is very complex and the method is trivial, so we omit it.

Next, we consider that the time delay induces switching of stability.

Consider the following characteristic equation:

$$P(\lambda) + Q(\lambda)e^{-\tau\lambda} = 0, \tag{3.9}$$

where P and Q are polynomials with real coefficients of degree n and m respectively, and τ is a nonnegative constant. For such a transcendental equation (3.9), Cooke et al. [7] obtained the following result.

Lemma 3.3. *Consider* Eq.(3.9), *where P and Q are analytic functions in a right half-plane* Re $z > -\vartheta$, $\vartheta > 0$, *which satisfy the following conditions.*

(1) *$P(\lambda)$ and $Q(\lambda)$ have no common imaginary zero.*

(2) $\overline{P(-iy)} = P(iy)$, $\overline{Q(-iy)} = Q(iy)$ *for real y ($^-$denotes a complex conjugate).*

(3) $P(0) + Q(0) \neq 0$.

(4) *There are at most a finite number of roots of* (3.9) *in the right half-plane when $\tau = 0$.*

(5) $F(y) = |P(iy)|^2 - |Q(iy)|^2$ *for real y, has at most a finite number of real zeros.*

Under these conditions, the following statements are true.

(a) *Suppose that the equation $F(y) = 0$ has no positive roots. Then if* (3.9) *is stable at $\tau = 0$ it remains stable for all $\tau \geq 0$, whereas if it is unstable at $\tau = 0$ it remains unstable for all $\tau \geq 0$.*

(b) *Suppose that the equation $F(y) = 0$ has at least one positive root and that each positive root is simple. As τ increases, stability switches may occur. There exists a positive number τ^* such that* Eq.(3.9) *is unstable for all $\tau > \tau^*$. As τ varies from 0 to τ^*, at most a finite number of stability switches may occur.*

We rewrite characteristic equation (3.3) in the following form:

$$P(\lambda) + Q(\lambda)e^{-\tau\lambda} = 0,$$

where $P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, $Q(\lambda) = B_1\lambda^2 + B_2\lambda + B_3$.

We state the following result.

Theorem 3.3. *Suppose the conditions* (H_1), (H_2) *and* (H_3) *are satisfied. Further assume that* (i) $P_3 < 0$ *and* (ii) *either* $P_1^2 < 3P_2$ *or both* $P_2 > 0$ *and* $P_1 > 0$. *Then stability switches may occur as τ increases and eventually interior equilibrium becomes unstable.*

Proof. In our model (1.4), $P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, $Q(\lambda) = B_1\lambda^2 + B_2\lambda + B_3$. Clearly, $P(\lambda)$ and $Q(\lambda)$ have no common imaginary root. Obviously $\overline{P(-iy)} = P(iy)$, $\overline{Q(-iy)} = Q(iy)$ for real y. Also $P(0) + Q(0) = 0$ since $A_3 + B_3 \neq 0$. Now $\limsup[|Q(\lambda)/P(\lambda)| : |\lambda| \to \infty, \mathrm{Re}\lambda \geq \infty] < 1$. We have $F(y) = |P(iy)|^2 - |Q(iy)|^2 = y^6 + P_1y^4 + P_2y^2 + P_3 = 0$, where $P_1 = A_1^2 - 2A_2 - B_1^2$, $P_2 = A_2^2 - B_2^2 - 2A_1A_3 + 2B_1B_3$, $P_3 = A_3^2 - B_3^2$. Since $F(y)$ is of even degree and the last term of $F(y)$ is negative, so $F(y)$ must have at least one positive root. Conditions of the theorem imply that each positive root is simple. This completes the proof. □

4. Estimation of the length of delay to preserve stability

We consider the system (1.4) and the space of all real valued continuous functions defined on $[-\tau, \infty)$ satisfying the initial conditions (1.5) on $[-\tau, 0]$. We linearize the system (1.4) about its interior equilibrium $E_2(S_2, I_2, y_2)$ and get

$$\begin{cases} \dot{S} = -\dfrac{rS_2}{K}S - \beta S_2 I(t-\tau), \\ \dot{I} = \beta I_2 S + \beta S_2 I(t-\tau) - \left[\dfrac{cmy_2^2}{(my_2+I_2)^2} + \mu\right] I - \dfrac{cI_2^2}{(my_2+I_2)^2} y, \\ \dot{y} = \dfrac{\delta}{h} I - \delta y. \end{cases} \tag{4.1}$$

Taking Laplace transform of the system given by (4.1), we get

$$\begin{cases} \left(\varsigma + \dfrac{rS_2}{K}\right) L_S(\varsigma) = -\beta S_2 e^{-\varsigma\tau} L_I(\varsigma) - \beta S_2 e^{-\varsigma\tau} K_1(\varsigma) + L_S(0), \\ \left(\varsigma + \dfrac{cmy_2^2}{(my_2+I_2)^2} + \mu - \beta S_2\right) L_I(\varsigma) = \beta I_2 L_S(\varsigma) + \beta S_2 e^{-\varsigma\tau} K_1(\varsigma) \\ \qquad\qquad - \dfrac{cI_2^2}{(my_2+I_2)^2} L_y(\varsigma) + L_I(0), \\ (\varsigma + \delta) L_y(\varsigma) = \dfrac{\delta}{h} L_I(\varsigma) + L_y(0), \end{cases} \tag{4.2}$$

where

$$K_1(\varsigma) = \int_{-\tau}^{0} e^{-\varsigma t} P_I(t) dt,$$

and L_T, L_I and L_y are the Laplace transform of $S(t)$, $I(t)$ and $y(t)$, respectively.

Following along the lines of [9] and using Nyquist criterion, it can be shown that the conditions for local asymptotic stability of $E_2(S_2, I_2, y_2)$ are given by

$$\text{Im } H(i\eta_0) > 0, \tag{4.3}$$

$$\text{Re } H(i\eta_0) = 0, \tag{4.4}$$

where $H(\varsigma) = \varsigma^3 + A_1\varsigma^2 + A_2\varsigma + A_3 + e^{-\varsigma t}(B_1\varsigma^2 + B_2\varsigma + B_3)$ and η_0 is the smallest positive root of (4.4).

In our case, (4.3) and (4.4) gives

$$A_3 - A_1\eta_0^2 = B_1\eta_0^2\cos(\eta_0\tau) - B_3\cos(\eta_0\tau) - B_2\eta_0\sin(\eta_0\tau), \tag{4.5}$$

$$A_2\eta_0 - \eta_0^3 > -B_1\eta_0^2\sin(\eta_0\tau) + B_3\sin(\eta_0\tau) - B_2\eta_0\cos(\eta_0\tau). \tag{4.6}$$

(4.5) and (4.6), if satisfied simultaneously, are sufficient conditions to guarantee stability. We shall utilize them to get an estimate on the length of delay. Our aim is to find an upper bound η_+ on η_0, independent of τ and then to estimate τ so that (4.6) holds for all values of η, $0 \le \eta \le \eta_+$ and hence in particular at $\eta = \eta_0$. We rewrite (4.5) as

$$A_1\eta_0^2 = A_3 + B_3\cos(\eta_0\tau) + B_2\eta_0\sin(\eta_0\tau) - B_1\eta_0^2\cos(\eta_0\tau). \tag{4.7}$$

Maximizing $A_3 + B_3\cos(\eta_0\tau) + B_2\eta_0\sin(\eta_0\tau) - B_1\eta_0^2\cos(\eta_0\tau)$ subject to $|\sin(\eta_0\tau)| \le 1$, $|\cos(\eta_0\tau)| \le 1$ we obtain

$$A_1\eta_0^2 \le A_3 + |B_3| + |B_2|\eta_0 + |B_1|\eta_0^2. \tag{4.8}$$

Hence, if

$$\eta_+ = \frac{|B_2| + \sqrt{B_2^2 + 4(A_1 - |B_1|)(A_3 + |B_3|)}}{2(A_1 + |B_1|)}, \tag{4.9}$$

then clearly from (4.8) we have $\eta_0 \le \eta_+$.

From (4.5) we obtain

$$\eta_0^2 < A_2 + B_1\eta_0\sin(\eta_0\tau) + B_2\cos(\eta_0\tau) - B_3\frac{\sin(\eta_0\tau)}{\eta_0}. \tag{4.10}$$

As E_2 is locally asymptotically stable for $\tau = 0$, therefore sufficiently small $\tau > 0$, (4.19) will continue to hold. Substituting (4.7) in (4.10) and rearranging we get,

(4.11)

$$(B_3 - A_1B_2 - B_1\eta_0^2)[\cos(\eta_0\tau) - 1] + \left[(B_2 - A_1B_1)\eta_0 + \frac{A_1B_3}{\eta_0}\right]\sin(\eta_0\tau)$$
$$< A_1A_2 - A_3 - B_3 + A_1B_2 + B_1^2\eta_0.$$

Using the bounds

$$\begin{aligned} &(B_3 - A_1B_2 - B_1\eta_0^2)[\cos(\eta_0\tau) - 1] \\ &= 2(B_3 - A_1B_2 - B_1\eta_0^2)\sin^2\left(\frac{\eta_0\tau}{2}\right) \\ &\le \frac{1}{2}|B_3 - A_1B_2 - B_1\eta_+^2|\eta_+^2\tau^2 \end{aligned}$$

and

$$\left[|B_2 - A_1B_1|\eta_0 + \frac{A_1B_3}{\eta_0}\right]\sin(\eta_0\tau) \le [(B_2 - A_1B_1)\eta_+^2 + A_1|B_3|]\tau,$$

we obtain from (4.11)

$$K_1\tau^2 + K_2\tau < K_3,$$

where

$$\begin{aligned} K_1 &= \tfrac{1}{2}|B_3 - A_1B_2 - B_1\eta_+^2|\eta_+^2, \\ K_2 &= (B_2 - A_1B_1)\eta_+^2 + A_1|B_3|, \\ K_3 &= A_1A_2 - A_3 - B_3 + A_1B_2 + B_1^2\eta_+. \end{aligned}$$

Hence, if $\tau_+ = \frac{1}{2K_1}(-K_2 + \sqrt{K_2^2 + 4K_1K_3})$, then stability is preserved for $0 \le \tau \le \tau_+$. Thus we are now in a position to state the following theorem.

Theorem 4.1. *If there exists a τ in $0 \le \tau \le \tau_+$ such that $K_1\tau^2 + K_2\tau < K_3$, then τ_+ is the maximum value* (*length of delay*) *of τ for which E_2 is asymptotically stable.*

5. Permanence

From biological point of view, persistence of a system means the survival of all populations of the system in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non-negative cone. Butler et al. [4], Freedman and Waltman [8, 10] developed the following definition of persistence:

Definition 5.1. System (1.4) is said to be permanence if there are positive constants m, M such that each positive solution $(S(t), I(t), y(t))$ of system (1.4) with initial conditions satisfies

$$m \le \lim_{t\to+\infty} \inf S(t) \le \lim_{t\to+\infty} \sup S(t) \le M,$$
$$m \le \lim_{t\to+\infty} \inf I(t) \le \lim_{t\to+\infty} \sup I(t) \le M,$$
$$m \le \lim_{t\to+\infty} \inf y(t) \le \lim_{t\to+\infty} \sup y(t) \le M.$$

In order to prove permanence of system (1.4), we present the permanence theory for infinite dimensional system from Theorem 4.1 in [11]. Let X be a complete metric space. Suppose that $X^0 \in X$, $X_0 \in X$, $X^0 \cap X_0 = \emptyset$. Assume that $T(t)$ is a C_0 semigroup on X satisfying

$$T(t) : X^0 \to X^0, \quad T(t) : X_0 \to X_0. \tag{5.1}$$

Let $T_b(t) = T(t) \mid_{X_0}$ and let A_b be the global attractor for $T_b(t)$.

Lemma 5.1 ([11])**.** *Suppose that $T(t)$ satisfies* (5.1) *and we have the following*:
(i) *there is a $t_0 \ge 0$ such that $T(t)$ is compact for $t > t_0$*;
(ii) *$T(t)$ is point dissipative in X*;
(iii) *$\overline{A}_b = \cup_{x\in A_b}\omega(x)$ is isolated and has an acyclic covering M, where*

$$\overline{M} = \{M_1, M_2, \dots, M_n\};$$

(iv) *$W^s(M_i) \cap X^0 = \emptyset$ for $i = 1, 2, \dots, n$.*
Then X_0 is a uniform repellor with respect to X^0, i.e., there is an $\epsilon > 0$ such that, for any $x \in X^0$, $\lim_{t\to+\infty} \inf d(T(t)x, X_0) \ge \epsilon$, where d is the distance of $T(t)x$ from X_0.

Theorem 5.1. *If $\beta K > \mu$, then system* (1.4) *is permanent.*

Proof. We begin by showing that the boundary planes of R^3_+ repel the positive solutions to system (1.4) uniformly. Let us define

$$C_0 = \{(\varphi_1, \varphi_2, \varphi_3) \in C([-\tau, 0], R^3_+) : \varphi_3(\theta) = 0, \varphi_1(\theta) \ne 0 \text{ and } \varphi_2(\theta) \ne 0\}.$$

If $C^0 = \text{int}C([-\tau, 0], R^3_+)$, it is suffices to show that there exists an ε_0 such that for all solution u_t of system (1.4) initiating from C^0, $\liminf_{t\to\infty} d(u_t, C_0) \ge \varepsilon_0$. To this end, we verify below that the conditions of Lemma 5.1 are satisfied. It is easy to see that C_0 and C^0 are positive invariant. Moreover, conditions

(i) and (ii) of Lemma 5.1 are clearly satisfied. Thus, we only need to verify conditions (iii) and (iv).

There is a constant solution E_1 in C_0. If $(S(t), I(t), y(t))$ is a solution of system (1.4) initiating C_0, to $S(t) = S_1$, $I(t) = I_1$, $y = 0$, where $S_1 = \frac{\mu}{\beta}, I_1 = \frac{r}{\beta}(1 - \frac{\mu}{\beta K})$. If $(S(t), I(t), y(t))$ is a solution of system (1.4) initiating from C_0, then $S(t) \to S_1$, $I(t) \to I_1$, $y \to 0$ as $t \to +\infty$. It is obvious that E_1 is isolated invariant. Now, we show that $W^s(E_1) \cap C^0 = \emptyset$. Assuming the contrary, then there exists a positive $(\widetilde{S}(t), \widetilde{I}(t), \widetilde{y}(t))$ of system (1.4) such that $(\widetilde{S}(t), \widetilde{I}(t), \widetilde{y}(t)) \to (S_1, I_1, 0)$ as $t \to +\infty$. Choosing $\xi > 0$ small enough such that $I_1 - \xi > 0$ when $\beta K > \mu$. Let $t_0 > 0$ be sufficiently large such that $I_1 - \xi < \widetilde{I}(t) < I_1 + \xi$ for $t \geq t_0 - \tau$. Then we have, for $t \geq t_0$,

$$\frac{d\widetilde{y}}{dt} \geq \delta\widetilde{y}\left(1 - \frac{h\widetilde{y}}{I_1 - \xi}\right).$$

It is easy to prove that $\widetilde{y}(t) \geq \frac{I_1-\xi}{h}$ when $I_1 - \xi > 0$. This is a contradiction. Hence, $W^s(E_1) \cap C^0 = \emptyset$.

Therefore, we are able to conclude from Lemma 5.1 that C_0 repels the positive solutions of system (1.4) uniformly, then the conclusion of Theorem 5.1 follows. □

6. Numerical study of the system behavior

We have gained analytical understanding of possible dynamics of this nonlinear delay differential equation model to some extent. We now perform some simulation work (using MATLAB dde23) with hypothetical set of parameters given in Table 1 and initial values $S(0) = 15$, $I(0) = 10$, $y(0) = 20$ for better understanding of our analytical treatment. In fact we have considered different values of the delay factor (τ) to observe biologically plausible different dynamical scenarios of the model, enough to merit the mathematical study.

Table 1: Parameter values used for simulation

Parameter	Values
r (intrinsic birth rate of the sound prey)	0.1
K (carrying capacity of the sound prey)	500
β (infection rate)	0.001
μ (death rate of the infected prey)	0.03
c (the maximum value of the per capita reduction rate of predator due to prey)	8
m (half saturation constant)	150
δ (intrinsic growth rate of the predator)	0.2
h (the maximum value of the per capita reduction rate of prey due to predator)	0.5

First we observe that without delay there exits a unique interior equilibrium point E_2 (83.15614618, 83.36877076, 166.7375415) with the set of parameter values from Table 1. Positive steady state E_2 is locally asymptotically stable, since the eigenvalues associated with the variational matrix of the system (1.4) at E_2, given by $(-0.2006433278$, $-0.007905651446 - 0.08379142400i$,

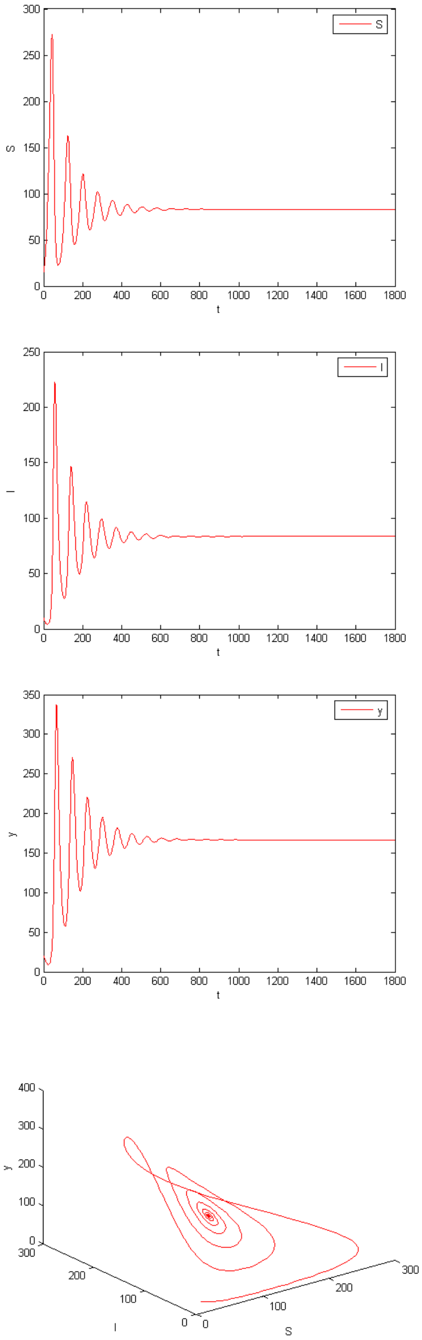

Fig.1: Time evolution of all the population for the model (1.4) with $\tau = 0$.

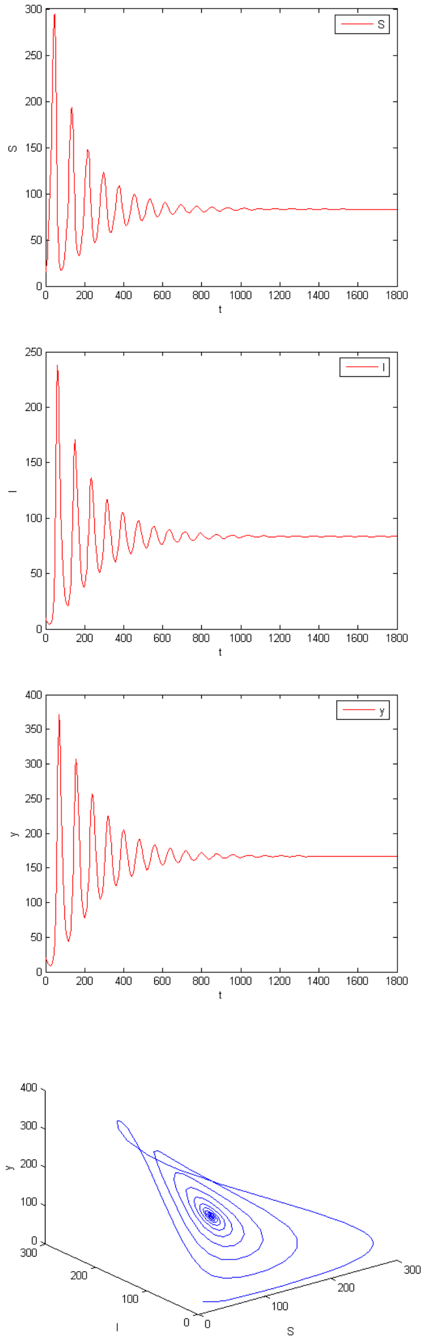

Fig.2: Time evolution of all the population for the model (1.4) with $\tau = 1$.

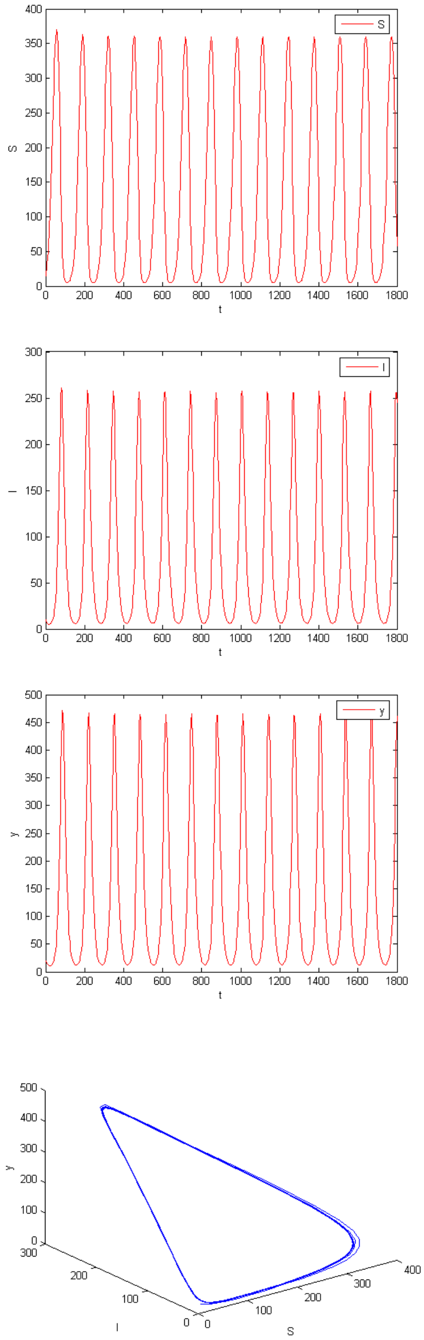

Fig.3: Time evolution of all the population for the model (1.4) with $\tau = 8$.

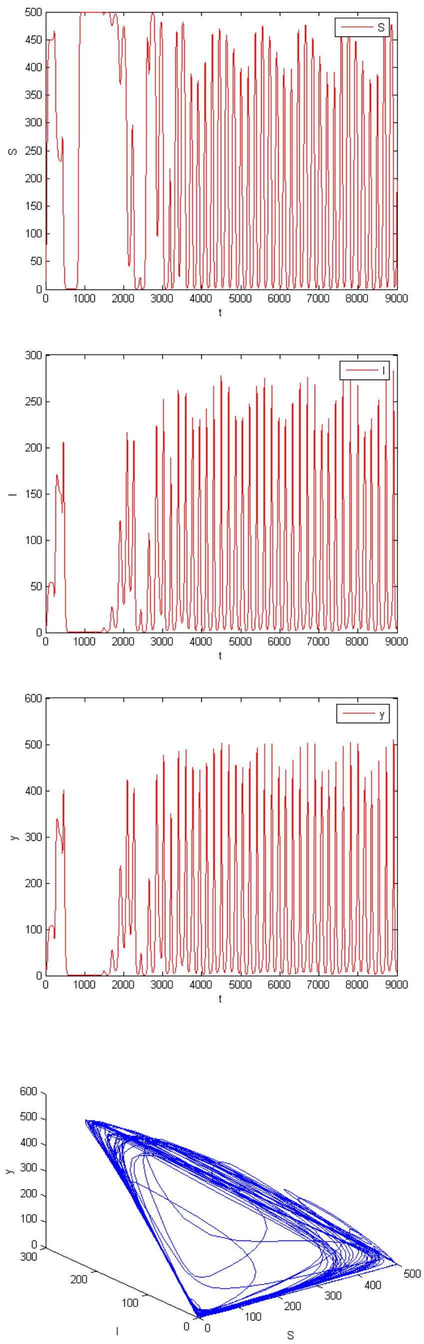

Fig.4: Time evolution of all the population for the model (1.4) with $\tau = 200$.

$-0.007905651446+0.08379142400i$) have negative real parts. Simulation of the model in this situation with $\tau = 0$, produce stable dynamics and is presented in Fig. 1. With the same set of parameters, we see that $P_3(-0.000001135012312) < 0$ and $P_1(0.04017661904) > 0$, which indicates the existence of a positive root. Solving (3.6) and (3.7) numerically, we see that there exist one simple positive root of, namely, $0.07369188011(= \omega_0)$. Hence, by Theorem 3.3, we can say that as τ increases, stability switch may occur. The value of τ where stability switch occurs (in our case) is $\tau_0 = 3.337326353$, which can be easily calculated using (3.6) and (3.7). Hence, by Butler's lemma, E_2 remains stable for $\tau < \tau_0$ $(= 3.337326353)$, which can be seen in Figs. 1 and 2 and which are the solutions of the system (1.4) for $\tau = 0$ and $\tau = 1$, respectively. As τ increases through $\tau = \tau_0 = 3.337326353$, a periodic solution occurs which is the case of Hopf bifurcation. The importance of Hopf bifurcation in this context is that at the bifurcation point a limit cycle (see Fig. 3) is formed around the fixed point, thus resulting in stable periodic solutions. No more stability switches occur and for $\tau > \tau_0 = 3.337326353$, E_2 is unstable, with increasing oscillations. It is interesting to observe that for sufficiently large τ, the system (1.4) remains unstable but show limit cycle with complex structure (Fig. 4).

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