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EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS WITH SUBLINEAR GROWTH NONLINEARITIES

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Abstract. In this paper, we establish a multiple existence result of T-periodic solutions for the semilinear parabolic boundary value problem with sublinear growth nonlinearities. We adapt sub-supersolution scheme and topological argument based on variational structure of functionals.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary ∂u. In this paper, we are concerned with the multiple existence result of T-periodic solutions for the semilinear parabolic boundary value problem

$$(P) \qquad \begin{cases} u_t - \triangle_x u + u = g(u) + h(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u(T) & \text{in } \bar{\Omega}. \end{cases}$$

We assume $u = u(t,x)$, $g : \mathbb{R} \to \mathbb{R}$ is continuous, and $h : \mathbb{R}\times\bar{\Omega} \to \mathbb{R}$ is a continuous function which is T-periodic with respect to the first variable and $h > 0$ on $\mathbb{R}\times\Omega$. There are many results for the multiple existence of T-periodic solutions for seminear parabolic equations with this type of nonlinearity in [6, 7, 8, 9], and for elliptic equations also in [2, 4, 10].

Here, we denote Q_T the open set $(0,T)\times\Omega$. For $q \geq 1$, we denote by $|\cdot|_q$ and $||\cdot||_q$ the norms of $L^q(\Omega)$ and $W^{1,q}(\Omega)$, respectively. $||\cdot||$ stands for the norm of $H_0^1(\Omega)$. We put $V = H_0^1(\Omega)$, $H = L^2(\Omega)$. The norm of the dual space V^* of V is denoted by $||\cdot||_*$. $\langle\cdot,\cdot\rangle$ stands for the paring of V and V^*. A function $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ is said to be a solution of (P) if u satisfies (P). Here, we assume

(H_1) g is Lipschitz continuous, nondecreasing, odd function and $g(0) = 0$,

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(H_2) there exist $C_1 > 0$ and $0 < \alpha < 1$ such that $|g(u)| \le C_1|u|^\alpha$ on $\mathbb{R}$,

(H_3) there exists $C_2 > 0$ such that

$$\liminf_{|u|\to 0} \frac{G(u)}{|u|} \ge C_2,$$

where $G(u) = \int_0^u g(s)ds$,

(H_4)

$$\lim_{|u|\to\infty} \frac{g(u)}{u} < \lambda_1,$$

(H_5)

$$\lambda_1 < \lim_{|u|\to 0} \frac{g(u)}{u},$$

where we denote by $\lambda_1 < \lambda_2 \le \cdots$ the eigenvalues of the problem

$$-\triangle u = \lambda u, \quad u \in H_0^1(\Omega)$$

and by ϕ_1 the normalized eigenfunction corresponding to λ_1.

Such a function exists; for example, we first fix a smooth function $\phi : (-\infty, \infty) \to [0, 1]$ such that $\phi'(t) \le a$, and

$$\phi(t) = \begin{cases} 0 & \text{for } t \in (-\infty, -1] \cup [1, \infty) \\ 1 & \text{for } t \in [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

Let $n \ge 1$ and $t_n^\pm$ be the numbers such that $t_n^- < 0 < t_n^+$ and $g(2t_n^\pm) = 2nt_n^\pm$. We put

$$g_n(t) = \begin{cases} n\phi_n^+(t)t + (1 - \phi_n^+(t))h(t) & \text{for } t \ge 0 \\ n\phi_n^-(t)t + (1 - \phi_n^-(t))h(t) & \text{for } t \le 0, \end{cases}$$

where $h(t) = |t|^{\alpha-1}t$, $\phi_n^+(t) = \phi(\frac{t}{2t_n^+})$ and $\phi_n^-(t) = \phi(\frac{-t}{2t_n^-})$. Then we have that $g_n(t) = nt$ on $[t_n^-, t_n^+]$ and $g_n(t) = h(t)$ on $(-\infty, 2t_n^-) \cup (2t_n^+, \infty)$. For Lipschitz continuity of g_n, let consider the case that $t > 0$. From the definition, we have $g_n(t) = nt$ on $[0, t_n^+]$. On the other hand, we have that for $t \in [t_n^+, 2t_n^+]$,

$$\begin{aligned} g_n'(t) &= n((\phi_n^+(t))'t + \phi_n^+(t)) + (1 - \phi_n^+(t))h'(t) - (\phi_n^+(t))'h(t) \\ &\le n\left(\frac{at}{2t_n^+} + 1\right) + \frac{\alpha}{t^{1-\alpha}} + \frac{at}{2t_n^+}t^\alpha \\ &\le n\left(\frac{a}{2} + 1\right) + \frac{\alpha}{(t_n^+)^{1-\alpha}} + \frac{a}{2}(2t_n^+)^\alpha. \end{aligned}$$

Then we find $g_n'(t) \le C \max\{n, h'(t)\}$ for some $C > 0$. Moreover recalling that $n(2t_n^+)^{1-\alpha} \cong 1$, we find that $h'(t) \le Cn$ on $[t_n^+, 2t_n^+]$ for some $C > 0$, and hence each g_n is Lipschitz continuous on $\mathbb{R}$. Therefore (H_1)-(H_5) follows from the definition.

2. Preliminary results

Let us consider a initial boundary value problem associated with (P)

$$(I) \qquad \begin{cases} u_t - \triangle_x u + u = g(u) + h & \text{in} \quad (0.\infty) \times \Omega \\ u(t) = 0 & \text{on } (0, \infty) \times \Omega \\ u(0) = u_0 & \text{in } \partial\Omega, \end{cases}$$

where $u_0 \in L^2(\Omega)$ and $h \in C^1(\bar{Q}_T)$. We denote by $t(u_0)$ the number such that $[0, t(u_0))$ is the maximal interval for $u(t)$ to exist. If u is a solution of problem (I) on $[0, t(u_0))$, u can be represented by the integral form

$$u(t) = S(t)u_0 + \int_0^t S(t-s)(g(u(s)) - u(s) + h(s,x))ds \tag{2.1}$$

for $0 < t < t(u_0)$. Here, $\{S(t)\}$ is the semigroup of linear operators generated by $-\triangle_x$. It is known that for each $q \geq 2$, there exists $c(q) > 0$ satisfying

$$||S(t)f||_q \leq c(q)t^{-1/2}|f|_q \quad \text{for all } f \in L^q(\Omega) \text{ and } t > 0 \tag{2.2}$$

(cf. Amann [1], Tanabe [12]). If we set $X_+ = \{u \in C_0^1(\bar{\Omega}); u \geq 0 \text{ on } \Omega\}$, then X_+ is a closed cone in $C_0^1(\bar{\Omega})$. We employ the standard order in $C_0^1(\bar{\Omega})$ as

$$u \geq v \Leftrightarrow u - v \in X_+, \quad u > v \Leftrightarrow u \geq v, u \neq v, \quad u \gg v \Leftrightarrow u - v \in \text{int}X_+.$$

For each $u, v \in C_0^1(\bar{\Omega})$, we put

$$[v, u] = \{w \in C_0^1(\bar{\Omega}); v \leq w \leq u\}.$$

A mapping $S : [u, v] \to C_0^1(\bar{\Omega})$ is said to be order preserving if $Sx \gg Sy$ for $x, y \in [u, v]$ with $x > y$. Here, we denote by S the Poincare mapping associated with problem (I). That is $Su_0 = u(T), u_0 \in H$. It is obvious that the Poincare mapping S is well defined only when $t(u_0) > T$. It follows from the parabolic maximal principle that S is strictly monotone with respect to the order defined above. That is, if $u > v$ in $C_0^1(\bar{\Omega})$ and Su, Sv exist, then $Su \gg Sv$. A function $u \in C^{1,2}((0.T) \times \Omega) \cap C^{0,1}((0,T) \times \bar{\Omega})$ is called subsolution (cf. Hess [5]) for the T-periodic problem (I) if

$$\begin{cases} u_t - \triangle_x u + u \leq g(u) + h & \text{in } (0, \infty) \times \Omega \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

A subsolution is said to be a strict subsolution if it is not a solution of (I). Similarly, a supersolution and strict supersolution are defined by the inequality sign, correspondingly.

3. Multiplicity result

We set

$$C([0,T]; u_0, H) = \{u \in C([0,T], H); u(0,x) = u_0(x) \text{ on } \Omega\}$$

for each $u_0 \in H$. For each $u_0 \in H$, we define a mapping $K_{u_0} : C([0,T]; u_0, H) \to C([0,T]; u_0, H)$ by

$$(K_{u_0}u)(t) = S(t)u_0 + \int_0^t S(t-s)(g(u(s)) - u(s) + h(s,x))ds$$

for each $u \in C([0,T]; u_0, H)$. Then we have:

Lemma 3.1. *For each $u_0 \in H$, K_{u_0} is compact and has a unique fixed point in $v_{u_0} \in C([0,T]; u_0, H)$.*

Proof. See the proofs of Theorems 1.7 and 2.1 in Chapter 6 of Pazy [11]. □

Remark. Since $Su_0 = v_{u_0}(T)$ and v_{u_0} is a solution of (I), v_{u_0} is a periodic solution of (I).

By (H_5), there exists $\mu_1 > 0$ such that $\frac{g(u)}{u} > \lambda_1$ for all $|u| \le \mu_1$.

Let $0 < \epsilon < 1$ be such that $h - \epsilon\phi_1 > 0$ and $|\epsilon\phi_1|_\infty \le \mu_1$ on Ω. Then we have

$$-\Delta(\epsilon\phi_1) + \epsilon\phi_1 = \epsilon\lambda_1\phi_1 + \epsilon\phi_1 < g(\epsilon\phi_1) + h \text{ on } \Omega.$$

Hence $\epsilon\phi_1$ is a strict subsolution of (I). Let $0 < \lambda < \lambda_1$. By (H_4), there exists $\mu_2 > 0$ such that $g(u) < \lambda u$ for all $|u| \ge \mu_2$. Put $c = \max\{g(u) : 0 \le u \le \mu_2\}$. Since $\lambda < \lambda_1$. Dirichlet boundary value problem

$$-\Delta_x u = \lambda u + c + h$$

has a solution $v \in H_0^1(\Omega)$. Note that $c + h > 0$, we have that $v \in C^1(\bar{\Omega})$ and $v > 0$ on Ω. Let $b > 0$ and put $\tilde{u} = b\phi_1 + v$. Then

$$\begin{aligned}\lambda v(x) + \lambda_1 b\phi_1(x) &> \lambda(v(x) + b\phi_1(x)) \\ &> g(v(x) + b\phi_1(x)) \text{ for } x \in \Omega \text{ with } \tilde{u}(x) \ge \mu_2\end{aligned}$$

and $c > g(\tilde{u}(x))$ for $x \in \Omega$ with $\tilde{u}(x) < \mu_2$.
Hence, we have

$$-\Delta_x\tilde{u} + \tilde{u} \ge \lambda v + \lambda_1 b\phi_1 + c + h > g(\tilde{u}) + h.$$

Therefore, $\tilde{u}$ is a strict supersolution of (I). Recall that $\partial\phi_1/\partial n < 0$ and $\partial v/\partial n < 0$ on $\partial\Omega$ by the maximal principle. Then we can choose $b > 0$ sufficiently large so that $\epsilon\phi_1 \ll \tilde{u}$ on Ω. We know that S is strongly order preserving on $[\epsilon\phi_1, \tilde{u}]$ and

$$S[\epsilon\phi_1, \tilde{u}] \subset [\epsilon\phi_1, \tilde{u}].$$

We know that $S[\epsilon\phi_1, \tilde{u}]$ is relatively compact in $C_0^1(\bar{\Omega})$ (cf. Proposition 21.2 of [5]). Hence, by Theorem 4.2 of [5], we have two sequences $u_n^{(1)} \equiv S^n(\epsilon\phi)$ and $u_n^{(2)} \equiv S^n(\tilde{u})$ which converges to a fixed point $u^{(1)}$ and $u^{(2)}$ of S as $n \to \infty$, respectively and $\epsilon\phi_1 < u^{(1)} \le u^{(2)} < \tilde{u}$. From Remark 21.3 of [5], the problem (P) has a solution $u_1 \in C^{1,2}([0,T] \times \tilde{\Omega})$ with $u_1(0) = u_1(T) = u^{(i)}$ for $i = 1, 2$ (cf. Lemma 20.1 of [5]). Therefore we have:

Lemma 3.2. *For each $h \in C^1(\bar{Q}_T)$ and $h > 0$, there exist a solution $u_1 \in C^{1,2}([0,T] \times \bar{\Omega})$ of (P) such that $\epsilon\phi_1 < u_1(t) < \bar{u}$ on $[0,T]$.*

Next, we prove the existence of the second solution.

By Lemma 3.1, we have:

Lemma 3.3. *If $\lim_{n\to\infty} |u_n^{(1)} - u_n^{(2)}|_{c_0^1(\bar{\Omega})} > 0$, then we have two solutions u_1, u_2 of (P) such that $\epsilon\phi_1 < u_1(0) = u^{(1)} < u_2(0) = u^{(2)} < \bar{u}$.*

Proof. cf. Lemma 3.1 and Remark 21.3 in [5]. □

To complete our assertion, we assume that

$$\lim_{n\to\infty} |u_n^{(1)} - u_n^{(2)}|_{c_0^1(\bar{\Omega})} = 0. \tag{3.1}$$

Now, we let $I : V \to R$ be a functional defined by

$$I(v) = \frac{1}{2}||v||_2^2 - \int_\Omega G(v)dx \quad \text{for} \quad v \in V.$$

By I^c, we denote the level set $I^c = \{v \in V : I(v) \leq c\}$. From the definition of I and (H_2), we can see that $\lim_{||v||_2\to\infty} I(v) = \infty$. Thus we have that

$$-\infty < m_1 = \min\{I(v) : v \in V\}.$$

(H_3) implies that for any nonzero $v \in V$, there is sufficiently small $t > 0$ that $I(tv) < 0$. That is $m_1 < 0$.

Lemma 3.4. *For any $\delta \in [m_1, 0]$, there exist $m \geq 1$ and a continuous function $h : S^m \to I^\delta$ such that $h(S^m)$ is not contractible in I^δ, where S^m denotes the unit sphere in R^m.*

Proof. We put $V_k = \text{span}\{\phi_1, \phi_2, \ldots, \phi_k\}$. Fix $\delta \in [m_1, 0]$. Let $v \in V$ with $||v||_2 = 1$. From (H_1), we have that the mapping $s \to I(sv)$ is decreases on interval $[0,t]$, where $t > 0$ satisfies

$$I(tv) = \min\{I(sv) : \ s \geq 0\}$$

and increases on $[t, \infty)$. From the definition of I, we have, by (H_2),

$$t^2||v||_2^2 = \int_\Omega g(tv)tvdx \leq C_1 \int_\Omega t^{\alpha+1}|v|^{\alpha+1}dx.$$

Suppose that $v \in V_{k-1}^\perp$ for some $k \geq 2$. Since $|v|_{\alpha+1}^{\alpha+1} \leq C_3|v|_2^{\alpha+1}$ for some $C_3 > 0$ and $\lambda_k|v|_2^2 \leq |\nabla v|_2^2$, we have that

$$\begin{aligned} t^{1-\alpha}||v||_2^2 &\leq C_1|v|_{\alpha+1}^{\alpha+1} \\ &\leq C_1C_3|v|_2^{\alpha+1} \\ &\leq C_1C_3\left(\frac{1}{\lambda_k}\right)^{\alpha+1}||v||_2^{2(\alpha+1)} \end{aligned}$$

and hence $0 < t \leq (C_1C_3)^{\frac{1}{1-\alpha}}(\frac{1}{\lambda_k})^{\frac{1+\alpha}{1-\alpha}}$. This implies that $t \to 0$ when $k \to \infty$. By (H_1) and (H_2),

$$\begin{aligned} I(tv) = \frac{t^2}{2}||v||_2^2 - \int_\Omega \int_0^{tv} g(s)ds &\geq \frac{t^2}{2} - C_1 t^{\alpha+1} \int_\Omega |v|^{\alpha+1} dx \\ &\geq \frac{t^2}{2} - C_1C_3 t^{\alpha+1} |v|_2^{\alpha+1} \\ &\geq \frac{t^2}{2} - C_1C_3 t^{\alpha+1} \left(\frac{|\nabla v|_2}{\sqrt{\lambda_k}} \right)^{\alpha+1} \\ &\geq \frac{t^2}{2} - C_1C_3 \left(\frac{t}{\sqrt{\lambda_k}} \right)^{\alpha+1}. \end{aligned}$$

Thus $I(tv) \to 0$ as $k \to \infty$. Therefore there exists $k_0 \geq 0$ such that $I^\delta \cap V_{k_0}^\perp = \phi$. Let $v_0 \in I^\delta$, then since I is an even function, $-v_0 \in I^\delta$. If $\{v_0, -v_0\}$ is contractible in I^δ, by Krasonalski's result (cf. Lemma 3.2 of Bahri [3]), we can define an odd continuous function $h_1 : S^1 \to I^\delta$ such that $h_1(S^1) \subset I^\delta$. By induction, if $h_{k_0-1}(S^{k_0-1})$ is contractible, we can construct an odd and continuous function $h_{k_0} : S_0^k \to I^\delta$. but since $h_{k_0}(S^{k_0}) \cap V_{k_0}^\perp \neq \phi$, this is impossible. Hence, this proves our theorem. □

By (H_4), there exists $c_3 > 0$ such that $\lambda u - c_3 < g(u)$ for all $u \leq 0$, where $0 < \lambda < \lambda_1$. Then there exists a negative solution $\underline{v} \in c^1(\bar{\Omega})$ of the Dirichlet problem

$$-\Delta_x u = \lambda u - c_3.$$

Let $a \geq 1$. If we put $\underline{u} = a\underline{v}$, then

$$-\Delta_x \underline{u} + \underline{u} = \lambda a \underline{v} - c_3 + a\underline{v} < \lambda a \underline{v} - c_3 < g(\underline{u}) + h.$$

That is $\underline{u}$ is a strict subsolution of (P).

Lemma 3.5. *For any $\delta < 0$, there exists $\delta_1, \delta_2 < 0$ such that $\delta < \delta_1 < \delta_2 < 0$ and the interval $[\delta_1, \delta_2]$ contains no critical point of I.*

Proof. Let $\delta_0 < 0$ and suppose contrary that there exists no interval in $(\delta_0, 0)$ satisfying the condition. Then, for any $\delta_0 < \delta < 0$, there exists a sequence $\{u_n\} \subset V$ such that $\nabla I(u_n) = 0$; i.e., $-\Delta u_n + u_n = g(u_n)$ and $\lim_{n\to\infty} I(u_n) = \delta$.
Then, by (H_2), we have

$$\begin{aligned} \delta = \lim_{n\to\infty} I(u_n) = &\lim_{n\to\infty} \left(\frac{1}{2}||u_n||_2^2 - \int_\Omega \int_0^{u_n(x)} g(t)dtdx \right) \\ \geq &\lim_{n\to\infty} \left(\frac{1}{2}||u_n||_2^2 - \frac{C_1}{1+\alpha}|u|_{1+\alpha}^{1+\alpha} \right). \end{aligned}$$

Hence $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$ and hence bounded in V. Therefore there exists a subsequence, say again $\{u_n\}$, such that $\{u_n\}$ converges to $u \in V$ strongly in H and weakly in V.

Since g is Lipschitz continuous and

$$||u_m - u_n||_2^2 \leq |g(u_m) - g(u_n)|_2 |u_m - u_n|_2 \leq L|u_m - u_n|_2$$

for some constant $L > 0$, $\{u_n\}$ converges to u strongly in V. Therefore, we have $\nabla I(u) = 0$ and $I(u) = \delta$. This is impossible and completes our assertion. □

Lemma 3.6. *Let $\delta_0 < 0$ and δ_1, δ_2 be constants, $\delta_0 < \delta_1 < \delta_2 < 0$, satisfying the assertion of Lemma 3.5. Then there exists m_0 such that, for each $h \in C^1(\bar{Q}_T)$ with $|h|_{C^1(\bar{Q}_T)} < m_0$, if v is the solution of (I) with $v(0) \in I^\delta$ for some $\delta \in [\delta_1, \delta_2]$, then $v(t) \in I^\delta$ for $t \geq 0$.*

Proof. Let δ_0 such that $I(\epsilon\phi_1) < \delta_0$. Let δ_1, δ_2 be constants such that $\delta_0 < \delta_1 < \delta_2 < 0$ and satisfying the assertion of Lemma 3.5. Then we define $\tilde{m}_0 = \inf\{||\nabla I(v)||_* : v \in I^{\delta_2} \setminus I^{\delta_1}\}$, then we have $\tilde{m}_0 > 0$. We put $m_0 = \tilde{m}_0/|\Omega|^{1/2}$. Now let $h \in C^1(\bar{Q}_T)$ with $|h|_{C^1(\bar{Q}_T)} < m_0$. Suppose $\delta \in [\delta_1, \delta_2]$, $v(0) \in I^\delta$ and $v(t) \in I^{\delta_2}$ on an interval $[0, t_{v(0)}]$. From the definition of m_0, we have that for $t \in [0, t_{v(0)}]$, using the Holder inequality,

$$\begin{aligned} I(v(t)) - I(v(0)) &= \int_0^t \nabla I(v(s)) \cdot \frac{dv}{ds} \\ &\leq \int_0^t (-||\nabla I(v)||_*^2 + ||h(s)|| ||\nabla I(v)||_*) \\ &\leq \int_0^t ||\nabla I||_* (-||\nabla I||_* + ||h(s)||) < 0. \end{aligned}$$

Then we have $I(v(t)) < I(v(0))$. Hence, we have that $v(t) \in I^\delta$ for all $t \geq 0$. This completes our assertion. □

Theorem. *There exists $m_0 > 0$ such that for each $h \in C^1(\bar{Q}_T)$ with $|h|_{C^1(\bar{Q}_T)} < m_0$, there exists a periodic solution u_2 in $V \setminus [\epsilon\phi_1, \bar{u}]$.*

Proof. Let δ_0, m_0 be as in Lemma 3.5. Let u_1 be the solution of (P) obtained in Lemma 3.2.

Suppose there in no fixed point of S in $V \setminus [\epsilon\phi_1, \bar{u}]$. Let δ_0, δ_2 be constants such that $\delta_0 < \delta_1 < \delta_2 < 0$ satisfying the assertion of Lemma 3.5. We recall Lemma 3.6. Since $\epsilon\phi_1 \in I^{\delta_0}$ and $u^{(1)} = \lim_{n\to\infty} u_n^{(1)} = \lim_{n\to\infty} S^n(\epsilon\phi_1)$, we find that $u^{(1)} \in I^{\delta_1}$. Let $\epsilon > 0$ be such that $\delta_1 + 2\epsilon < \delta_2$. Then, by (3.1), there exists n_0 such that for all $n \geq n_0$, such that

$$u_n^{(1)}, u_n^{(2)} \in I^{\delta_1 + \epsilon/2}$$

and

$$\left| \int_\Omega G(v)dx - \int_\Omega G(z)dx \right| < \epsilon \text{ for all } v, z \in [u_n^{(1)}, u_n^{(2)}]. \tag{3.2}$$

Since $|| \cdot ||_2^2$ is a convex function, by (3.2), $I(\alpha v + (1-\alpha)u_n^{(1)}) < \delta_1 + 2\epsilon$ for all $v \in [u_n^{(1)}, u_n^{(2)}] \cap I^{\delta_1+\epsilon}$ and $\alpha \in [0, 1]$, and hence $[u_n^{(1)}, u_n^{(2)}] \cap I^{\delta_1+\epsilon}$ is star convex with respect to $u_n^{(1)}$ in $I^{\delta_1+2\epsilon}$. Therefore, for any sufficiently small $r > 0$,

$$[v,z] \cap I^{\delta_1+\epsilon} \text{ is cotractible in } I^{\delta_2}, \tag{3.3}$$

where $v, z \in C_0^1(\bar{\Omega})$ such that $||v - u_n^{(1)}||_2 < r$ and $||z - u_n^{(2)}||_2 < r$.

By Lemma 3.4, there exist $m > 0$ and a continuous function $h : S^m \to I^{\delta_2}$ such that $h(S^m)$ is not contractible in I^{δ_2}. Since $C_0^1(\bar{\Omega}) \cap I^{\delta_2}$ is dense in I^{δ_2}, we may have $h(S^m) \subset C_0^1(\bar{\Omega}) \cap I^{\delta_2}$. Let u_z be the solution of (I) with $u_z(0) = h(z), z \in S^m$. By choosing $b > 0$ sufficiently large in the definition of $\tilde{u}$, we have another strict supersolution $\bar{u} > \tilde{u}$ such that $v < \bar{u}$ for all $v \in h(S^m)$. Similarly, by choosing $a > 0$ in the definition of $\underline{u}$, we have that $\underline{u} < \epsilon\phi_1$ and $\underline{u} < v$ for all $v \in h(S^m)$. We recall that $\underline{u}$ and $\bar{u}$ are strict sub and supersolution of (I), respectively and that $\underline{u} < \epsilon\phi_1, \tilde{u} < \bar{u}$. Since S has no fixed point in $V \setminus [\epsilon\phi_1, \tilde{u}]$, we have that $S^n(\underline{u}) \to u^{(1)}$ and $S^n(\bar{u}) \to u^{(2)}$ as $n \to \infty$. Therefore, there exists $n \geq n_0$ such that $||S^n(\underline{u}) - u_n^{(1)}||_2 < r$ and $||S^n(\bar{u}) - u_n^{(2)}||_2 < r$. Since $S^n(\underline{u}) \leq u_z(nT) \leq S^n(\bar{u})$ for all $z \in S^m$, by $(3,3)$, $\{u_z(nT) : z \in S^m\}$ is contractible in I^{δ_2}.

By Lemma 3.6, we can define a homotopy

$$\rho : [0, nT] \times h(S^m) \to C_0^1(\Omega) \cap I^{\delta_2}$$

by

$$\rho(s, h(z)) = u_z(s) \text{ for } 0 \leq s \leq nT \text{ and } z \in S^m.$$

Then $h(S^m)$ is contractible in I^{δ_2}. This is a contraction. Hence, S has a fixed point u_2 in $V \setminus [\epsilon\phi_1, \tilde{u}]$. This proves assertion. $\square$

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