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THE GROUP OF UNITS OF SOME FINITE LOCAL RINGS III

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Abstract. As a sequel to the papers [2, 3], we will complete our identification of the groups of units of the finite local rings $\mathbb{Z}_4[X]/(X^k + 2t(X), 2X^r)$ which is the most general type of finite local rings with a single nilpotent generator over $\mathbb{Z}_4$.

1. Introduction

Consider the ring $R=\mathbb{Z}_4[X]/(X^k+2u(X)X^a, 2X^r)$ where $u(X)=\sum_{i=0}^{s} a_iX^i$ with $u(0) = 1$, and $\deg(u) < k-a$. We will adopt the convention that $X^{-\infty} = 0$ so that our ring R can be $R = \mathbb{Z}_4[X]/(X^k)$ if $a = r = -\infty$; or $R = \mathbb{Z}_4[X]/(X^k+2X^a)$ if $r = -\infty$. If $a > 0$, then the elements of the form $1 + Xf(X)$, where $f \in \mathbb{Z}_4[X]$ form a subgroup of $U(R)$ of the group of units which we denote by $U_1(R)$ and we call such an element a 1-*unit*. In [2, XVIII.2] the group $U_1(R)$ is called the *one group* of R. If $a = 0$, then the set of 1-units do not form a subgroup and in that case we will consider the group of units $U(R)$ of R.

In [2] any finite $\mathbb{Z}_4$-algebra which is generated by a single element is of the form $R = \mathbb{Z}_4[X]/(X^k+2u(X)X^a, 2X^r)$ with $a < r < k$ and a polynomial $u(X)$ such that $u(0) = 1$ and $\deg(u) < r - a$.

In this paper, we will identify the group of units $U(R)$ of R which is a finite abelian 2-group by decomposing into a direct sum of cyclic subgroups thereby completing the identification of the groups of units of the finite rings which is generated by a single nilpotent element. For this we need to find the “natural” generators of the cyclic subgroups.

As in [3], there is a natural surjective ring homomorphism

$$\phi : \mathbb{Z}_4[X]/(X^k + 2u(X)X^a, 2X^r) \to \mathbb{F}_2[X]/(X^k)$$

which induces a surjective group homomorphism which we still call ϕ,

$$\phi : U(\mathbb{Z}_4[X]/(X^k + 2u(X)X^a, 2X^r)) \to U(\mathbb{F}_2[X]/(X^k))$$

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on the groups of units with $\ker(\phi) = T_0$. Hence we have an exact sequence

$$(1) \to T_0 \to U(\mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)) \xrightarrow{\phi} U(\mathbb{F}_2[X]/(X^k)) \to (1),$$

where T_0 is generated by $\Delta = \{-1 + 2X^i \mid 0 \le i < n\}$ with $n = k$ if $r = -\infty$; and $n = r$ if $r > 0$. Note that we can choose the generators of T_0 by $\{-1\} \cup \{1 + 2X^i \mid 0 < i < n\}$.

When $a > 0$, we can construct a similar exact sequence for 1-units

$$(1) \to T \to U_1(\mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)) \xrightarrow{\phi_1} U(\mathbb{F}_2[X]/(X^k)) \to (1),$$

where $T = \ker(\phi_1)$ which is generated by $\Delta_1 = \{1 + 2X^i \mid 0 < i < n\}$ with $n = k$ if $r = -\infty$; and $n = r$ if $r > 0$.

We let

$$G_i = \langle 1 + X^i \rangle \text{ where } i \text{ is odd with } 0 < i < k,$$
$$H_j = \langle 1 + 2X^j \rangle \text{ where } j \text{ is an integer with } 0 < j < k \quad\text{and}$$
$$H_0 = \langle -1 \rangle$$

and let

$$G = \sum_{1 \le i:\text{odd} < k} G_i, \; H_{(l)} = \sum_{0 < i < l} H_i \quad\text{and}\quad H^+_{(l)} = \sum_{0 \le i < l} H_i.$$

Also note that if $U_1(R)$ is well defined, then $U(R) = H_0 \oplus U_1(R)$ and $T_0 = H_0 \oplus T$.

In Section 2, we consider the rings of the form $R = \mathbb{Z}_4[X]/(X^k + 2u(X)X^a)$ and find a direct sum decomposition of the group of units $U(R)$ of R into a sum of cyclic subgroups. As we pointed out in [3, Proposition 2.6] the generators for the group of units of the ring $R = \mathbb{Z}_4[X]/(X^k + 2X^a)$ do not give rise to a direct sum decomposition. It turns out that we need to modify the generators for the group of units of the ring $R = \mathbb{Z}_4[X]/(X^k + 2X^a)$ to get the generators of the group of units of the ring $R = \mathbb{Z}_4[X]/(X^k + 2u(X)X^a)$.

In Section 3, we consider the ring $R = \mathbb{Z}_4[X]/(X^k + 2u(X)X^a, 2X^r)$ and we identify the group of units. When $u = 1$ a complete description of the group of units of R as a direct sum of cyclic groups is given in [3].

We will maintain our notations of [2, 3] which we recall briefly: For a finite set S of positive integers and a nonnegative integer α we will write $S + \alpha = \{i + \alpha \mid i \in S\}$. And $X^S = \sum_{i \in S} X^i$ as an element of $\mathbb{Z}_4[X]$. If $S = \emptyset$, then we define $X^S = 0$.

For a rational number r let $\lfloor r \rfloor_2$ to be the smallest integer greater than or equal to $\log_2(a)$. Hence $2^{\lfloor r \rfloor_2}$ is the smallest 2-power which is greater than or equal to r. If the order $o(G)$ of a group G is 2^n, then we will say the 2-*logarithmic order* of G is n and we will write $lo_2(G) = n$ or simply $lo(G) = n$. For $x \in G$ we will write $lo_2(x)$ for the 2-logarithmic order of the subgroup generated by x.

2. The group of units of the ring $R = \mathbb{Z}_4[X]/(X^k + 2uX^a)$

In this section we consider the group of units of the ring $R = \mathbb{Z}_4[X]/(X^k + 2u(X)X^a)$, where $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$. Also write $t(X) = u(X)X^a = X^{a_0} + \cdots + X^{a_s}$ with $a_0 < \cdots < a_s < k$ and $a_0 = a$.

Now we look closely at the group structure of T. Let $\mathfrak{A} = \{i \mid 0 < i < n\}$, where $n = k$ when $r = -\infty$; and $n = r$ when $r > 0$. First note that $(1+2X^i)(1+2X^j) = 1+2X^i+2X^j$. Hence we can identify the multiplicative group T with the additive group

$$T^0 = \{2X^S \in \mathbb{Z}_4[X] \mid S \subset \mathfrak{A}\}$$

which also can be identified with $2\mathbb{Z}_4^{n-1}$. For a polynomial $f(X) = 1 + 2X^S$ we let $f^0(X) = f(X) - 1 = 2X^S$.

As in [3], we will make use of the following simple observation.

Lemma 2.1. *Let G be a finite abelian group. Let G_i be subgroups of G such that $\sum G_i = G$ and $\prod o(G_i) = o(G)$. Then $G = \oplus G_i$.*

In [3, Proposition 2.6] we showed that the sum $\sum_{0<i:\text{odd}<k} G_i$ is not a direct sum if $u \neq 1$ and $a_0 \leq \frac{k}{2}, a_0 \neq 1$. The next theorem shows how to modify the generators of G_i to get a direct sum decomposition of $U(R)$.

The cases $a = 1, 0$ being treated in [3], we will assume that $a_0 \neq 1$ and $a_0 \neq 0$.

Theorem 2.2. *Let $R = \mathbb{Z}_4[X]/(X^k + 2t(X))$, where $t(X) = X^{a_0} + \cdots + X^{a_s}$ with $a_0 < \cdots < a_s$ and $a_s < k$. Using the notations above assume $u \neq 1$ and $a_0 \leq \frac{k}{2}, a_0 \neq 0, 1$. Then there is $g'_{i(s)} \in \sum\limits_{k-a\leq j<k} G_{i(j)}$ with $k - a \leq i(s) < k$ such that*

$$U_1(R) = \bigoplus_{0<i:\text{odd}<k} G'_i \oplus \bigoplus_{i\in S} H_j$$

with

$$G'_i = \begin{cases} \langle 1 + X^i \rangle \text{ with } lo(1 + X^i) = k_i + 1 \text{ for } i \neq i(s), \\ \langle g'_{i(s)} \rangle \text{ with } lo(g_{i(s)}) = k_i \quad \text{for } i = i(s), \end{cases}$$

and

$$S = \{i \,|\, 0 < i < a\} \cup \{a + i \,|\, 0 < i : \text{odd} < k - 2a\} \cup \{k - a\} \text{ if } k \text{ is even.}$$

Proof. By [3, Lemma 2.1], we know $\{i2^{k_i-1} \,|\, 0 < i : \text{odd} < k\} = \{j \,|\, \frac{k}{2} \leq j < k\}$. For each j with $\frac{k}{2} \leq j < k$ let

$$g_j(X) = (1 + X^{i(j)})^{2^{k_{i(j)}}}$$

so that $j = i(j)2^{k_{i(j)}}$.

Hence if $j = i2^{k_i}$, then

$$\begin{aligned}(1+X^i)^{2^{k_i}} &= 1 + 2X^{i2^{k_i-1}} + X^{i2^{k_i}} \\ &= 1 + 2X^{i2^{k_i-1}} + 2u(X)X^{i2^{k_i}-k+a} \\ &= 1 + 2X^j + 2uX^{2j-k+a}\end{aligned}$$

which is a generator of $G_i \cap T$ which is cyclic of order 2. Hence we see that the coefficients of $g_j^0 = (1+X^i)^{2^{k_i}} - 1$ for $0 < i$ odd $< k$ looks like

X	X^2	$\cdots$	X^a	X^{a+1}	X^{a+2}	$\cdots$	X^{k-a}	X^{k-a+1}	X^{k-a+2}	$\cdots$	X^{k-1}
			2	$*$	$*$						
					2	$*$					
						$\ddots$					
							0	$*$	$*$	$*$	$*$
								2	$*$	$*$	$*$
									2	$*$	$*$
										$\ddots$	
											2

The matrix above shows that the row containing 0 is a linear combination of the rows below. But this means that

$$(1+X^{i(k-a)})^{2^{k_{i(k-a)}}} = \prod_{j>k-a} (1+X^{i(j)})^{2^{k_{i(j)}}}.$$

Let $k_{i(s)}$ be the minimum of $k_{i(j)}$'s which appear as the exponents in the equality above. Then

$$\begin{aligned}&[(1+X^{i(k-a)})^{2^{k_{i(k-a)}-k_{i(s)}}}]^{2^{k_{i(s)}}} \\ &= (1+X^{i(s)})^{2^{k_{i(s)}}} \prod_{k-a<j\neq s<k} [(1+X^{i(j)})^{2^{k_{i(j)}-k_{i(s)}}}]^{2^{k_{i(s)}}}.\end{aligned}$$

Let

$$g'_{i(s)} = \begin{cases} (1+X^{i(s)}) \prod\limits_{k-a<j\neq s<k} [(1+X^{i(j)})^{2^{k_{i(j)}-k_{i(s)}}}](1+X^{i(k-a)})^{-2^{k_{i(k-a)}-k_{i(s)}}} & \\ & \text{if } s \neq k-a, \\ (1+X^{i(k-a)}) \prod\limits_{k-a<j<k} (1+X^{i(j)})^{-2^{k_{i(j)}-k_{i(k-a)}}} & \text{if } s = k-a, \end{cases}$$

and let $g'_i = g_i$ for $i \neq i(s)$. If we let $G'_i = \langle g'_i \rangle$, then the consideration above shows that

$$G'_i \cap T = \begin{cases} \langle (1+X^i)^{2^{k_i}} \rangle & \text{if } \ 0 < i : \text{odd} < k, i \neq i(s), \\ (1) \ \text{ if } \ i = i(s), \end{cases}$$

which means, the matrix formed by the coefficients of $(g'_i)^0$ is the same as the above except that all the entries of the s-th row ($k-a \le s < k$) are made to be 0 (by using elementary row operations).

Let

$$\mathcal{F} = \{1+2X^i \mid 0 < i < a\} \cup \{1+2X^{a+i} \mid 0 < i : \text{odd} < k-2a\} \cup \{1+2X^{k-a}\}.$$

Let M be the matrix formed by the coefficients of g_i^0 with $0 < i$: odd$< k$ together with f_i^0 ($f_i \in \mathfrak{F}$); and M' be the matrix formed by the coefficients of $(g_i')^0$ together with f_i^0 ($f_i \in \mathcal{F}$). Then the subgroup of $2\mathbb{Z}_4^{k-1}$ generated by the rows of M is the same as the subgroup generated by the rows of M' since M' is obtained by performing elementary row operations on M. Now it is clear that the subgroup generated by the rows of M is the whole group $2\mathbb{Z}_4^{k-1}$. Therefore if we let $G' = \sum G_i'$, then $G' \cap T_1$ together with $\mathcal{F}$ generate T_1. And obviously, $\phi(G')$ generate $U(\mathbb{F}_2[X]/(X^k))$.

Finally we need to check that $\sum_{0<i:\text{odd}<k} G_i' + \sum_{i\in S} lo(H_i)$ is a right number. But the situation is the same as [3, Theorem 3.4(ii)] and we skip our computation. □

Example 2.3. Let $R = \mathbb{Z}_4[X]/(X^5 + 2uX^2)$, where $u(X) = 1 + X + X^2$. Then $lo(U_1(R)) = 8$. We observe $X^8 = 2uX^5 = 0$. Then $lo(1+X) = 4$, $lo(1+X^3) = 2$. Further

$$\begin{aligned}(1+X)^8 &= 1+2X^4,\\ (1+X^3)^2 &= 1+2X^3+X^6\\ &= 1+2X^3+2X^3(1+X+X^2)\\ &= 1+2X^4.\end{aligned}$$

Then $G_1 \cap G_3 = \langle 1+2X^4\rangle$ and $(1+X^3)^2 = ((1+X)^4)^2$. And if we take

$$\begin{aligned}G_1' &= G_1,\\ G_3' &= \langle (1+X^3)(1+X)^{-4}\rangle,\end{aligned}$$

then $G_1' \cap G_3' = 1$ and $lo(G_3') = 1$.

If we let $H_1 = \langle 1+2X\rangle$, $H_2 = \langle 1+2X^2\rangle$, $H_1 = \langle 1+2X^3\rangle$, then

$$U(R) = G_1' \oplus G_3' \oplus H_1 \oplus H_2 \oplus H_3.$$

Example 2.4. Let $R = \mathbb{Z}_4[X]/(X^{10} + 2uX^4)$, where $u(X) = 1 + X + X^3$. Then $lo(U_1(R)) = 18$. We observe $X^{2k-4} = X^{16} = 2uX^{10} = 0$ and $k - a = 6 = 3 \cdot 2^{k_3-1}$. Now we compute

$1+X^i$	k_i+1	$lo(G_i)$	$(1+X^i)^{2^{k_i}}$	
$1+X$	5	5	$(1+X)^{2^4} = 1+2X^8$	8
$1+X^3$	3	3	$(1+X^3)^4 = 1+2X^7+2X^9$	$6 = k-a = 3\cdot 2^{k_3-1}$
$1+X^5$	2	2	$(1+X^5)^2 = 1+2X^4+2X^5+2X^7$	4
$1+X^7$	2	2	$(1+X^7)^2 = 1+2X^7+2X^8+2X^9$	7
$1+X^9$	2	2	$(1+X^9)^2 = 1+2X^9$	9

The coefficients of $(1+X^i)^{2^{k_i}} - 1$ looks like;

X	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	
			2	0	0	0	0	0	$(1+X^5)^2$
					0	2	0	2	$(1+X^3)^4$
						2	2	2	$(1+X^7)^2$
							2	0	$(1+X)^{16}$
								2	$(1+X^9)^2$

The line joining the coefficient 2 of X^4 and 0 of X^6 has slope $-\frac{1}{2}$ and the line joining the coefficient 0 of X^6 and 2 of the coefficient of X^9 has slope -1 and the slope changes at the coefficient 0 of X^6.

Hence we see that

$$\begin{aligned}(1+X^3)^4 &= 1+2X^7+2X^9 \\ &= (1+2X^7+2X^8+2X^9)(1+2X^8) = (1+X^7)^2(1+X)^{16}.\end{aligned}$$

Now let

$$\begin{array}{ll} g_1' = 1+X & lo(g_1') = 5 \\ g_3' = 1+X^3 & lo(g_3') = 3 \\ g_5' = 1+X^5 & lo(g_5') = 2 \\ g_7' = (1+X^7)(1+X^3)^{-2}(1+X)^{-8} & lo(g_7') = 1 \\ g_9' = 1+X^9 & lo(g_9') = 2. \end{array}$$

Let $G_i' = \langle g_i' \rangle$. Then $lo(G_i') = k_i + 1$ for $i \neq 7$ and $lo(G_7') = 1$. Hence $\sum lo(G_i') = 5+3+2+1+2 = 13$. Note also that $G_7' \cap T = (1)$ (g_7' is of order 2 and $g_7' \notin \mathrm{Ker}(\phi)$). Further, $G' \cap T$ together with $\{1+2X, 1+2X^2, 1+2X^3, 1+2X^5, 1+2X^6\}$ generate T and $\sum_{i=1,2,3,5,6} lo(H_i) = 5$. Hence we conclude that

$$U_1(R) = \bigoplus_{0<i:\text{odd}<10} G_i' \oplus \bigoplus_{i=1,2,3,4,6} H_i.$$

3. Decomposing the group of units of $\mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)$

Throughout this section we let $R = \mathbb{Z}_4[X]/(X^k + 2u(X)X^a, 2X^r)$, where $0 \le a < r < k$ and $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$ with $a + s < r$. Let $t(X) = X^{a_0} + \cdots + X^{a_s}$ with $a_0 < \cdots < a_s$, $a_0 = a$ and $b_i + a = a_i$ for $0 \le i \le s$ so that $t(X) = X^a u(X)$.

Lemma 3.1. *Let $R = \mathbb{Z}_4[X]/(X^k + 2u(X)X^a, 2X^r)$. Then the number of elements of $U(R)$ is 2^{k+r-1} and the group of 1-units $U_1(R)$ has order 2^{k+r-2}.*

Proof. The proof is the same as [1, Lemma 6.1]. □

Lemma 3.2. *Let $R = \mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)$, where $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$. Suppose $k \ge r + a$ and let $\alpha = k + r - a$ and $\alpha_i = \lfloor \frac{k+r-a}{i} \rfloor_2$. Then $lo(1 + X^i) = \alpha_i$ for all odd integer i ($0 < i < k$).*

Proof. We know that $(1+X^i)^{2^{\alpha_i}} = 1$. We need to check whether $(1+X^i)^{2^{\alpha_i-1}} = 1$. By [3, Lemma 2.1], we see that $\{i2^{a_i-1} \mid 0 < i : \text{odd} < \alpha\} = \{j \mid \frac{\alpha}{2} \le j < \alpha\}$. Now

$$\begin{aligned} &(1+X^i)^{2^{\alpha_i-1}} \\ &= 1 + 2X^{i2^{\alpha_i-2}} + X^{i2^{\alpha_i}} \\ &= 1 + 2X^{i2^{\alpha_i-2}} + 2(X^{i2^{\alpha_i-1}-k+a} + X^{i2^{\alpha_i-1}-k+a+b_1} + \cdots + X^{i2^{\alpha_i-1}-k+a+b_s}) \end{aligned}$$

only if $j = i2^{\alpha_i} \ge k$. (Otherwise $(1+X^i)^{2^{\alpha_i}} \ne 1$.)

On the other hand, $k \ge r + a$ implies that $a < \frac{k}{2}$. Therefore $k - a > \frac{k}{2} \ge \frac{j}{2}$. Hence $\frac{j}{2} > j-k+a$ and consequently $(1+X^i)^{2^{\alpha_i-1}} = 1+2X^{i2^{\alpha_i-1}-k+a}+(\text{hdt}) \ne 1$. Thus $lo(1+X^i) = \alpha_i$. □

Remark. Note that if we let $\alpha_i = \lfloor \frac{k+r-a}{i} \rfloor_2$, then since $k < k + r - a < 2k$ we have

$$k_i \le \alpha_i \le k_i + 1.$$

Theorem 3.3. *Let $R = \mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)$, where $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$ with $0 < a < r < k$. If $k \ge r + a$, then the group of 1-units $U_1(R)$ decomposes into the direct sum*:

$$U_1(R) = \bigoplus_{1 \le i:\text{odd} < k} G_i \oplus \bigoplus_{i \in S} H_i,$$

where $S = \{i \mid 0 < i < a\} \cup \{a + l \,|\, l : \text{odd} > 0, a + l < r\}$. Here, G_i is the cyclic subgroup generated by $1 + X^i$ of order 2^{α_i} and H_i is the cyclic subgroup generated by $1 + 2X^i$ of order 2.

Proof. First we look at $G \cap T$. Since the elements of T are order 2, the only possible elements in G_i which are in T are of the form $(1 + X^i)^{\alpha_i - 1} = 1 + 2X^{i2^{\alpha_i-2}} + X^{i2^{\alpha_1-1}}$. This will belong to T only if $i2^{\alpha_i-1} \ge k$. Now $\{i2^{\alpha_i-1} \mid 1 \le i : \text{odd} < \alpha\} = \{j \mid \frac{\alpha}{2} \le j < \alpha\}$, by Lemma 2.1, which we will call $\mathfrak{S}$. If j is odd such that $j \ge k$, then $\alpha_j = 1$ and $(1 + X^j)^{\alpha_j - 1} = (1 + X^j) \notin T$. Hence the even numbers $\ge k$ in $\mathfrak{S}$ is of the form $S' = \{i2^{\alpha_i-1} \mid i : \text{odd}, \alpha_i > 1, i2^{\alpha_i-1} \ge k\} = \{j : \text{even} \mid k \le j < \alpha\}$. Thence $G \cap T$ is generated by $\{1 + 2X^{\frac{j}{2}} + 2u(X)X^{j-k+a} | j \in S'\}$.

As in the proof of [3, Theorem 4.1], $\frac{j}{2} > j - k + a$ and hence $G \cap T$ is generated by $\{1 + 2X^{a+2i} + (\text{hdt}) \,|\, i = 0, 1, \dots\}$. Therefore we are reduced to the situation of the proof of [3, Theorem 4.1] and we safely omit the proof. □

Example 3.4. Let $R = \mathbb{Z}_4[X]/(X^{20} + 2u(X)X^5, 2X^{12})$, where $u(X) = 1 + X^2 + X^3$. Here we have $k+r-a = 27$, $k \ge r+a$ and $lo(U_1(R)) = k+r-2 = 30$. Also $X^{27} = 2uX^{12} = 0$. Let $\alpha = k + r - a = 27$ and $\alpha_i = \lfloor \frac{\alpha}{i} \rfloor_2 = \lfloor \frac{27}{i} \rfloor_2$.

We look at the possible elements of T of the form $(1 + X^i)^{2^{\alpha_i-1}}$.

$1 \le i : \text{odd} < \alpha$	α_i	$i2^{\alpha_i - 1}$	$(1+X^i)^{2^{\alpha_i - 1}}$	
1	5	$2^4 = 16$	$1 + 2X^8 + X^{16}$	not in T
3	4	$3 \cdot 2^3 = 24$	$1 + 2(X^9 + X^{11})$	
5	3	$5 \cdot 2^2 = 20$	$1 + 2X^{10} + 2(X^5 + X^7 + X^8)$	
7	2	$7 \cdot 2 = 14$	$1 + 2X^7 + X^{14}$	not in T
9	2	$9 \cdot 2 = 18$	$1 + 2X^9 + X^{18}$	not in T
11	2	$11 \cdot 2 = 22$	$1 + 2X^{11} + 2(X^7 + X^9 + X^{10})$	
13	2	$13 \cdot 2 = 26$	$1 + 2X^{11}$	
15	1	15	$1 + X^{15}$	not in T
17	1	17	$1 + X^{17}$	not in T
19	1	19	$1 + X^{19}$	not in T
21		21		
23		23		
25		25		$i \ge k$
27		27		
29		29		

Now we list the coefficients of $(1+X^i)^{2^{\alpha_i}} - 1$;

X^5	X^6	X^7	X^8	X^9	X^{10}	X^{11}	$(1+X^i)^{2^{\alpha_i}}$
2		2	2		2		$(1+X^5)^4$
		2		2	2	2	$(1+X^{11})^2$
				2		2	$(1+X^3)^8$
						2	$(1+X^{13})^2$

Therefore $T \cap G$ together with $\{1+2X, 1+2X^3, 1+2X^5, 1+2X^7, 1+2X^9, 1+2X^{11}\}$ generate T. Of course $\phi(G)$ generate $U(\mathbb{F}_2[X]/(X^k))$.

Now

$$\sum_{1 \le i:\text{odd} < 20} lo(G_i) = \sum_{1 \le i:\text{odd} < k} \alpha_i = 5 + 4 + \cdots + 1 = 24,$$

and

$$\sum \{lo(H_i) \mid i = 1, 3, 5, 7, 9, 11\} = 6.$$

Hence

$$U_1(R) = \bigoplus_{1 \le i:\text{odd} < 20} G_i \oplus \bigoplus \{H_i | i = 1, 3, 5, 7, 9, 11\},$$

where G_i are cyclic generated by $1 + X^i$ for each odd i with $lo(G_i) = \alpha_i$; H_i are cyclic with $lo(H_i) = 1$ generated by $1 + 2X^i$.

Lemma 3.5. *Let $R = \mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)$, where $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$. Suppose $k < r + a$ and $a > \frac{k}{2}$. Then $lo(1+X^i) = r_i + 1$ for all odd integer i $(0 < i < k)$.*

Proof. We know that $(1+X^i)^{2^{r_i+1}} = 1$. We need to check whether $(1+X^i)^{2^{r_i}} = 1$. By [3, Lemma 2.1], we see that $\{i2^{r_i} \mid 0 < i : \text{odd} < 2r\} = \{j \mid r \le j < 2r\}$.

Now

$$\begin{aligned}&(1+X^i)^{2^{r_i}}\\ &= 1+2X^{i2^{r_i-1}}+X^{i2^{r_i}}\\ &= 1+2X^{i2^{r_i-1}}+2(X^{i2^{r_i}-k+a}+X^{i2^{r_i}-k+a+b_1}+\cdots+X^{i2^{r_i}-k+a+b_s})\end{aligned}$$

only if $j = i2^{r_i} \geq k$. (Otherwise $(1+X^i)^{2^{r_i}} \neq 1$.) On the other hand, $a > \frac{k}{2}$ together with $j = i2^{r_i} \geq k$ implies that $\frac{j}{2} < j-k+a$. Therefore $(1+X^i)^{2^{r_i}} = 1+2X^{i2^{r_i-1}} + (\mathrm{hdt}) \neq 1$. Hence $lo(1+X^i) = r_i + 1$. □

Remark. Note that since $k \leq r+a \leq 2r \leq 2k$ we have

$$k_i \leq \lfloor \frac{2r}{i} \rfloor_2 \leq k_i + 1.$$

Theorem 3.6. *Let $R = \mathbb{Z}_4[X]/(X^k + 2uX^a, 2X^r)$, where $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$ with $0 < a < r < k$. If $k < r+a$ and $a > \frac{k}{2}$, then the group of 1-units $U_1(R)$ decomposes into the direct sum*:

$$U_1(R) = \bigoplus_{0<i:\mathrm{odd}<k} G_i \oplus \bigoplus_{0<i<\frac{k}{2}} H_i,$$

where G_i is cyclic generated by $1+X^i$ with $lo(G_i) = r_i + 1$ and H_i is cyclic generated by $1+2X^i$ with $lo(H_i) = 1$.

Proof. As before consider the exact sequence

$$(1) \to T \to U_1(\mathbb{Z}_4[X]/(X^k+2X^a, 2X^r)) \xrightarrow{\phi} U(\mathbb{F}_2[X]/(X^k)) \to (1),$$

where T is generated by $1+2X^i$ where $0 < i < r$. As usual we need to show that $G \cap T$ together with $\bigoplus_{0<i<\frac{k}{2}} H_i$ generate T when $a > \frac{k}{2}$.

By Lemma 2.1, we see that

$$\{i2^{r_i} \mid 0 < i : \mathrm{odd} \ < 2r\} = \{j \mid r \leq j < 2r\}$$

which we call $\mathfrak{R}$. If $i \in \mathfrak{R}$ is odd such that $i \geq k$, then $r_i = 0$ and hence $(1+X^i)^{2^{r_i}} \notin T$. And therefore if $n \in \mathfrak{R}$ is even such that $k \leq n < 2r$, then it is of the form $n = i2^{r_i}$ with $r_i \geq 1$ and i is odd such that $0 < i < 2r$ (we only need i for which $0 < i < k$). For an odd i with $k \leq j = i2^{r_i}$ and $0 < i < k$ we have

$$\begin{aligned}(1+X^i)^{2^{r_i}} &= 1+2X^{i2^{r_i-1}}+X^{i2^{r_i}}\\ &= 1+2X^{i2^{r_i-1}}+2uX^{i2^{r_i}-k+a} \in T.\end{aligned}$$

Since $\frac{k}{2} < a$ we see that $k-a < \frac{k}{2}$ and as $k \leq n$ we conclude that $k-a < \frac{n}{2}$. This in turn implies that $n-k+a > \frac{n}{2}$. Thus $(1+X^i)^{2^{r_i}} = 1+2X^{\frac{n}{2}} + (\mathrm{hdt})$ for all $n \geq \frac{k}{2}$.

Thus we conclude that $G \cap T$ together with $\{1+2X^i \mid 0 < i < \frac{k}{2}\}$ generate T by Lemma 2.4.

To finish our proof we need to show that the sum of the logarithmic order of our subgroups is the right number. But this is the same as [3, Theorem 4.2]. $\square$

Example 3.7. Let $R = \mathbb{Z}_4[X]/(X^{20} + 2uX^{11}, 2X^{17})$, where $u(X) = 1 + X^2 + X^3$. Here we have $r + a = 28$, $k < r + a$ and $a = 11 > 10 = \frac{k}{2}$. Also $lo(U_1(R)) = k + r - 2 = 35$ and $X^{26} = 2uX^{17} = 0$. Let $r_i = \lfloor \frac{r}{i} \rfloor_2$ so that $r_i + 1 = \lfloor \frac{2r}{i} \rfloor_2$.
We look at the possible elements of T of the form $(1 + X^i)^{2^{r_i}}$.

$1 \leq i : \text{odd} < 34$	r_i	$i2^{r_i}$	$(1+X^i)^{2^{r_i}}$	
1	5	32	$1 + 2X^{16}$	
3	3	$3 \cdot 2^3 = 24$	$1 + 2X^{12} + 2uX^{15}$	
5	2	$5 \cdot 2^2 = 20$	$1 + 2X^{10} + 2uX^{11}$	
7	2	$7 \cdot 2^2 = 28$	$1 + 2X^{14}$	
9	1	$9 \cdot 2 = 18$	$1 + 2X^9 + X^{18}$	not in T_1
11	1	$11 \cdot 2 = 22$	$1 + 2X^{11} + 2uX^{13}$	
13	1	$13 \cdot 2 = 26$	$1 + 2X^{13}$	
15	1	$15 \cdot 2 = 30$	$1 + 2X^{15}$	
17	0	17	$1 + X^{17}$	not in T_1
19	0	19	$1 + X^{19}$	not in T_1
21		21		
23		23		
25		25		
27		27		$i \geq k$
29		29		
31		31		
33		33		

Now we list the coefficients of $(1 + X^i)^{2^{r_i}}$;

X^{10}	X^{11}	X^{12}	X^{13}	X^{14}	X^{15}	X^{16}	$(1+X^i)^{2^{\alpha_i}}$
2	2		2	2			$(1+X^5)^4$
	2		2		2	2	$(1+X^{11})^2$
		2			2		$(1+X^3)^8$
			2				$(1+X^{13})^2$
				2			$(1+X^7)^2$
					2		$(1+X^{15})^2$
						2	$(1+X)^{32}$

Therefore $G \cap T$ together with H_i, $(i = 1, 2, \ldots, 9)$ generate T.

Now $lo(G_i) = r_i + 1$ and $\sum_{1 \leq i:\text{odd} < 20} lo(G_i) = 26$. Further, $\sum lo(G_i) + \sum_{i=1}^{9} lo(H_i) = 35$. Hence

$$U_1(R) = \bigoplus_{1 \leq i:\text{odd} < 20} G_i \oplus \bigoplus \{H_i | i = 1, 2, \ldots, 9\},$$

where G_i are cyclic generated by $1+X^i$ for each odd i with $lo(G_i)=\alpha_i$; H_i are cyclic with $lo(H_i)=1$ generated by $1+2X^i$.

Lemma 3.8. *Let $R=\mathbb{Z}_4[X]/(X^k+2uX^a, 2X^r)$, where $u(X)=1+X^{b_1}+\cdots+X^{b_s}$. Assume $k<a+r$ and $a\le\frac{k}{2}$. Then there is an odd i $(0<i<k)$ with $k-a=i2^{r_i-1}$, $r_i\ge 1$. For such an i we have*

$$lo(1+X^i)=\begin{cases} r_i+1 & \text{if } i2^{r_i-1}+b_1<r,\\ r_i & \text{if } i2^{r_i-1}+b_1\ge r.\end{cases}$$

Proof. By [3, Lemma 2.1], we see that $\{i2^{r_i}\mid 0<i:\text{odd}<2r\}=\{j\mid r\le j<2r\}$. If j is even and $j\ge k$, then it is of the form $i2^{r_i}$ with $r_i\ge 1$. Our conditions $k<a+r$ and $a\le\frac{k}{2}$ imply that $k\le 2(k-a)<2r$. Hence there is an odd i such that $2(k-a)=i2^{r_i}$ with $r_i\ge 1$.

Now suppose $2(k-a)=i2^{r_i}$ with $r_i\ge 1$. Then

$$\begin{aligned}&(1+X^i)^{2^{r_i}}\\ &=1+2X^{i2^{r_i-1}}+X^{i2^{r_i}}\\ &=1+2X^{i2^{r_i-1}}+2(X^{i2^{r_i}-k+a}+X^{i2^{r_i}-k+a+b_1}+\cdots+X^{i2^{r_i}-k+a+b_s})\\ &=1+2(X^{i2^{r_i-1}+b_1}+\cdots+X^{i2^{r_i-1}+b_s}).\end{aligned}$$

Now it is clear that the logarithmic order of $1+X^i$ is as stated. □

Theorem 3.9. *Let $R=\mathbb{Z}_4[X]/(X^k+2uX^a, 2X^r)$, where $u(X)=1+X^{b_1}+\cdots+X^{b_s}$ with $0<a<r<k$ and $u\ne 1$. If $k<r+a$ and $a\le\frac{k}{2}$, then there is an odd integer j $(0<j<k)$ such that $k-a=j2^{r_j-1}$ and*

$$U_1(R)=\bigoplus_{0<i:\text{odd}<k} G_i\oplus\bigoplus_{i<\in S} H_i,$$

where

$$S=\begin{cases}\{0<i<a\}\cup\{a+l|0<l:\mathit{odd}<k-2a\}\cup\{k-a\} & \text{when } k \text{ is even},\\ \{0<i\le a\}\cup\{a+l|1<l:\mathit{even}<k-2a\}\cup\{k-a\} & \text{when } k \text{ is odd}.\end{cases}$$

Here, G_i is cyclic generated by $1+X^i$ with $lo(G_i)=r_i+1$ for $i\ne j$ and $lo(G_j)=r_j$; and H_i is cyclic generated by $1+2X^i$ with $lo(H_i)=1$.

Proof. The proof will be similar to the proof of Theorem 2.2. First we need to investigate $G\cap T$. By [3, Lemma 2.1] we see that $\{i2^{r_i}\mid 0<i:\text{odd}<2r\}=\{n\mid r\le n<2r\}$ which we call $\mathfrak{R}$. We define $n(i)$ and $i(n)$ by

$$n(i)=i2^{r_i}\quad\text{and}\quad n=i(n)2^{r_{i(n)}}.$$

But our assumption implies that $r<k-a$. On the other hand, since $a<r$ we have $k<a+r<2r$. Therefore we see $r<k-a<2r$ which shows that there is j such that $k-a=j2^{r_j-1}$.

The possible elements of $G_i\cap T$ are $(1+X^i)^{2^{r_i}}$ with $i2^{r_i}\ge k$. (It has to be order 2 and this is the only one of order 2 in G_i and the inequality $i2^{r_i}\ge k$

guarantees that it is in $\mathrm{Ker}(\phi)$.) If $n = i2^{r_i} \in \mathfrak{R}$ is odd with $n \geq k$, then $r_i = 0$. Hence $(1+X^i)^{2^{r_i}} \notin T$. Therefore $(1+X^i)^{2^{r_i}} \in T$ exactly when $i2^{r_i} \geq k$ with $r_i \geq 1$ and $0 < i : \text{odd} < k$ which are precisely $\{n : \text{even} \mid k \leq n < 2r\}$ which we will call $\mathfrak{R}'$.

If $n \in \mathfrak{R}'$ with $\frac{n}{2} < k - a$ (resp. $\frac{n}{2} = k - a$, resp. $\frac{n}{2} > k - a$), then $\frac{n}{2} > n - k + a$ (resp. $\frac{n}{2} = n - k + a$, resp. $\frac{n}{2} < n - k + a$). Hence $G \cap T$ is generated by the following elements

$$\begin{cases} 1 + 2(X^{n-k+a} + (\mathrm{hdt})) & \text{when } k \leq n < 2(k-a), \\ 1 + 2(X^{a_1} + (\mathrm{hdt})) & \text{when } \frac{n}{2} = k - a, \\ 1 + 2(X^{\frac{n}{2}} + (\mathrm{hdt})) & \text{when } \frac{n}{2} > k - a. \end{cases}$$

As in the proof of Theorem 2.2, the second elements is a linear combination of the elements if the third type. Let

$$1 + 2(X^{a_1} + (\mathrm{hdt})) = \prod_i (1 + 2(X^{\frac{n(i)}{2}} + (\mathrm{hdt}))).$$

But this means that

$$(1 + X^{i(\frac{n}{2})})^{2^{r_{i(\frac{n}{2})}}} = \prod_{\frac{n}{2} > k-a} (1 + X^{i(n)})^{2^{r_{i(n)}}}.$$

Let $r_{i(s)}$ be the minimum of r_i's. Then

$$\begin{aligned} &[(1 + X^{i(\frac{n}{2})})^{2^{r_{i(\frac{n}{2})} - r_{i(s)}}}]^{2^{r_{i(s)}}} \\ &= (1 + X^{i(s)})^{2^{r_{i(s)}}} \prod_{\frac{n}{2} > k-a,\, n \neq s} [(1 + X^{i(n)})^{2^{k_{i(n)} - r_{i(s)}}}]^{2^{r_{i(s)}}}. \end{aligned}$$

Let

$$g'_{i(s)} = \begin{cases} (1 + X^{i(s)}) \prod_{\frac{n}{2} > k-a,\, n \neq s} [(1 + X^{i(n)})^{2^{r_{i(n)} - r_{i(s)}}}](1 + X^{i(\frac{n}{2})})^{-2^{r_{i(\frac{n}{2})} - r_{i(s)}}} \\ \hfill \text{if } s \neq 0, \\ (1 + X^{i(\frac{n}{2})}) \prod_{k-a < \frac{n}{2} < r} (1 + X^{i(n)})^{-2^{r_{i(n)} - r_{i(\frac{n}{2})}}} \quad \text{if } s = 0, \end{cases}$$

and let $g'_i = g_i$ for $i \neq i(s)$. If we let $G'_i = \langle g'_i \rangle$, then the consideration above shows that

$$G'_i \cap T_1 = \begin{cases} \langle (1 + X^i)^{2^{r_i}} \rangle & \text{if } \ 0 < i : \text{odd} < k, i \neq i(s), \\ (1) \ \text{ if } \ i = i(s). \end{cases}$$

Let

$$\mathcal{F} = \{1 + 2X^i \mid 0 < i < a\} \cup \{1 + 2X^{a+l} \mid 0 < l : \text{odd} < k - 2a\} \cup \{1 + 2X^{k-a}\}$$

when k is even; and

$$\mathcal{F}' = \{1 + 2X^i \mid 0 < i \leq a\} \cup \{1 + 2X^{a+l} \mid 1 < l : \text{even} < k - 2a\} \cup \{1 + 2X^{k-a}\}$$

when k is odd. As before if we let $G' = \sum G'_i$, then $G' \cap T$ together with $\mathcal{F}$ generate T. And obviously, $\phi(G')$ generate $U(\mathbb{F}_2[X]/(X^k))$.

Finally we need to check that $\sum_{0<i:\text{odd}<k} G'_i + \sum_{i\in S} lo(H_i)$ is a right number. But this is similar to [3, Theorem 4.2(ii)] and we omit the proof safely. $\square$

Example 3.10. Let $R = \mathbb{Z}_4[X]/(X^{20} + 2uX^7, 2X^{17})$, where $u(X) = 1 + X^2 + X^3$. Then we have $X^{30} = 2uX^{17} = 0$. Here we have $r + a = 24$, $k \leq r + a$ and $7 = a \leq \frac{k}{2} = 10$; and $lo(U_1(R)) = k + r - 2 = 35$. Let $r_i = \lfloor \frac{r}{i} \rfloor_2$ so that $r_i + 1 = \lfloor \frac{2r}{i} \rfloor_2$.
We look at the possible elements of T of the form $(1 + X^i)^{2^{r_i}}$.

$1 \leq i : \text{odd} < 34$	r_i	$i2^{r_i}$	$(1+X^i)^{2^{r_i}}$	
1	5	32	$1 + 2X^{16} + 2uX^{19}$	
3	3	$3 \cdot 2^3 = 24$	$1 + 2X^{12} + 2uX^{11}$	
5	2	$5 \cdot 2^2 = 20$	$1 + 2X^{10} + 2uX^7$	
7	2	$7 \cdot 2^2 = 28$	$1 + 2X^{14} + 2uX^{15}$	
9	1	$9 \cdot 2 = 18$	$1 + 2X^9 + X^{18}$	not in T
11	1	$11 \cdot 2 = 22$	$1 + 2X^{11} + 2uX^9$	
13	1	$13 \cdot 2 = 26$	$1 + 2X^{13} + 2uX^{13}$	
15	1	$15 \cdot 2 = 30$	$1 + 2X^{15} + 2uX^{17}$	
17	0	17	$1 + X^{17}$	not in T
19	0	19	$1 + X^{19}$	not in T
21		21		
23		23		
25		25		
27		27		$i \geq k$
29		29		
31		31		
33		33		

Now we list the coefficients of $(1 + X^i)^{2^{r_i}}$;

X^7	X^8	X^9	X^{10}	X^{11}	X^{12}	X^{13}	X^{14}	X^{15}	X^{16}	$(1+X^i)^{2^{\alpha_i}}$
2		2								$(1+X^5)^4$
		2			2					$(1+X^{11})^2$
				2	2	2	2			$(1+X^3)^8$
						0		2	2	$(1+X^{13})^2$
							2	2		$(1+X^7)^4$
								2		$(1+X^{15})^2$
									2	$(1+X)^{32}$

The table above shows that

$$(1 + X^{13})^2 = (1 + X^{15})^2 (1 + X)^{32}.$$

Hence we let

$$\begin{cases} g'_i = 1 + X^i \text{ if } i \text{ is odd } < k, \\ g'_{13} = (1 + X^{13})(1 + X^{15})^{-1}(1 + X)^{-16}, \end{cases}$$

and let $G'_i = \langle g'_i \rangle$. Then

$$\begin{cases} lo(G'_i) = r_i + 1 \text{ if } i \text{ is odd } < k, \\ lo(G'_{13}) = r_i, \end{cases}$$

and $G'_{13} \cap T = (1)$. If we let $G' = \oplus G'_i$, then it is easy to show that $\phi(G'_i)$ generate $U_1(\mathbb{F}_2[X]/(X^k))$. Further, the table above shows that $G' \cap T$ together with

$$H_S = \bigoplus_{i \in S} H_i = \bigoplus_{i \in S} \{1 + 2X^i\}, \quad \text{where } S = \{1, 2, 3, 4, 5, 6; 8, 10, 12; 13\}$$

generate T. Now $\sum lo(G'_i) = 25$ and $lo(H_S) = 10$.

Hence

$$U_1(R) = \bigoplus_{1 \le i:\text{odd} < 20} G'_i \oplus \bigoplus_{i \in S} H_i,$$

where G_i are cyclic generated by $1 + X^i$ for each odd i; H_i are cyclic with $lo(H_i) = 1$ generated by $1 + 2X^i$ for $i \in S$.

Theorem 3.11. *Let $R = \mathbb{Z}_4[X]/(X^k + 2u(X), 2X^r)$ where $u(X) = 1 + X^{b_1} + \cdots + X^{b_s}$. Then the group of units $U(R)$ of R is isomorphic to*

$$U(R) = \bigoplus_{1 \le i:\text{odd} < k} G_i \oplus \bigoplus_{i \in S} H_i,$$

where

$$S = \begin{cases} \{i : \text{even} \mid 0 \le i < r\} \text{ if } k \text{ is odd}, \\ \{i : \text{odd} \mid 0 < i < r\} \cup \{0\} \text{ if } k \text{ is even}. \end{cases}$$

Here, G_i is the cyclic group generated by $1 + X^i$ with $lo(G_i) = \alpha_i$, where $\alpha = k+r$ and $\alpha_i = \lfloor \frac{\alpha}{i} \rfloor_2$; and H_i is cyclic generated by $1+2X^i$ with $lo(H_i) = 1$.

Proof. By the exact sequence

$$(1) \to T_0 \to U(\mathbb{Z}_4[X]/(X^k + 2u(X), 2X^r)) \xrightarrow{\phi} U(\mathbb{F}_2[X]/(X^k)) \to (1),$$

where $T_0 = \sum_{0 \le i < r} H_i$, we need to show that $\{1 + 2X^i | i \in S\}$ together with $G \cap T_0$ with odd i generate T_0 and the sum of our subgroups has the right order (Recall $H_0 = \langle -1 \rangle$.)

First we look at $G \cap T_0$. Let $\alpha = k + r$ and $\alpha_i = \lfloor \frac{\alpha}{i} \rfloor_2$. Since the elements of T_0 are of order 2, the only possible elements in G which are in T_0 are of the form $(1 + X^i)^{\alpha_i - 1} = 1 + 2X^{i2^{\alpha_i - 2}} + X^{i2^{\alpha_1 - 1}}$ which belong to T_0 only if $i2^{\alpha_i - 1} \ge k$. Now we know that $\{i2^{\alpha_i - 1} \mid 1 \le i : \text{odd} < \alpha\} = \{j \mid \frac{\alpha}{2} \le j < \alpha\}$ which we will call $\mathfrak{S}$. If i is odd such that $i \ge k$, then $\alpha_i = 1$; $i2^{\alpha_i - 1} = i$ and $(1 + X^i)^{\alpha_i - 1} = (1 + X^i) \notin T_0$. Therefore the even numbers $\ge k$ in $\mathfrak{S}$

is of the form $S' = \{i2^{\alpha_i - 1} \mid i : \text{odd}, \alpha_i > 1, i2^{\alpha_i - 1} \geq k\} = \{j : \text{even} \mid k \leq j < \alpha\}$. Since $j < 2k$ for $j \in S$ we have $\frac{j}{2} > j - k$. And hence $G \cap T_0 = \{1 + 2X^{j-k}u(X) + 2X^{\frac{j}{2}} | j \in S'\}$. Now by [3, Lemma 2.4] we see that $G \cap T_0$ together with $\{1 + 2X^i \mid i \in S\}$ generate T_0.

As we already know that $\phi_2(G)$ generate $\mathbb{F}_2[X]/(X^k)$ we conclude that G together with $\oplus_{i \in S} H_i$ generate $U(R)$.

Now we need to show that $\sum_{0<i:\text{odd}<k} lo(G_i) + \sum_{i \in S} lo(H_i) = k + r - 1$. But this is the same as [3, Theorem 4.2]. □

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