A Simple Method to Reduce the Splitting Error in the LOD-FDTD Method

Ki-Bok Kong[1] · Myung-Hun Jeong[1] · Hyung Soo Lee[2] · Seong-Ook Park[1]

Abstract

This paper presents a new iterative locally one-dimensional finite-difference time-domain(LOD-FDTD) method that has a simpler formula than the original iterative LOD-FDTD formula[1]. There are fewer arithmetic operations than in the original LOD-FDTD scheme. This leads to a reduction of CPU time compared to the original LOD-FDTD method while the new method exhibits the same numerical accuracy as the iterative ADI-FDTD scheme. The number of arithmetic operations shows that the efficiency of this method has been improved approximately 20 % over the original iterative LOD-FDTD method.

Key words : Alternating-Direction-Implicit Finite-Difference Time-Domain(ADI-FDTD), Iterative Method, Locally-One-Dimensional Finite-Difference Time-Domain(LOD-FDTD) Technique, Maxwell's Equations.

Ⅰ. Introduction

The finite-difference time-domain(FDTD) method is widely used to solve electromagnetic problems due to its low computational complexity and easy implementation [2]. However, certain types of problems require a small grid size, bringing about a large Courant-Friedrichs-Lewy(CFL) number. To overcome the CFL condition, the ADI-FDTD method has been applied to the discrete scheme of Maxwell's equations[3]. More recently, the LOD-FDTD scheme has been applied to another unconditionally stable method, which provides simple implementation and reduces the computational time[4]. However, the ADI-FDTD and LOD-FDTD schemes generate a large numerical dispersion error due to the truncation term in their two-step factorization[5]. The iterative scheme has been used to reduce the splitting error associated with the ADI and LOD-FDTD discrete formulas[1],[6]. In [7], the error vector between the Crank-Nicolson FDTD (CN-FDTD) and ADI-FDTD schemes has been applied to obtain a more efficient iterative version of the ADI-FDTD method.

In this paper, we propose a new 2-D iterative LOD-FDTD scheme using the error vector between the CN-FDTD and LOD-FDTD methods. The proposed scheme has a simpler discrete formulation and reduces the computational time while maintaining the same accuracy of the numerical results.

Ⅱ. Derivation of New LOD-FDTD Method

The 2-D Maxwell's equations of the TEz case can be written as the following matrix system divided into two matrices of x and y:

$$\frac{\partial}{\partial t}\vec{u} = \mathbf{A}\vec{u} + \mathbf{B}\vec{u}, \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} -\sigma/\varepsilon & 0 & \frac{\partial}{\varepsilon\partial y} \\ 0 & 0 & 0 \\ \frac{\partial}{\mu\partial y} & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sigma/\varepsilon & -\frac{\partial}{\varepsilon\partial x} \\ 0 & -\frac{\partial}{\mu\partial x} & 0 \end{bmatrix}$$

and $\vec{u} = \begin{bmatrix} E_x & E_y & H_z \end{bmatrix}^T$.

Replacing the time derivative of equation (1) with its corresponding difference operation generates the following CN-FDTD equation:

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} - \frac{\Delta t}{2}\mathbf{B}\right)\vec{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} + \frac{\Delta t}{2}\mathbf{B}\right)\vec{u}^{n}, \tag{2}$$

where $\mathbf{I}$ denotes an identity matrix. The two sub-step LOD-FDTD procedures are given by

$$\left[\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right]\vec{u}_{LOD}^{n+1/2} = \left[\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right]\vec{u}^{n} \tag{3a}$$

$$\left[\mathbf{I} - \frac{\Delta t}{2}\mathbf{B}\right]\vec{u}_{LOD}^{n+1} = \left[\mathbf{I} + \frac{\Delta t}{2}\mathbf{B}\right]\vec{u}_{LOD}^{n+1/2}, \tag{3b}$$

where $\vec{u}_{LOD}^{n+1/2}$ is an intermediate solution. To reduce the

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splitting error, the iterative LOD-FDTD scheme has been introduced [1], and it has two iterative processing steps with respect to k:

$$\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}\right]\vec{u}^{n+1/2}=\left[\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right]\vec{u}^{n}+\frac{\Delta t^2}{4}\mathbf{AB}(\vec{u}_k^{n+1}+\vec{u}^{n+1/2}) \tag{4a}$$

$$\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right]\vec{u}_{k+1}^{n+1}=\left[\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}\right]\vec{u}^{n+1/2}-\frac{\Delta t^2}{4}\mathbf{BA}(\vec{u}^{n+1/2}+\vec{u}^{n}). \tag{4b}$$

This method is less costly than the iterative ADI-FDTD method due to the reduced number of arithmetic operations, and the same numerical accuracy is maintained [1]. In this paper, a new iterative LOD-FDTD scheme for 2D Maxwell's equations is introduced and shown to be more efficient than the original iterative LOD-FDTD scheme. For each ($n+1$)th time step, we first solve $\vec{u}_{LOD}^{n+1}$ from the LOD-FDTD equation, and the equations of (3) can be written equivalently as

$$\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}\right]\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right]\vec{u}_{LOD}^{n+1}=\left[\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}\right]\left[\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right]\vec{u}^{n}+\frac{\Delta t^2}{4}(\mathbf{BA}-\mathbf{AB})\vec{u}_{LOD}^{n+1/2}, \tag{5}$$

where $\vec{u}^{n}$ is the solution in the previous time step. To investigate the error between the LOD-FDTD and CN-FDTD schemes, we redescribe the linear system CN-FDTD (2), including the LOD-FDTD equation and the temporal second-order spatial derivative terms, giving us the following:

$$\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}\right]\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right]\vec{u}^{n+1}=\left[\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}\right]\left[\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right]\vec{u}^{n}+\frac{\Delta t^2}{4}\mathbf{AB}\vec{u}^{n+1}-\frac{\Delta t^2}{4}\mathbf{BA}\vec{u}^{n}. \tag{6}$$

Since LOD-FDTD solves (3) instead of (6), a splitting error depending on the truncation error terms occurs. The truncation error is related to the terms of the spatial derivatives with the second-order time step size factor Δt^2, *i.e.*, $\frac{\Delta t^2}{4}(\mathbf{BA}-\mathbf{AB})\vec{u}_{LOD}^{n+1/2}$, $\frac{\Delta t^2}{4}\mathbf{AB}\vec{u}^{n-1}$, and $\frac{\Delta t^2}{4}\mathbf{BA}\vec{u}^{n}$. The iterative scheme efficiently reduces the splitting error in ADI-FDTD and LOD-FDTD[1],[6]. To apply the iterative scheme, we first define a new vector $\vec{e}^{n+1}=\vec{u}^{n+1}-\vec{u}_{LOD}^{n+1}$ that describes the error difference between the LOD-FDTD and CN-FDTD schemes. After sub-tracting (5) from (6), we obtain the following equation related to $\vec{e}^{n+1}$:

$$\mathbf{M}\vec{e}^{n+1}=\mathbf{N}\vec{e}^{n+1}+\vec{b}^{n+1}, \tag{7}$$

where $\mathbf{M}=[\mathbf{I}-(\Delta t/2)\mathbf{A}][\mathbf{I}-(\Delta t/2)\mathbf{B}]$, $\mathbf{N}=(\Delta t^2/4)\mathbf{AB}$, and $\vec{b}^{n+1}=(\Delta t^2/4)\mathbf{AB}(\vec{u}_{LOD}^{n+1}+\vec{u}_{LOD}^{n+1/2})-(\Delta t^2/4)\mathbf{BA}(\vec{u}_{LOD}^{n+1/2}+\vec{u}^{n})$. Here, $\mathbf{M}-\mathbf{N}$ is a splitting of the matrix $(\mathbf{I}-(\Delta t/2)\mathbf{A}-(\Delta t/2)\mathbf{B})$. To apply the iterative scheme to error vector equation (7), we highlight the iterative nature of equation (7):

$$\mathbf{M}\vec{e}_{k+1}^{n+1}=\mathbf{N}\vec{e}_{k}^{n+1}+\vec{b}^{n+1}, \tag{8}$$

where the subscript k denotes the kth iterative solution [7]. If $\mathbf{M}$ is nonsingular and the spectral radius of $\mathbf{M}^{-1}\mathbf{N}$ is less than one, then iterate $\vec{e}_k^{n+1}$ converges into $\vec{e}^{n+1}$ for any initial guess[8]. The convergence of iterative equation (8) can be proved by a method similar to that of [7]. The solution $\vec{u}^{n+1}$ for each time step $n+1$ is given by $\vec{u}^{n+1}\approx\vec{e}_{final\ k}^{n+1}+\vec{u}_{LOD}^{n+1}$ after the iterative execution of (8). If the LOD concept is applied to implement this iterative solver, the system in (8) can be divided into two steps,

$$\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}\right]\vec{e}^{tmp}=\frac{\Delta t^2}{4}\mathbf{AB}\vec{e}_k^{n+1}+\frac{\Delta t^2}{4}\mathbf{AB}(\vec{u}_{LOD}^{n+1}+\vec{u}_{LOD}^{n+1/2}) \tag{9a}$$

$$\left[\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right]\vec{e}_{k+1}^{n+1}=\vec{e}^{tmp}-\frac{\Delta t^2}{4}\mathbf{BA}(\vec{u}_{LOD}^{n+1/2}+\vec{u}^{n}), \tag{9b}$$

where $\vec{e}^{tmp}=(e_x^{tmp},e_y^{tmp},e_z^{tmp})$ denotes an intermediate solution. Equations (9a) and (9b) resemble the original iterative scheme because they contain the common spatial second-order terms in each time sub-step. To recover (8) from the two-step implementation in (9), we use the matrix multiplication of $\mathbf{ABA}$=0 in the 2D TE$_z$ case.

In the first sub-step ($n+1/2$), (9a) can be expanded into the following:

$$\left(1+\frac{\sigma\Delta t}{2\varepsilon}\right)e_x^{tmp}=\frac{\Delta t}{2\varepsilon}\frac{\partial}{\partial y}e_z^{tmp}+\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial x\partial y}e_{y,k}^{n+1}-\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial x\partial y}\left(E_{y,LOD}^{n+1}+E_{y,LOD}^{n+1/2}\right) \tag{10a}$$

$$e_y^{tmp}=0 \tag{10b}$$

$$e_z^{tmp}=\frac{\Delta t}{2\mu}\frac{\partial}{\partial y}e_x^{tmp}, \tag{10c}$$

and in the second sub-step ($n+1$), we obtain the following equations from (9b):

$$e_{x,k+1}^{n+1}=e_x^{tmp} \tag{11a}$$

$$\left(1+\frac{\sigma\Delta t}{2\varepsilon}\right)e_{y,k+1}^{n+1}=e_y^{tmp}-\frac{\Delta t}{2\varepsilon}\frac{\partial}{\partial x}e_{z,k+1}^{n+1}+\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial x\partial y}\left(E_{x,LOD}^{n+1/2}+E_x^{n}\right) \tag{11b}$$

$$e^{n+1}_{z,k+1} = e^{tmp}_{z} - \frac{\Delta t}{2\mu}\frac{\partial}{\partial x} e^{n+1}_{y,k+1}. \tag{11c}$$

In each iterative sub-step, the performance of solving the equation requires one implicit tridiagonal matrix solver and one explicit update.

The whole procedure of this scheme is illustrated in Fig. 1. Note that e^{tmp}_{y} is always zero and $\vec{e}^{\,n+1}$ is zero on the boundary of a domain, which simplifies the tridiagonal matrix in comparison to the original iterative scheme.

Moreover, Table 1 illustrates that the number of arithmetic operations on the right side of each sub-step equation is less than the original iterative LOD-FDTD, leading to a reduction of CPU time for computational executions.

Ⅲ. Numerical Implementations

As a comparative study of numerical error over the ADI-FDTD method, two 2-m-long parallel conducting plates with a separation distance of 0.02 m in free space surrounded by a perfect magnetic conductor(PMC) are investigated. The grid sizes are $\Delta h = \Delta x = \Delta y = 0.02$ m in the x and y directions. The numerical experiment is conducted with a 750-kHz raised cosine, which held constant after reaching its maximum of 1 V. In the region between the plates (4 m$\le$x$\le$6 m), the field amplitude is constant. Unlike the original iterative LOD-FDTD, execution of the iterative process will be conducted in a different way because of the convergent speed of iterative equation (9). If the iteration number is small, then solution $\vec{e}^{\,n+1}_{k}$ of (9) is large, leading to a divergence of $\vec{u}^{\,n+1}$. Thus, the iteration number needs to be sufficiently large. Since good agreement between the iterative LOD-FDTD and iterative ADI-FDTD has been previously observed and described by K. Jung[1], we investigate the agreement between the proposed iterative LOD-FDTD and iterative ADI-FDTD.

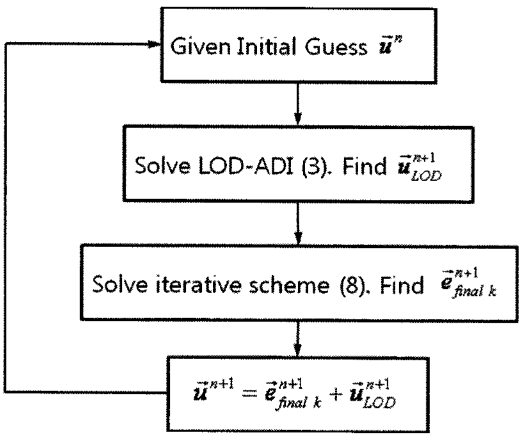

Fig. 1. The proposed iterative method.

The L^2 norm $\|\eta^{n}_{k}\|$ of $e^{n}_{y,k}$ for each time step n and iterative step k is defined as

$$\|\eta^{n}_{k}\| = \sqrt{\sum\nolimits_{x} \left| e^{n}_{y,k} - e^{n}_{y,k-1} \right|^{2}}, \tag{12}$$

where $e^{n}_{y,0}$ is an initial guess. Since the iterative equation of (7) converges, the norm $\|\eta^{n}_{k}\|$ goes to zero as $k \to \infty$ [8]. We use $\|\eta^{n}_{k}\|$ to determine the iteration number for convergence of the iterative scheme. The iterative step is executed until the norm $\|\eta^{n}_{k}\|$ arrives at a sufficiently small value. The numerical implementation of (9) in Fortran can be simply expressed as follows:

do while ($\|\eta^{n}_{k}\|$ .gt. *LIMIT(n)*)

execute (9)

calculate $\|\eta^{n}_{k}\|$

k=k+1

end while.

Here *LIMIT(n)* is a lower limit for interrupting the *while* loop, and is related to the previous limit value $\|\eta^{n-1}_{final}\|$ because of the stability of the iterative scheme. Thus, a numerical example of this chapter takes *LIMIT (n)* as a value similar to $\|\eta^{n-1}_{final}\|$. The total iteration number is obtained by summing every final k for each time step n. The iteration number of iterative ADI-FDTD is calculated by dividing the total iteration number of the new iterative LOD-FDTD by the final n.

Table 1. Number of arithmetic operations.

Method	Implicit		Explicit		Total	
	M/D	A/S	M/D	A/S	M/D	A/S
Original LOD	4	12	2	8	6	20
New LOD	2	14	2	3	4	17

M/D: multiplication/division, A/S: addition/subtraction.

The CFL number is given as $s = c_0 \Delta t / \Delta h$ where c_0 is the speed of light in a vacuum. The L_2 relative error norm for E_y along the x-axis is defined as follows:

$$\sqrt{\frac{\sum_x |E_y - E_y^{ref}|^2}{\sum_x |E_y^{ref}|^2}} \times 100 , \qquad (13)$$

where E_y is a measured field at y=2 where the upper plate is located, and E_y^{ref} is a reference field with a value of 1 at the feeding plate and 0 at the other area.

Fig. 2 shows the E_y field along the x-axis by the new iterative LOD-FDTD and iterative ADI-FDTD method with the same total iteration number. This illustrates that the new iterative LOD-FDTD reduces the splitting error in a manner similar to the original LOD-FDTD case.

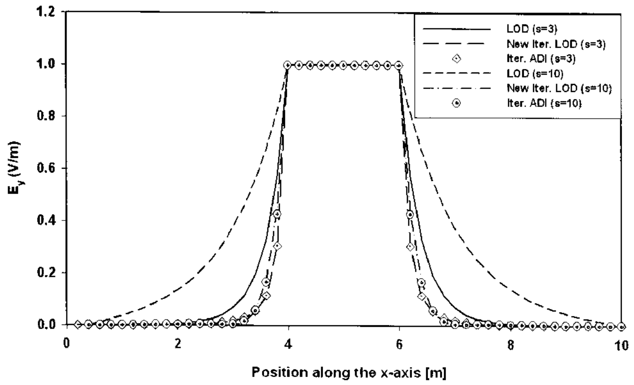

Fig. 2. E_y along the x-axis calculated by LOD-FDTD, the proposed iterative LOD-FDTD method, and iterative ADI-FDTD method with the same total iteration number.

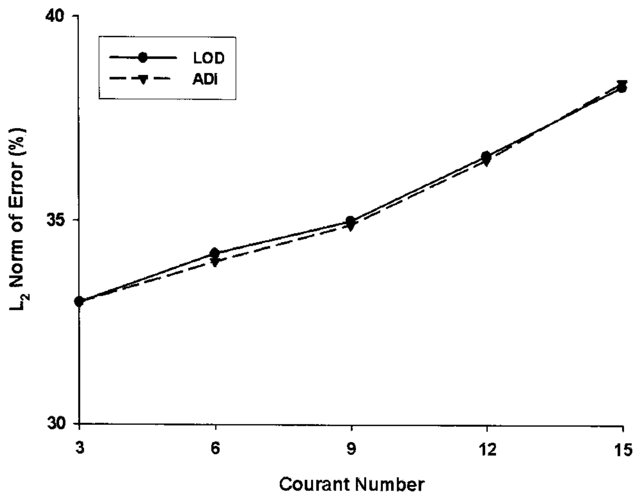

Fig. 3. L_2 norm of the error of the new iterative LOD-FDTD method and the iterative ADI-FDTD method for different courant number s.

Fig. 3 shows that the L_2 relative error norm for E_y along the x-axis of the new iterative LOD-FDTD method agrees with the iterative ADI-FDTD method with the same total iteration number.

Ⅳ. Conclusions

A new iterative LOD-FDTD method has been introduced using an error vector between the CN-FDTD and LOD-FDTD. A numerical example shows that the new iterative LOD-FDTD scheme efficiently reduces the splitting error in a manner similar to that of the original iterative LOD-FDTD scheme. The new iterative LOD-FDTD scheme uses a simpler numerical discrete formula compared with the original iterative LOD-FDTD, and due to the reduced number of arithmetic operations, the computational time is lower than that of the original iterative LOD-FDTD.

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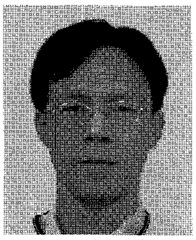

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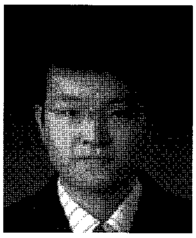

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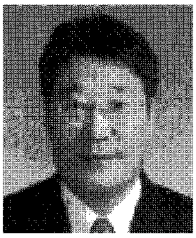

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