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NOTES ON GENERALIZED DERIVATIONS ON LIE IDEALS IN PRIME RINGS

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Abstract. Let R be a prime ring, H a generalized derivation of R and L a noncommutative Lie ideal of R. Suppose that $u^sH(u)u^t = 0$ for all $u \in L$, where $s \geq 0, t \geq 0$ are fixed integers. Then $H(x) = 0$ for all $x \in R$ unless char $R = 2$ and R satisfies S_4, the standard identity in four variables.

Let R be an associative ring with center $Z(R)$. For $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$. An additive mapping d from R to R is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $d(x) = [a, x]$ holds for all $x \in R$. An additive subgroup L of R is said to be a Lie ideal of R if $[u, r] \in L$ for all $u \in L$, $r \in R$. The Lie ideal L is said to be noncommutative if $[L, L] \neq 0$. Hvala [8] introduced the notion of generalized derivation in rings. An additive mapping H from R to R is called a generalized derivation if there exists a derivation d from R to R such that $H(xy) = H(x)y + xd(y)$ holds for all $x, y \in R$. Thus the generalized derivation covers both the concepts of derivation and left multiplier mapping. The left multiplier mapping means an additive mapping F from R to R satisfying $F(xy) = F(x)y$ for all $x, y \in R$.

Throughout this paper R will always present a prime ring with center $Z(R)$, extended centroid C and U its Utumi quotient ring. It is well known that if ρ is a right ideal of R such that $u^n = 0$ for all $u \in \rho$, where n is a fixed positive integer, then $\rho = 0$ [7, Lemma 1.1]. In [2], Chang and Lin consider the situation when $d(u)u^n = 0$ for all $u \in \rho$ and $u^n d(u) = 0$ for all $u \in \rho$, where ρ is a nonzero right ideal of R. More precisely, they proved the following:

Let R be a prime ring, ρ a nonzero right ideal of R, d a derivation of R and n a fixed positive integer. If $d(u)u^n = 0$ for all $u \in \rho$, then $d(\rho)\rho = 0$ and if $u^n d(u) = 0$ for all $u \in \rho$, then $d = 0$ unless $R \cong M_2(F)$, the 2×2 matrices over a field F of two elements.

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Recently, for noncommutative Lie ideal L of R, Dhara and Sharma obtained results [4] that if $a \in R$ such that $au^s d(u)^n u^t = 0$ for all $u \in L$, where $s(\geq 0), t(\geq 0), n(\geq 1)$ are fixed integers, then either $a = 0$ or $d(R) = 0$ unless char $R = 2$ and R satisfies S_4, the standard identity in four variables.

From this line of investigation, our aim in this paper is to study the situation when $u^s H(u)u^t = 0$ for all $u \in L$, where L a noncommutative Lie ideal of R, H a generalized derivation of R and $s \geq 0, t \geq 0$ are fixed integers.

Remark 1. It is well known that if L is a noncommutative Lie ideal of a prime ring R and I is the ideal of R generated by $[L, L]$, then $I \subseteq L+L^2$ and $[I, I] \subseteq L$ (see [11, Lemma 2 (i),(ii)]).

Proof. To give its brief proof, let $a, b \in L$ and $r \in R$. We have $[a, b]r = [ar, b] - a[r, b] \in L + L^2$. For $s \in R$, we get commuting both sides by s that $s[a, b]r = [a, b]rs + [[ar, b], s] - [a[r, b], s] \in L + L^2$, since $[a[r, b], s] = a[[r, b], s] + [a, s][r, b] \in L^2$. Thus $I \subseteq L + L^2$. Now since $[L^2, I] \subseteq L$ holds true by using the identity $[xy, z] = [x, yz] + [y, zx]$ for $x, y \in L$ and $z \in I$, we have $[I, I] \subseteq L$. □

Remark 2. Let R be a prime ring and U be the Utumi quotient ring of R and $C = Z(U)$, the center of U (see [1] for more details). It is well known that any derivation of R can be uniquely extended to a derivation of U. In [13, Theorem 3], Lee proved that every generalized derivation H on a dense right ideal of R can be uniquely extended to a generalized derivation of U and assume the form $H(x) = ax + d(x)$ for all $x \in U$, for some $a \in U$ and a derivation d of U.

Lemma 1. *Let $R = M_k(F)$, the ring of $k \times k$ matrices over a field F and $a, b \in R$ such that $[x_1, x_2]^s(a[x_1, x_2] + [x_1, x_2]b)[x_1, x_2]^t = 0$ for all $x_1, x_2 \in R$, where $s \geq 0, t \geq 0$ are fixed integers. If char $F = 2$, then $a = b$ and if char $R \neq 2$, then $a \in F \cdot I_k$, $b \in F \cdot I_k$ and $a + b = 0$.*

Proof. Let $a = (a_{ij})_{k\times k}$ and $b = (b_{ij})_{k\times k}$. Now in our assumption

$$[x_1, x_2]^s(a[x_1, x_2] + [x_1, x_2]b)[x_1, x_2]^t = 0,$$

we may assume that s and t both are even integers, because if they are not even, we multiply $[x_1, x_2]$ from left or right in both sides to make them even. Now putting $x_1 = e_{ij}$, $x_2 = e_{ji}$ for any $i \neq j$, we have

$$\begin{aligned} 0 &= [e_{ij}, e_{ji}]^s(a[e_{ij}, e_{ji}] + [e_{ij}, e_{ji}]b)[e_{ij}, e_{ji}]^t \\ &= (e_{ii} + e_{jj})(a(e_{ii} - e_{jj}) + (e_{ii} - e_{jj})b)(e_{ii} + e_{jj}). \end{aligned}$$

Left multiplying by e_{ii}, we get

$$\begin{aligned} 0 &= e_{ii}(a(e_{ii} - e_{jj}) + (e_{ii} - e_{jj})b)(e_{ii} + e_{jj}) \\ &= a_{ii}e_{ii} - a_{ij}e_{ij} + b_{ii}e_{ii} + b_{ij}e_{ij} \\ &= (a_{ii} + b_{ii})e_{ii} + (-a_{ij} + b_{ij})e_{ij} \end{aligned}$$

implying $a_{ii} + b_{ii} = 0$ and $a_{ij} = b_{ij}$ for any $i, j (i \neq j)$. This gives $a - b$ is diagonal. Let $a - b = \sum_{i=1}^{k} w_{ii}e_{ii}$. For some F-automorphism θ of R,

$(a-b)^\theta$ enjoys the same property as $a-b$ does, namely, $[x_1,x_2]^s(a^\theta[x_1,x_2]+[x_1,x_2]b^\theta)[x_1,x_2]^t=0$ for all $x_1,x_2\in R$. Hence $a^\theta-b^\theta=(a-b)^\theta$ must be diagonal. For each $j\neq 1$, we have $(1+e_{1j})(a-b)(1-e_{1j})=\sum_{i=1}^k w_{ii}e_{ii}+(w_{jj}-w_{11})e_{1j}$ diagonal. Therefore, $w_{jj}=w_{11}$ and so $a-b$ is central that is $a-b\in F\cdot I_k$. Clearly $a-b=w_{11}\cdot I_k=(a_{11}-b_{11})\cdot I_k=2a_{11}\cdot I_k$. If char $F=2$, then $a=b$. Let char $F\neq 2$. Then $a=b+2a_{11}\cdot I_k$. Now $w_{11}=w_{22}=\cdots=w_{kk}$ and $a_{ii}+b_{ii}=0$ for $i=1,\ldots,k$ together implies $a_{11}=a_{22}=\cdots=a_{kk}$ and $b_{11}=b_{22}=\cdots=b_{kk}$. Therefore the identity becomes,

$$[x_1,x_2]^s(b[x_1,x_2]+[x_1,x_2]b)[x_1,x_2]^t+2a_{11}[x_1,x_2]^{s+t+1}=0.$$

Now, putting $x_1=e_{ii},x_2=e_{ij}-e_{ji}$ $(i\neq j)$, we obtain,

$$(e_{ij}+e_{ji})^s(b(e_{ij}+e_{ji})+(e_{ij}+e_{ji})b)(e_{ij}+e_{ji})^t+2a_{11}(e_{ij}+e_{ji})^{s+t+1}=0$$

which implies

$$(e_{ii}+e_{jj})(b(e_{ij}+e_{ji})+(e_{ij}+e_{ji})b)(e_{ii}+e_{jj})+2a_{11}(e_{ij}+e_{ji})=0.$$

Left multiplying by e_{ii} yields

$$b_{ii}e_{ij}+b_{ij}e_{ii}+b_{ji}e_{ii}+b_{jj}e_{ij}+2a_{11}e_{ij}=0.$$

Since $b_{ii}+b_{jj}+2a_{11}=0$, above relation implies that $(b_{ij}+b_{ji})e_{ii}=0$ and so $b_{ij}+b_{ji}=0$ for any $i\neq j$.

Now, putting $x_1=e_{ii},x_2=e_{ij}+e_{ji}$ $(i\neq j)$, we obtain $[x_1,x_2]^n=(-1)^{n/2}(e_{ii}+e_{jj})$ if n is even and $(-1)^{(n-1)/2}(e_{ij}-e_{ji})$ if n is odd. Thus we have

$$\begin{aligned}(-1)^{s/2}(e_{ii}+e_{jj})(b(e_{ij}-e_{ji})+(e_{ij}-e_{ji})b)(-1)^{t/2}(e_{ii}+e_{jj})\\ +(-1)^{(s+t)/2}2a_{11}(e_{ij}-e_{ji})=0.\end{aligned}$$

Left multiplying by e_{ii}, we get

$$(-1)^{(s+t)/2}\{b_{ii}e_{ij}-b_{ij}e_{ii}+b_{ji}e_{ii}+b_{jj}e_{ij}+2a_{11}e_{ij}\}=0.$$

Again, since $b_{ii}+b_{jj}+2a_{11}=0$, we have $(-b_{ij}+b_{ji})e_{ii}=0$ and so $-b_{ij}+b_{ji}=0$ for any $i\neq j$. Addition and subtraction of $b_{ij}+b_{ji}=0$ and $-b_{ij}+b_{ji}=0$ yields that $b_{ij}=0=b_{ji}$ for any $i\neq j$. Therefore, b is central in R that is $b=b_{11}\cdot I_k\in F\cdot I_k$ and so $a=b_{11}\cdot I_k+2a_{11}\cdot I_k=a_{11}\cdot I_k\in F\cdot I_k$. Thus the identity becomes $(a+b)[x_1,x_2]^{s+t+1}=0$ for all $x_1,x_2\in R$. Since $a+b\in F\cdot I_k$, either $a+b=0$ or $[x_1,x_2]^{s+t+1}=0$ for all $x_1,x_2\in R$. But $[x_1,x_2]^{s+t+1}=0$ gives contradiction by choosing $x_1=e_{12}$ and $x_2=e_{21}$. Thus $a+b=0$. □

Lemma 2. *Let R be a prime ring with extended centroid C and $a,b\in R$. If $[x_1,x_2]^s(a[x_1,x_2]+[x_1,x_2]b)[x_1,x_2]^t=0$ for all $x_1,x_2\in R$, then either R satisfies a nontrivial generalized polynomial identity (GPI) or $a\in C$, $b\in C$ and $a+b=0$.*

Proof. Suppose on contrary that R does not satisfy any nontrivial GPI. Let $T = U *_C C\{X_1, X_2\}$, the free product of U and $C\{X_1, X_2\}$, the free C-algebra in noncommuting indeterminates X_1 and X_2. Then, since $[x_1, x_2]^s(a[x_1, x_2] + [x_1, x_2]b)[x_1, x_2]^t$ is a GPI for R, we see that

$$[X_1, X_2]^s(a[X_1, X_2] + [X_1, X_2]b)[X_1, X_2]^t$$

is zero element in $T = U *_C C\{X_1, X_2\}$. If $a \notin C$, then a and 1 are linearly independent over C. Thus,

$$[X_1, X_2]^s a[X_1, X_2]^{t+1} = 0$$

and

$$[X_1, X_2]^{s+1} b[X_1, X_2]^t = 0$$

in T, which implies $a = 0$, a contradiction. Therefore, we conclude that $a \in C$ and hence

$$[X_1, X_2]^s(a[X_1, X_2] + [X_1, X_2]b)[X_1, X_2]^t = [X_1, X_2]^{s+1}(a + b)[X_1, X_2]^t$$

is zero element in T, again implying $a + b = 0$ that is $b = -a \in C$. □

Lemma 3. *Let R be a prime ring with extended centroid C and $a, b \in R$. Suppose that $[x_1, x_2]^s(a[x_1, x_2] + [x_1, x_2]b)[x_1, x_2]^t = 0$ for all $x_1, x_2 \in R$. Then*

(i) *if char $R \neq 2$, $a \in C$, $b \in C$ and $a + b = 0$;*
(ii) *if char $R = 2$, $a = b \in C$ unless R satisfies S_4.*

Proof. By assumption, R satisfies generalized polynomial identity

$$f(x_1, x_2) = [x_1, x_2]^s(a[x_1, x_2] + [x_1, x_2]b)[x_1, x_2]^t.$$

If R does not satisfy any nontrivial GPI, by Lemma 2, $a \in C$, $b \in C$ and $a + b = 0$ which gives conclusion (i) and (ii). Next assume that R satisfies a nontrivial GPI. Since R and U satisfy same generalized polynomial identity (see [3]), U satisfies $f(x_1, x_2)$. In case C is infinite, we have $f(x_1, x_2) = 0$ for all $x_1, x_2 \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $f(x_1, x_2) = 0$ for all $x_1, x_2 \in R$. By Martindale's theorem [15], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [9, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C, and H consists of the linear transformations in R of finite rank.

Let $\dim_C V = k$. Then the density of R on V implies that $R \cong M_k(C)$. If char $R \neq 2$, then by Lemma 1, we have that, $a \in C$, $b \in C$ and $a + b = 0$ which is conclusion (i). If char $R = 2$, then by Lemma 1, $a = b$ and so R satisfies the generalized identity $f(x_1, x_2) = [x_1, x_2]^s[a, [x_1, x_2]][x_1, x_2]^t$. Suppose that $\dim_C V \geq 3$. Then we show that for any $v \in V$, v and av are linearly C-dependent. Suppose that v and av are linearly C-independent for some $v \in V$.

Since $\dim_C V \geq 3$, there exists $w \in V$ such that v, av, w are linearly independent over C. By density there exist $x_1, x_2 \in R$ such that

$$x_1 v = 0, \qquad x_1 av = v, \qquad x_1 w = v$$

$$x_2 v = av, \qquad x_2 av = w, \qquad x_2 w = 0.$$

Then $[x_1, x_2]v = (x_1x_2 + x_2x_1)v = v$, $[x_1, x_2]av = (x_1x_2 + x_2x_1)av = x_1w + x_2v = v + av$ and so $[a, [x_1, x_2]]v = v$. Hence

$$0 = [x_1, x_2]^s[a, [x_1, x_2]][x_1, x_2]^t v = v,$$

a contradiction.

Thus v and av are linearly C-dependent. Hence for each $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $av = v\alpha$ for all $v \in V$ and $\alpha \in C$ fixed.

Now, let $r \in R$, $v \in V$. Since $av = v\alpha$,

$$[a, r]v = (ar)v + (ra)v = a(rv) + r(av) = (rv)\alpha + r(v\alpha) = 0$$

that is $[a, r]V = 0$. Hence $[a, r] = 0$ for all $r \in R$, implying $a \in C$. Now, if $\dim_C V = 2$, then $R \cong M_2(C)$ that is R satisfies S_4. Thus we obtain $a = b \in C$ unless R satisfies S_4, which is conclusion (ii).

If $\dim_C V = \infty$, then for any $e^2 = e \in H = soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Assume that either $a \notin C$ or $b \notin C$. Then one of them does not centralize the nonzero ideal $H = soc(R)$. Hence there exist $h_1, h_2 \in H$ such that either $[a, h_1] \neq 0$ or $[b, h_2] \neq 0$. By Litoff's theorem [6], there exists idempotent $e \in H$ such that $ah_1, h_1a, bh_2, h_2b, h_1, h_2 \in eRe$. We have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since R satisfies generalized identity $f(ex_1e, ex_2e) = [ex_1e, ex_2e]^s(a[ex_1e, ex_2e] + [ex_1e, ex_2e]b)[ex_1e, ex_2e]^t$, the subring eRe satisfies $f(x_1, x_2) = [x_1, x_2]^s(eae[x_1, x_2] + [x_1, x_2]ebe)[x_1, x_2]^t$. Then by the above finite dimensional case, eae, ebe are central elements of eRe. Thus $ah_1 = (eae)h_1 = h_1eae = h_1a$ and $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$, a contradiction.

Thus we conclude that $a, b \in C$. Then we have that R satisfies

$$f(x_1, x_2) = (a + b)[x_1, x_2]^{s+t+1}$$

implying $a + b = 0$. In case char $R = 2$, $a = b \in C$. Thus we get conclusion (i) and (ii). □

Theorem 1. *Let R be a prime ring, H a generalized derivation of R and L a noncommutative Lie ideal of R. Suppose that $u^sH(u)u^t = 0$ for all $u \in L$, where $s \geq 0, t \geq 0$ are fixed integers. Then $H(x) = 0$ for all $x \in R$ unless char $R = 2$ and R satisfies S_4, the standard identity in four variables.*

Proof. Since L is noncommutative, by Remark 1, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence without loss of generality we may assume $L = [I, I]$. By our assumption we have

$$[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in I$. Since I and U satisfy the same differential identities [14], we may assume that

$$[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t = 0$$

for all $x \in U$. As we have already remarked in Remark 2, we may assume that for all $x \in U$, $H(x) = bx + d(x)$ for some $a \in U$ and a derivation d of U. Hence U satisfies

$$[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t = 0.$$

Assume first that d is inner derivation of U, i.e., there exists $p \in U$ such that $d(x) = [p, x]$ for all $x \in U$. Then

$$[x_1, x_2]^s (b[x_1, x_2] + [p, [x_1, x_2]])[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$ that is

$$[x_1, x_2]^s ((b + p)[x_1, x_2] - [x_1, x_2]p)[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. By Lemma 3, if char $R \neq 2$, $b+p \in C$, $p \in C$ and $b+p-p = 0$ implying that $b = 0$. Hence $H(x) = 0$ for all $x \in U$ and so for all $x \in R$. Now if char $R = 2$, by Lemma 3, $b + p = -p \in C$ implying $b = 0$ unless R satisfies S_4. Hence $H(x) = 0$ for all $x \in U$ and so for all $x \in R$ unless R satisfies S_4.

If d is not Q-inner, then by Kharchenko's theorem [10]

$$[x_1, x_2]^s (b[x_1, x_2] + [x_3, x_2] + [x_1, x_4])[x_1, x_2]^t = 0$$

for all $x_1, x_2, x_3, x_4 \in U$. In particular U satisfies its blended component

$$[x_1, x_2]^s ([x_3, x_2] + [x_1, x_4])[x_1, x_2]^t.$$

This is a polynomial identity and hence there exists a field F such that $U \subseteq M_k(F)$ with $k > 1$ and U and $M_k(F)$ satisfy the same polynomial identity [12, Lemma 1]. But by choosing $x_1 = x_3 = e_{12},\ x_2 = e_{21},\ x_4 = 0$, we get

$$0 = [x_1, x_2]^s ([x_3, x_2] + [x_1, x_4])[x_1, x_2]^t = \left(e_{11} + (-1)^{s+t+1} e_{22}\right),$$

which is a contradiction. □

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