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ON IDEALS, FILTERS AND CONGRUENCES IN INCLINES

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Abstract. This paper studies the relations between ideals, filters, regular congruences and normal congruences in inclines. It is shown that for any incline, there are a one-to-one correspondence between all ideals and all regular congruences and a one-to-one correspondence between all filters and all normal congruences.

1. Introduction and preliminaries

Inclines are additively idempotent semirings in which products are less than or equal to either factors. The concept of inclines was first introduced and studied by Z. Q. Cao in a series of his papers and a monograph [4] *Incline Algebra and Applications* coauthored with Kim and Roush in the first half of 1980's. Inclines and incline matrices are useful tools in diverse areas such as automata theory, design of switching circuits, graph theory, medical diagnosis, Markov chains, informational systems, complex systems modeling, decision-making theory, dynamical programming, control theory, nervous system, probable reasoning, psychological measurement, clustering and so on [5, 8]. As an algebraic system, an incline can be viewed as a generalization of both distributive lattices and fuzzy algebras. A lot of scholars have been interested in and researched the theory of incline matrices, while a relative few people studied the algebraic structure of inclines [1, 2, 6, 7, 9].

Let $+$ and $\cdot$ be two binary operations on a nonempty set K. An algebraic system $(K, +, \cdot)$ is called an *incline* if it satisfies the following axioms: for all $x, y, z \in K$,

(K1) $x + y = y + x$;
(K2) $x + (y + z) = (x + y) + z$;
(K3) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(K4) $x \cdot (y + z) = x \cdot y + x \cdot z$;
(K5) $(y + z) \cdot x = y \cdot x + z \cdot x$;

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(K6) $x + x = x$;
(K7) $x + x \cdot y = x$;
(K8) $y + x \cdot y = y$.

There are many examples of inclines.

Example 1.1. (1) Every distributive lattice, of course every Boolean algebra, $(L, \vee, \wedge)$ is an incline.

(2) Each fuzzy algebra $([0, 1], \vee, T)$ is an incline, where T is a t-norm on the real interval $[0, 1]$.

(3) For each t-conorm on the real interval $[0, 1]$, $([0, 1], \wedge, S)$ is an incline.

(4) The tropical algebra $(\mathbb{R}_0^+ \cup \{\infty\}, \wedge, +)$ is an incline, where $\mathbb{R}_0^+$ is the set of all nonnegative real numbers.

(5) For each residuated lattice $(L, \wedge, \vee, *, \to, 0, 1)$ [10] (not necessarily commutative), $(L, \vee, *)$ is an incline.

(6) Let $K = \{0, a, b, c, d, 1\}$ be a lattice ordered by the following Hasse graph. Define $\cdot : K \times K \longrightarrow K$ by $x \cdot y = d$ for all $x, y \in \{1, b, c, d\}$ and 0 otherwise. Then $(K, \vee, \cdot)$ is an incline which is not a distributive lattice.

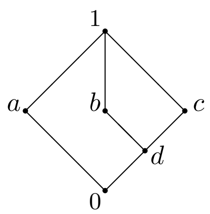

Figure 1

Usually, two operations $+$ and $\cdot$ on an incline K are called an *addition* and a *multiplication*, respectively. For the multiplication of two elements x and y, the symbol $\cdot$ is always omitted. In an incline K, we define a binary relation $\leq$ by $x \leq y \Leftrightarrow x + y = y$. It is easy to see that $\leq$ is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$, i.e., $x + y = x \vee y$ in the poset $(K, \leq)$. We say that $\leq$ is induced by operation $+$. It follows that

(1) $xy \leq x$ and $yx \leq x$ for all $x, y \in K$;
(2) $y \leq z$ implies $xy \leq xz$ and $yx \leq zx$ for any $x, y, z \in K$;
(3) if $x \leq y,\ a \leq b$, then $x + a \leq y + b,\ xa \leq yb$.

Given an algebraic system, ideals and congruences are often used to study its structure, for example the substructures and quotient structures. Sometimes we use ideals and sometimes we use congruences. Thus one-to-one correspondence between ideals and congruences is always an interesting topic for us.

The aim of this paper is to study the relations between ideals, filters and congruences in inclines. In Section 2, regular congruences are defined and it is shown that there is a one-to-one correspondence between the set of all ideals and the set of all regular congruences. In Section 3, filters and normal congruences are defined and it is shown that there is a one-to-one correspondence between the set of all filters and the set of all normal congruences. In Section 4, some remarks are given.

Firstly, we shall make a preparation related with congruences.

An equivalence relation θ on an incline K is called a *congruence* if (a,b), $(c,d) \in \theta$ always implies $(a+c, b+d)$, $(ac, bd) \in \theta$. The set of all congruences on K is denoted by $\mathcal{C}(K)$. Let K/θ be the set of all congruence classes in K with respect to θ and for $x \in K$, $[x]_\theta$ denotes the congruence class of x.

We define a binary relation $\prec$ on K/θ as

$$\forall x, y \in K,\ [x]_\theta \prec [y]_\theta \text{ if and only if } (x+y, y) \in \theta.$$

Proposition 1.2. *The relation $\prec$ defined above is a partial order on K/θ.*

Proof. (1) $\prec$ is well defined. For all $x, y \in K$, suppose that $(x+y,\ y) \in \theta$. For all $a \in [x]_\theta,\ b \in [y]_\theta$, we have $(a,x),\ (b,y) \in \theta$. Then $(a+b, x+y) \in \theta$ and $(a+b, y) \in \theta$ and $(a+b, b) \in \theta$.

(2) The reflexivity and antisymmetry of $\prec$ are obvious. Transitivity: suppose that $[x]_\theta \prec [y]_\theta \prec [z]_\theta$. Then $(x+y,y), (y+z,z) \in \theta$ and $(x+y+z, y+z),\ (x+y+z, x+z) \in \theta$. Hence $(x+z, z) \in \theta$ and $[x]_\theta \prec [z]_\theta$. □

Remark 1.3. The relation $\prec$ defined above on K/θ just is the partial order induced by the quotient incline K/θ [1]. In this paper, we don't want to study quotients induced by congruences.

2. A one-to-one correspondence between ideals and regular congruences

Definition 2.1 ([1])**.** A subincline I of K is called an *ideal* if it is a lower set, i.e., for any $x \in I$ and $y \in K$, $y \leq x$ implies $y \in I$.

Proposition 2.2. *If I is a nonempty subset of K, then the following statements are equivalent*:

(1) *I is an ideal*;

(2) *I is a lower set and closed under $+$.*

We denote by $\mathcal{I}(K)$ the family of all ideals of K. Then $\mathcal{I}(K)$ is a poset under the set-inclusion. The largest element of $\mathcal{I}(K)$ is K.

Definition 2.3. (see in [3] for distributive lattices) A congruence θ on K is said to be *regular* if

(R1) There exists a least element in the poset $(K/\theta,\ \prec)$, denoted by m_θ;

(R2) If $(x,y) \in \theta$, then there exists $a \in m_\theta$ such that $a + x = a + y$.

The family of all regular congruences on K is denoted by $\mathcal{RC}(K)$.

Proposition 2.4. *Suppose that θ is a regular congruence on K. Then m_θ is an ideal, denoted by $I(\theta)$.*

Proof. (1) m_θ is a lower set. Suppose that $x \in m_\theta$, $y \in K$ and $y \leq x$. Then $(y+x,x)=(x,x)\in\theta$ and $[y]_\theta \prec [x]_\theta = m_\theta$. Thus $[y]_\theta = m_\theta$ and $y \in m_\theta$ by the minimality of m_θ in K/θ.

(2) m_θ is closed under $+$. For all $x, y \in m_\theta$, we have $[x]_\theta \prec [y]_\theta$ and $(x+y,y)\in\theta$. Then $(x+y+y,y)=(x+y,y)\in\theta$ and $[x+y]_\theta \prec [y]_\theta = m_\theta$ and $x+y\in m_\theta$ by the minimality of m_θ in K/θ.

By Proposition 2.2, m_θ is an ideal. □

For an ideal $I \in \mathcal{I}(K)$, we define a binary relation $\theta(I)$ on K by

$$(x,y)\in\theta(I) \text{ if and only if } \exists a \in I,\ a+x=a+y.$$

Obviously, $\theta(I)$ is an equivalence relation on K.

Lemma 2.5. *$(x,y)\in\theta(I)$ if and only if $a+x=b+y$ for some $a,b\in I$.*

Proof. The necessity is obvious. If $a+x=b+y$ for some $a,b\in I$, then $c=a+b\in I$ and $c+x=a+b+x=a+b+a+x=a+b+b+y=a+b+y=c+y$. So $(x,y)\in\theta(I)$. □

Proposition 2.6. *If $I\in\mathcal{I}(K)$, then $\theta(I)\in\mathcal{RC}(K)$.*

Proof. (C) Suppose that $(x,y),(u,v)\in\theta(I)$. Then there exist $a,b\in I$ such that $a+x=a+y$ and $b+u=b+v$. Putting $c=a+b$, we have $c\in I$ and $c+(x+u)=c+(y+v)$. Thus $(x+u,y+v)\in\theta(I)$. Also, we have $(a+x)(b+u)=(a+y)(b+v)$ and $(ab+xb+au)+xu=(ab+yb+av)+yv$. Since I is a lower set, $ab+xb+au\leq a+b+a=a+b=c\in I$ and $ab+yb+av\leq a+b+a=a+b=c\in I$, we have $ab+xb+au, ab+yb+av\in I$. By Lemma 2.5, $(xu,yv)\in\theta(I)$.

(R1) I is the least congruence class of $\theta(I)$. In fact, let $x\in I$. For all $y\in K$ with $(x,y)\in\theta(I)$, we have $a+x=a+y$ for some $a\in I$. It follows that $a+y\in I$ and $y\in I$ since I is a lower set. Thus $[x]_{\theta(I)}\subseteq I$. Conversely, $\forall y\in I$, we have $x+y\in I$ and $(x+y)+x=x+y=(x+y)+y$. Then $(x,y)\in\theta(I)$ and $y\in[x]_{\theta(I)}$, $I\subseteq[x]_{\theta(I)}$. Hence $I=[x]_{\theta(I)}$. For all $y\in K$, we have $x+(x+y)=x+y$ and $(x+y,y)\in\theta(I)$. Therefore $[x]_{\theta(I)}\prec[y]_{\theta(I)}$. Thus I is the least congruence class of $\theta(I)$, i.e., $I=m_{\theta(I)}$.

(R2) obviously holds by the definition of $\theta(I)$. □

Proposition 2.7. *If $I\in\mathcal{I}(K)$, then $I(\theta(I))=I$.*

Proof. Immediately by the proof of Proposition 2.6(R1). □

Theorem 2.8. *If $\theta\in\mathcal{C}(K)$ is regular, then $\theta(I(\theta))=\theta$.*

Proof. For all $(x,y)\in\theta$, there exists $a\in m_\theta=I(\theta)$ such that $a+x=a+y$. By the definition of $\theta(I(\theta))$, we have $(x,y)\in\theta(I(\theta)$. Thus $\theta\subseteq\theta(I(\theta))$. Conversely, suppose that $(x,y)\in\theta(I(\theta))$, there exists $a\in I(\theta)=m_\theta$ such that $a+x=a+y$. Since $[a]_\theta=m_\theta\prec[x]_\theta$, we have $(a+x,x)\in\theta$. Similarly $(a+y,y)\in\theta$ and thus $(x,y)\in\theta$. Hence $\theta(I(\theta))\subseteq\theta$. □

Theorem 2.9. *There is a one-to-one correspondence between* $\mathcal{I}(K)$ *and* $\mathcal{RC}(K)$.

Example 2.10. (1) Let $K = \{0, a, b, c, d, 1\}$ be the incline in Example 1.1(6). $I = \{0, a\}$ is an ideal and

$$\theta_I = (\{0, a\} \times \{0, a\}) \cup (\{1, b, c, d\} \times \{1, b, c, d\})$$

is a nontrivial regular congruence.

(2) Let $K = \{0, a, b, 1\}$ be a Boolean algebra, where $0 < a, b < 1$ and $a//b$. $I = \{0, a\}$ is an ideal of $(K, \vee, \wedge)$ and $\theta_I = (\{0, a\} \times \{0, a\}) \cup (\{1, b\} \times \{1, b\})$ is a nontrivial regular congruence.

3. A one-to-one correspondence between filters and normal congruences

Definition 3.1. A subincline F of K is called a *filter* if it is an upper set, i.e., $x \in F,\ y \in K,\ x \leq y$ implies $y \in F$.

Proposition 3.2. *If F is a nonempty subset of K, then the following statements are equivalent*:

(1) *F is a filter*;
(2) *F is an upper set and closed under $\cdot$* ;
(3) *F is closed under $\cdot$ and $a + b \in F$ for all $a \in F,\ b \in K$.*

Proof. Trivial since the condition "$a+b \in F$ for all $a \in F,\ b \in K$" is equivalent to that "F is an upper set". □

We denote by $\mathcal{F}(K)$ the family of all filters of K. Then $\mathcal{F}(K)$ is a poset under the set-inclusion. The largest element of $\mathcal{F}(K)$ is K.

Definition 3.3. A congruence $\theta \in \mathcal{C}(K)$ is said to be *normal* if

(N1) There exists a largest element in $(K/\theta,\ \prec)$, denoted by M_θ;
(N2) $(x, xy) \in \theta$ for all $x,\ y \in M_\theta$;
(N3) $(x, y) \in \theta$ if and only if $a + x \in M_\theta \Leftrightarrow a + y \in M_\theta$ for all $a \in K$.

The family of all normal congruences on K is denoted by $\mathcal{NC}(K)$.

For a filter $F \in \mathcal{F}(K)$, we define a binary relation $\theta(F)$ on K by

$$(x, y) \in \theta(F) \text{ if and only if } \forall a \in K,\ a + x \in F \Leftrightarrow a + y \in F.$$

Obviously, $\theta(F)$ is an equivalence relation on K.

Theorem 3.4. *If $F \in \mathcal{F}(K)$, then $\theta(F) \in \mathcal{NC}(K)$.*

Proof. (C) $\theta(F)$ is a congruence. Suppose that $(x, y), (u, v) \in \theta(F)$. For any $a \in K$, $a+x \in F$ if and only if $a+y \in F$, and $a+u \in F$ if and only if $a+v \in F$. So $a + (x + u) = (a + u) + x \in F$ if and only if $(a + y) + u = (a + u) + y \in F$ if and only if $a + (y + v) = (a + y) + v \in F$. Hence, $(x + u, y + v) \in \theta(F)$. If $a + xu \in F$, then $a + x, a + u \in F$ since $a + xu \leq a + x$ and $a + xu \leq a + u$ and F is an upper set. It follows that $a + y, a + v \in F$ and $(a + y)(a + v) \in F$. Since

$(a+y)(a+v) = a^2+ya+av+yv \leq a+a+a+yv = a+yv$, we have $a+yv \in F$. Analogously, if $a+yv \in F$, then $a+xu \in F$. Therefore, $(xu, yv) \in \theta(F)$.

(N1) F is the largest element in $(K/\theta, \prec)$. Firstly, it is easy to verify that $F = [a]_{\theta(F)}$ for any $a \in F$. For all $x \in F$, $y \in K$, we have $y+x \in F$ and $(y+x, x) \in \theta$. Then $[y]_{\theta(F)} \prec [x]_{\theta(F)} = F$. Hence $F = M_{\theta(F)}$.

(N2) For all $x, y \in M_{\theta(F)} = F$, we have $xy \in F$ since F is a filter. Then $[x]_{\theta(F)} \prec [xy]_{\theta(F)}$ by (N1). Then $(x+xy, xy) = (x, xy) \in \theta(F)$.

(N3) holds by the definition of $\theta(F)$. □

For a normal congruence θ, we define $F(\theta) = M_\theta$. Then $F(\theta)$ is a filter by (N1) and N(2).

Corollary 3.5. (1) $F = M_{\theta(F)}$.

(2) *Let θ be a normal congruence. Then $\theta(F(\theta)) = \theta$.*

Proof. (1) is immediately from the proof of Theorem 3.4(N1) and (2) is from (N3). □

Lemma 3.6. *If $F \in \mathcal{F}(K)$, then $F(\theta(F)) = F$.*

Proof. $F(\theta(F)) = M_{\theta(F)} = F$. □

Theorem 3.7. *There is a one-to-one correspondence between $\mathcal{F}(K)$ and $\mathcal{NC}(K)$.*

Example 3.8. (1) Let K be the incline in Example 1.1(6). $F = \{1, b, c, d\}$ is a filter. Then $\theta_F = (\{1, b, c, d\} \times \{1, b, c, d\}) \cup (\{0, a\} \times \{0, a\})$, which is a nontrivial normal congruence.

(2) Let $K = \{0, a, b, 1\}$ be the Boolean algebra in Example 2.10(2). Then $\theta = (\{1, a\} \times \{1, a)\}) \cup (\{0, b\} \times \{0, b)\})$ is a nontrivial normal congruence.

4. Remarks

Remark 4.1. There is no direct implication relation between regular congruences and normal congruences even if K is a finite distributive lattice.

Example 4.2. Let $K = \{0, a, b, c, 1\}$ ordered by Figure 2. Then K is a distributive lattice. A congruence $\theta = (\{1, a, b, c\} \times \{1, a, b, c\}) \cup \{(0, 0)\}$ is normal (which is induced by the filter $F = \{1, a, b, c\}$), but not regular. In fact, $m_\theta = \{0\}$, $(a, b) \in \theta$, while $a \vee 0 \neq b \vee 0$.

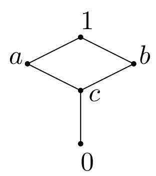

Figure 2

Example 4.3. Let $K = \{0, a, b, c, 1\}$ ordered by Figure 3. Then K is a distributive lattice. A congruence Δ_K is regular (which is induced by the ideal $\{0\}$), but not normal. In fact, $M_\theta = \{1\}$ and (a, b) is in $\theta(\{1\})$, not in θ.

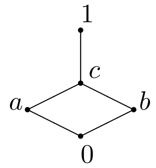

Figure 3

Example 4.4. All congruences in Example 2.10 and Example 3.8 are normal and regular simultaneously.

Example 4.5. In Figure 4, a congruence

$$\{1, a', b', c'\} \times \{1, a', b', c'\} \cup \{0, a, b, c\} \times \{0, a, b, c\}$$

is neither normal nor regular. For any $x \in K$, $x+c \in [1]_\theta$ if and only if $c' \le x+c$ if and only if $c' \le x$ if and only if $c' \le x + c'$ if and only if $x + c' \in [1]_\theta$, but $(c, c') \notin \theta$. Hence, θ is not normal. For any $x \in [0]_\theta = \{0, a, b, c\}$, $x+a' \ne x+b'$, but $(a', b') \in \theta$. Thus θ is not regular.

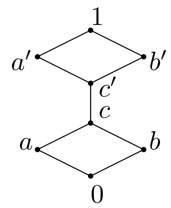

Figure 4

Remark 4.6. An ideal I of a poset is a directed (i.e., $\forall a,\ b \in I,\ \exists\ c \in I,$ $a \le c,\ b \le c$) lower set. An ideal I in a $\vee$-semilattice is a lower set and closed under operation $\vee$. We can see that an ideal of an incline K is an ideal of K as a $\vee$-semilattice, and it is not directly related to operation $\cdot$. But a filter of an incline is directly related to two operations $+$ and $\cdot$ as well as to the induced partial order $\le$. Hence, we may say that in studying of algebraic structures of inclines, filters play more important role than ideals.

References

[1] S. S. Ahn, Y. B. Jun, and H. S. Kim, *Ideals and quotients of incline algebras*, Commun. Korean Math. Soc. **16** (2001), no. 4, 573–583.

[2] S. S. Ahn and H. S. Kim, *On r-ideals in incline algebras*, Commun. Korean Math. Soc. **17** (2002), no. 2, 229–235.

[3] B. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Mo., 1974.

[4] Z. Q. Cao, K. H. Kim, and F. W. Roush, *Incline Algebra and Applications*, John Wiley & Sons, New York, 1984.
[5] S. C. Han, *A Study on Incline Matrices with Indices*, Ph.D. Dissertation, Beijing Normal University, China, 2005.
[6] Y. B. Jun, S. S. Ahn, and H. S. Kim, *Fuzzy subinclines* (*ideals*) *of incline algebras*, Fuzzy Sets and Systems **123** (2001), no. 2, 217–225.
[7] K. H. Kim and F. W. Roush, *Inclines of algebraic structures*, Fuzzy Sets and Systems **72** (1995), no. 2, 189–196.
[8] ———, *Inclines and incline matrices: a survey*, Linear Algebra Appl. **379** (2004), 457–473.
[9] K. H. Kim, F. W. Roush, and G. Markowsky, *Representation of inclines*, Algebra Colloq. **4** (1997), no. 4, 461–470.
[10] E. Turunen, *Mathematics Behind Fuzzy Logic*, Physica-Verlag, Heidelberg, 1999.

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